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Off-Shell Solution of Scattering Equations

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A method of solving scattering equations is discussed which enables one to exhibit explicitly the off-shell solutions. Results are derived for nonrelativistic scattering from superpositions of exponential and Yukawa potentials. As a byproduct, we make use of our representation for the partial-wave amplitude to prove meromorphy in the left half l plane, as well as giving simple derivations of some well-known results by Regge. The technique is easily generalized to other equations and potentials; in particular, we state the result obtained for the Blankenbecler-Sugar equation.

I. INTRODUCTION

THIS paper is concerned with the development of a method of solving integral equations of the type commonly written for scattering amplitudes. The method is applicable to a large class of scattering equations, and enables one to construct explicit representations for the off-shell solutions of these equations. The solutions are expressed as the ratio of two convergent series in the coupling parameter, the terms of which are generated by a simple iterative procedure. As we shall show, the form of the solutions is particularly convenient for discussing analyticity properties in the energy and angular momentum.

The basic idea behind our approach lies in the simple observation that an analytic function is completely determined by its singularities, provided that it approaches zero at infinity. The integral equations which we shall consider are written in terms of a single real variable: the absolute value of the off-shell center-of-mass (c.m.) momentum. Their solution is therefore some function defined on the positive real axis. By working directly with the integral equation, we shall show that it is possible to analytically continue this function into the complex plane. Furthermore, in a large number of cases it is possible to do so in such a way that we can explicitly determine all its singularities, i.e., the positions of all its cuts and poles, and the values of the corresponding discontinuities and residues. Having done so, it is trivial to write a representation for the function.

We will illustrate this approach in Sec. II by considering the s-wave Lippmann-Schwinger (LS) equation for an exponential potential. By a series of simple arguments we will obtain an explicit form for the off-shell solution. The on-shell version of this solution coincides with the well-known analytic result. In Secs. III and IV we again consider the LS equation, extending the result of Sec. II to superpositions of exponential and Yukawa potentials and arbitrary l.

In Sec. V we discuss the properties of these solutions when l is allowed to become complex. The form of our solutions is such that we easily prove and extend some well-known results of Regge. In particular, we show that the partial-wave amplitude is meromorphic in the entire l plane. This is to be compared with the results of Regge,[1] who proved meromorphy for $\mathrm{Re}\,l \geqslant -\frac{1}{2}$, and Mandelstam,[2] who extended this domain to include the entire left-hand plane, provided that the superposition weight function decreases exponentially. Our condition on the weight function is simply that it approaches zero at infinite mass.

It was also shown by Regge that the partial-wave amplitude approaches the Born term as $|l| \to \infty$, for $\mathrm{Re}\,l \geqslant -\frac{1}{2}$, and physical values of the energy. We extend this result to a domain in the complex energy plane, and allow l to approach infinity in any direction such that $|\arg(l-l_0)| \leqslant \frac{1}{2}\pi$, where l_0 is any finite complex number.

Finally, in Sec. VI we discuss applications in addition to those of potential scattering. In particular, we give a result obtained for the Blankenbecler-Sugar equation.

II. EXPONENTIAL POTENTIAL

The technique mentioned above is best illustrated by applying it to a particular example. We consider nonrelativistic potential scattering for an exponential potential $V(r)=\mu^2 G e^{-\mu r}$. The $l=0$ partial-wave LS equation then has the form

$$a_0(p',p;\,s)=v_0(p',p)-\int_0^\infty \frac{dp''p''^2}{p''^2-s-i\epsilon}\times v_0(p',p'')a_0(p'',p;\,s)\,, \quad (1)$$

where

$$v_0(p',p)=\frac{\mu^3 G}{\pi p p'}\left[\frac{1}{(p'-p)^2+\mu^2}-\frac{1}{(p'+p)^2+\mu^2}\right], \quad (2)$$

and $a_0(\sqrt{s},\sqrt{s};\,s)$ is the s-wave scattering amplitude with normalization such that $a_0(\sqrt{s},\sqrt{s};\,s)=-2e^{i\delta_0}\times(\sin\delta_0)/\pi\sqrt{s}$. The solution of this problem is well known,[3] and δ_0 is expressible in terms of Bessel functions of imaginary argument and order. Our method, however, does not rest upon the special features which make this problem exactly solvable. We will work directly with Eq. (1) for the off-shell amplitude $a_0(p',p;\,s)$, assuming only that a solution exists for real p', which is a fact that has been well established. In obtaining our solution, we assume that $0<\mathrm{Im}\sqrt{s}<\mu$. Since our result will be analytic in s, this imposes no restriction.

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[1] T. Regge, Nuovo Cimento 14, 951 (1959).
[2] S. Mandelstam, Ann. Phys. (N. Y.) 19, 254 (1959).
[3] H. A. Bethe and R. Bacher, Rev. Mod. Phys. 8, 111 (1936).

We begin by assuming that $a_0(p'',p;s)$ is a solution of Eq. (1) with p fixed and positive. We may then regard the right side of (1) as a representation of $a_0(p',p;s)$, and use it to extend p' to complex and negative values. Since $v_0(p',p)=v_0(-p',p)$ it follows that

$$a_0(-p',p;s)=a_0(p',p;s)\,, \tag{3}$$

which defines a_0 for negative real values of p'.

We may rewrite (1) in the form

$$a_0(p',p;s)=\frac{\mu^3 G}{\pi p p'}\left[\frac{1}{(p'-p^2)+\mu^2}-\frac{1}{(p'+p)^2+\mu^2}\right]-\frac{\mu^3 G}{\pi p'}\int_{-\infty}^{\infty}\frac{dp''p''a_0(p'',p;s)}{(p''^2-s)[(p'-p'')^2+\mu^2]}\,. \tag{4}$$

Using Eq. (4), we may determine the singularities of $a_0(p',p;s)$ in the entire p' plane, for fixed p and s. Considering first the inhomogeneous term, it is clear that a_0 will have four poles at $p'=\pm(\pm p+i\mu)$, with the respective residues $\pm(\mu^2G/2\pi ip)\times[1/(p\pm i\mu)]$. Additional singularities arise due to the integral term in Eq. (4). It is sufficient to consider only singularities for which $\mathrm{Im}p'>0$; from Eq. (3) we may later determine all singularities in the lower half-plane. The integral term is analytic for p' in the strip $-\mu<\mathrm{Im}p'<\mu$, thus $a_0(p',p;s)$ is also analytic in this strip. We define

$$I_\pm(p')=\int_{-\infty}^{\infty}\frac{dp''p''}{p''^2-s}\frac{a_0(p'',p;s)}{(p'-p''-i\mu\pm i\epsilon)(p'-p''+i\mu)} \tag{5}$$

and

$$\Delta(p')=-\frac{\pi}{\mu}\frac{p'-i\mu}{(p'-i\mu)^2-s}a_0(p'-i\mu,\,p;s)\,.$$

The function $I_-(p')$ is analytic for $\mathrm{Im}p'<\mu$, while $I_+(p')$ is analytic for $\mu<\mathrm{Im}p'$. From the analyticity of $a_0(p',p;s)$ in $0\leqslant\mathrm{Im}p'<\mu$, we see that $\Delta(p')$ is analytic except for poles in the strip $\mu\leqslant\mathrm{Im}p'<2\mu$.

It is easy to verify that the function $F(p')$, defined by

$$\begin{aligned} F(p')&\equiv I_-(p') \quad \text{for} \quad 0\leqslant\mathrm{Im}p'<\mu\,,\\ F(p')&\equiv I_+(p')+\Delta(p') \quad \text{for} \quad \mu<\mathrm{Im}p'<2\mu\,, \end{aligned} \tag{6}$$

is continuous across the line $\mathrm{Im}p'=\mu$. From the above it then follows that $F(p')$ is analytic, except for poles, in the strip

$$0\leq\mathrm{Im}p'<2\mu\,. \tag{7}$$

We may now define the analytic continuation of $a_0(p',p;s)$ across the line $\mathrm{Im}p'=\mu$, and throughout strip (7), by writing

$$a_0(p',p;s)=\frac{\mu^3 G}{\pi p p'}\left[\frac{1}{(p'-p)^2+\mu^2}-\frac{1}{(p'+p)^2+\mu^2}\right]-\frac{\mu^3 G}{\pi p'}F(p')\,. \tag{8}$$

Within the original strip $0\leqslant\mathrm{Im}p'<\mu$, Eq. (8) coincides with Eq. (4). Analyticity of a_0 in the extended strip (7) then follows from the analyticity of $F(p')$ and the inhomogeneous term.

The singularities of $a_0(p'p;s)$ in this extended strip consist of the inhomogeneous term poles at $p'=\pm p+i\mu$ and the singularities of $\Delta(p')$. From Eq. (5) we see that $\Delta(p')$ has a pole at $p'=i\mu+\sqrt{s}$ with residue $(\pi/2\mu)\times a_0(\sqrt{s},p;s)$. It follows that $a_0(p',p;s)$ has a pole at $p'=i\mu+\sqrt{s}$ with residue

$$[-\mu^2G/2(i\mu+\sqrt{s})]a_0(\sqrt{s},p;s)\,.$$

As a result of the above procedure, we have extended our knowledge of the analytic structure of $a_0(p',p;s)$ from the strip $0\leqslant\mathrm{Im}p'<\mu$ to the strip $0\leqslant\mathrm{Im}p'<2\mu$. Returning to Eq. (5), we define

$$\Delta(p')=-\frac{\pi}{\mu}\frac{p'-i\mu}{(p'-i\mu)^2-s}a_0(p'-i\mu,\,p;s) \tag{9}$$

for the additional region $2\mu\leqslant\mathrm{Im}p'<3\mu$ and extend Eq. (8) to this region. The function $F(p')$ is now analytic in the strip $0\leqslant\mathrm{Im}p'<3\mu$ except for the singularities already mentioned and additional poles at $p'=\pm p+2i\mu$ and $p'=(\sqrt{s})+2i\mu$. The latter arise from the previously determined singularities of $a_0(p'-i\mu,p;s)$ through Eq. (9). By Eq. (8), $a_0(p',p;s)$ is now analytic in the extended strip $0\leqslant\mathrm{Im}p'<3\mu$ except for known poles.

It is clear that we may repeat this procedure indefinitely, obtaining exact knowledge of the singularities of $a_0(p',p;s)$ in the entire p' plane.

Combining (8) and (6) we may write

$$a_0(p',p;s)=v_0(p',p)-\frac{\mu^3G}{\pi p'}I_+(p')-\frac{\mu^2G}{p'}\frac{p'-i\mu}{(p'-i\mu)^2-s}a_0(p'-i\mu,\,p;s)\,, \tag{10}$$

which is valid for all p' such that $\mathrm{Im}p'>\mu$. Equation (10) provides an iterative formula from which we may determine the singularities of $a_0(p',p;s)$ strip by strip. We find that a_0 is analytic everywhere except for poles at

$$\begin{aligned} &p'=\pm(im\mu+p), \quad \text{with residue} \quad \pm R_m(-p;s)\,,\\ &p'=\pm(im\mu-p), \quad \text{with residue} \quad \pm R_m(p;s)\,,\\ &m=1,2,3,\cdots \end{aligned} \tag{11}$$

and

$$p'=\pm(im\mu+\sqrt{s})\,,$$

with residue

$$\mp i\pi(\sqrt{s})a_0(\sqrt{s},p;s)R_m(-\sqrt{s};s)\,.$$

The residue functions $R_m(p;s)$ are defined by

$$R_1(p;s)=\frac{\mu^2G}{2\pi ip}\frac{1}{p-i\mu} \tag{12}$$

and

$$R_m(p;s)=-\frac{(-\mu^2G)^m}{2\pi ip}\frac{1}{p-im\mu}\frac{1}{[p-i(m-1)\mu]^2-s}\times\frac{1}{[p-i(m-2)\mu]^2-s}\cdots\frac{1}{[p-i\mu]^2-s}$$

for $m\geqslant 2$.

It is clear from Eq. (4) and the resulting development that $|a_0(p',p;s)|\xrightarrow[|p'|\to\infty]{}0$ at least as fast as $1/|p'|$. With the above information it is then trivial to write the solution of Eq. (1). We obtain

$$a_0(p',p;s)=-2\sum_{m=1}^{\infty}\frac{R_m(-p;s)(im\mu+p)}{(im\mu+p)^2-p'^2}-2\sum_{m=1}^{\infty}\frac{R_m(p;s)(im\mu-p)}{(im\mu-p)^2-p'^2}+2\pi i(\sqrt{s})a_0(\sqrt{s},p;s)\times\sum_{m=1}^{\infty}\frac{R_m(-\sqrt{s};s)(im\mu+\sqrt{s})}{(im\mu+\sqrt{s})^2-p'^2}. \quad (13)$$

If we define

$$r(p',p;s)\equiv-2\sum_{m=1}^{\infty}\frac{(im\mu-p)R_m(p;s)}{(im\mu-p)^2-p'^2}, \quad (14)$$

we may write (13) more compactly in the form

$$a_0(p',p;s)=r(p',p;s)+r(p',-p;s)-i\pi(\sqrt{s})r(p',-\sqrt{s};s)a_0(\sqrt{s},p;s), \quad (15)$$

where, by setting $p'=\sqrt{s}$, we have

$$a_0(\sqrt{s},p;s)=\frac{r(\sqrt{s},p;s)+r(\sqrt{s},-p;s)}{1+i\pi(\sqrt{s})r(\sqrt{s},-\sqrt{s};s)}. \quad (16)$$

The off-shell solution is thus

$$a_0(p',p;s)=r(p',p;s)+r(p',-p;s)-i\pi(\sqrt{s})\times r(p',-\sqrt{s};s)\frac{r(\sqrt{s},p;s)+r(\sqrt{s},-p;s)}{1+i\pi(\sqrt{s})r(\sqrt{s},-\sqrt{s};s)}. \quad (15')$$

Defining $\hat{f}_0(\sqrt{s})\equiv r(\sqrt{s},\sqrt{s};s)$, the on-shell solution takes the simple form

$$a_0(\sqrt{s},\sqrt{s};s)=\frac{\hat{f}_0(\sqrt{s})+\hat{f}_0(-\sqrt{s})}{1+i\pi(\sqrt{s})\hat{f}_0(-\sqrt{s})}\equiv\frac{N(s)}{D(s)}. \quad (17)$$

This result can be put in more familar form by noting that

$$\frac{(im\mu-\sqrt{s})R_m(\sqrt{s};s)}{(im\mu-\sqrt{s})^2-s}=\frac{1}{2\pi i\sqrt{s}}\frac{G^m}{m!}\times\sum_{n=1}^{m}\frac{(-1)^{n+1}}{(m-n)!}\frac{1}{(n-1)!}\frac{1}{n+i2(\sqrt{s})/\mu}. \quad (18)$$

It then follows that

$$D(s)=1+\sum_{m=1}^{\infty}\frac{G^m}{m!}\sum_{n=1}^{m}\frac{(-1)^{n+1}}{(m-n)!}\times\frac{1}{(n-1)!}\frac{1}{n-i2(\sqrt{s})/\mu}. \quad (19)$$

Equation (19), for $\sqrt{s}=k>0$, is simply the series expansion for the well-known result[4]

$$D(k)=J_{2ik}(-2i\sqrt{G})(-i\sqrt{G})^{-2ik}/\Gamma(1-2ik). \quad (20)$$

From the form of Eq. (17) and the fact that

$$\lim_{k\to\infty}[k\hat{f}_0(-k)]=0, \quad (21)$$

which is easily proved, we can make the identification

$$f(k,0)=1-i\pi k\hat{f}_0(k), \quad (22)$$

where $f(k,0)$ is the Jost function.

Application of our technique to the exponential potential thus results in an explicit form for the scattering amplitude which coincides with the known solution. In deriving this result, we have not made use of any information other than the integral equation itself. As we shall demonstrate, the technique employed may be applied equally well to a wide range of potentials and integral equations.

In addition to determining $a_0(\sqrt{s},\sqrt{s};s)$, we have also determined explicit forms for the off-shell and half-on-shell amplitudes, as well as completely determining their analytic properties. This information is essential in formulating many-body equations such as those proposed for three particles by Faddeev.[5]

III. SUPERPOSITION OF YUKAWA POTENTIALS

With only minor modifications, the derivation given above can be applied to a considerably more general situation. We consider a superposition of Yukawa potentials, such that

$$V(r)=2\pi^2\int_{\mu}^{\infty}d\alpha\,\sigma(\alpha)\frac{e^{-\alpha r}}{r}. \quad (23)$$

Below, we shall put some mild restrictions on $\sigma(\alpha)$. The integral equation for partial-wave amplitude $a_l(p',p;s)$ is given by

$$a_l(p',p;s)=V_l(p',p)-\int_0^{\infty}\frac{dp''p''^2}{p''^2-s}\times V_l(p',p'')a_l(p'',p;s) \quad (24)$$

with

$$V_l(p',p)=\frac{2\pi}{pp'}\int_{\mu}^{\infty}d\alpha\,\sigma(\alpha)Q_l\left(\frac{p'^2+p^2+\alpha^2}{2pp'}\right).$$

[4] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

[5] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

We now have

$$a_l(-p',\, p;\, s)=(-1)^l a_l(p',p;\, s). \qquad (25)$$

Keeping p fixed, as before, we see that $V_l(p',p)$ is analytic in the p' plane, except for cuts along the lines $p'=\pm(p\pm i\alpha)$ for $\mu\leqslant\alpha<\infty$. Comparing this with the exponential potential, we see that we now have branch points where we previously had poles. As we shall see, this analogy will continue to hold when we consider the analytic continuation of a_l into the entire complex p' plane. To avoid repetition, we will sketch a derivation which exploits this similarity.

We first note that the Q_l function appearing in Eq. (24) may be represented in the form[6]

$$Q_l(\eta[p',p])=\tfrac{1}{2}P_l(\eta[p',p])\times\ln\left[\frac{(p'+p)^2+\alpha^2}{(p'-p)^2+\alpha^2}\right]+R_{l-1}(\eta), \qquad (26)$$

where $\eta[p',p]\equiv(p'^2+p^2+\alpha^2)/2pp'$ and R_{l-1} is a polynomial of degree $l-1$ in $\eta[p',p]$. It follows that we may represent $V_l(p',p)$ in the following form:

$$V_l(p',p)=\frac{2\pi}{p'p}\int_\mu^\infty d\alpha\,\alpha\rho_l(\alpha;\,p',p)\left[\frac{1}{(p'-p)^2+\alpha^2}-\frac{1}{(p'+p)^2+\alpha^2}\right]+\frac{2\pi}{pp'}\int_\mu^\infty d\alpha\,\sigma(\alpha)R_{l-1}(\eta[p',p])+\frac{\pi}{p'p}\left\{\rho_l(\alpha;\,p',p)\ln\left[\frac{(p'+p)^2+\alpha^2}{(p'-p)^2+\alpha^2}\right]\right\}_{\alpha=\infty}, \qquad (27)$$

where we defined

$$\rho_l(\alpha;\,p',p)\equiv\int_\mu^\alpha d\beta\,\sigma(\beta)P_l\left(\frac{p'^2+p^2+\beta^2}{2pp'}\right). \qquad (28)$$

So far, however, we have said nothing about conditions on $\sigma(\alpha)$, although we have tacitly assumed it to be such that Eq. (24) for $V_l(p',p)$ is defined for all real p' and p. For this to be true, it is sufficient that $\sigma(\alpha)$ satisfies the two conditions:

$$\begin{aligned}&\text{(a)} && \sigma(\alpha)\text{ is integrable},\\ &\text{(b)} && |\sigma(\alpha)|<K\alpha^{-\epsilon},\end{aligned} \qquad (29)$$

with $\epsilon>0$ but arbitrarily small. It is not our purpose here to determine in what way condition (b) may be weakened, and we will henceforth assume that Eq. (29) is satisfied.

Although this guarantees that the integral defining $V_l(p',p)$ converges, it is by no means certain that each term of Eq. (27) will be separately finite. To avoid this difficulty, we will approximate $V_l(p',p)$ by replacing ∞ everywhere in Eq. (27) by a large but finite number M. The function $V_l{}^M(p',p)$ so defined has analytic properties which are identical to those of $V_l(p',p)$ within the strip $-M<\text{Im}p'<M$. Once having obtained the solution for fixed M, we will take the limit $M\to\infty$ and show that it converges to the correct result.

We thus rewrite (27) in the form

$$V_l{}^M(p',p)\equiv\frac{2\pi}{p'p}\int_\mu^M d\alpha\,\alpha\rho_l(\alpha;\,p',p)\times\left[\frac{1}{(p'-p)^2+\alpha^2}-\frac{1}{(p'+p)^2+\alpha^2}\right]+\phi_l{}^M(p',p). \qquad (30)$$

The function $\phi_l{}^M$, so defined, has no singularities within the strip $-M<\text{Im}p'<M$, and we may therefore regard $V_l{}^M(p',p)$ formally as being a superposition of exponential s-wave potentials with respect to its analytic properties.

Applying the procedure of Sec. II, we find that $V_l{}^M(p',p)$ is analytic in the strip $0\leqslant\text{Im}p'<M$, except for cuts along the lines $p'=\pm p+i\alpha$, with $\alpha\geqslant\mu$;

$$V_l{}^M(p+i\alpha+\epsilon,\,p)-V_l{}^M(p+i\alpha-\epsilon,\,p)\underset{\epsilon\to0}{\longrightarrow}\times[-2\pi^2 i/(p+i\alpha)p]\rho_l(\alpha;\,p+i\alpha,\,p), \qquad (31)$$

while

$$V_l{}^M(-p+i\alpha+\epsilon,\,p)-V_l{}^M(-p+i\alpha-\epsilon,\,p)\underset{\epsilon\to0}{\longrightarrow}\times[2\pi^2 i/(-p+i\alpha)p]\rho_l(\alpha;\,-p+i\alpha,\,p).$$

Define

$$F_l(\alpha;\,p)'\equiv\int_{-\infty}^\infty\frac{dp''p''}{p''^2-s}\frac{\rho_l(\alpha;\,p',p'')a_l(p'',p;\,s)}{(p'-p''-i\alpha-i\epsilon)(p'-p''+i\alpha)},$$

for

$$0\leqslant\text{Im}p'<\alpha; \qquad (32)$$

and

$$F_l(\alpha;\,p')\equiv\int_{-\infty}^\infty\frac{dp''p''}{p''^2-s}\times\frac{\rho_l(\alpha;\,p',p'')a_l(p'',p;\,s)}{(p'-p''-i\alpha+i\epsilon)(p'-p''+i\alpha)}+\Delta_l(\alpha;\,p'),$$

for $\alpha<\text{Im}p'<\mu+\alpha$, with

$$\Delta_l(\alpha;\,p')\equiv-\frac{\pi}{\alpha}\frac{p'-i\alpha}{(p'-i\alpha)^2-s}\times\rho_l(\alpha;\,p',\,p'-i\alpha)a_l(p'-i\alpha,\,p;\,s). \qquad (33)$$

$F_l(\alpha;\,p')$ is analytic in the strip $0\leqslant\text{Im}p'<\alpha+\mu$, except for the pole of $\Delta_l(\alpha;\,p')$ at $p'=i\alpha+\sqrt{s}$. The analytic continuation of $a_l(p',p;\,s)$ across the line $\text{Im}p'=\mu$, and throughout the region $0\leqslant\text{Im}p'<2\mu$, is then defined by

$$a_l(p',p;\,s)=V_l{}^M(p',p)-\frac{2\pi}{p'}\int_\mu^M d\alpha\,\alpha F_l(\alpha;\,p')-\frac{1}{2}\int_{-\infty}^\infty\frac{dp''p''^2}{p''^2-s}\phi_l{}^M(p',p'')a_l(p'',p;\,s)+C_l{}^M(p',p), \qquad (34)$$

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[6] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., Inc., New York, 1953), p. 1328.

where $C_l^M(p',p)$ is a correction term which arises from using V_l^M in place of V_l. From the previous discussion, $C_l^M(p',p)$ is analytic in the region $-M<\text{Im}p'<M$. All functions appearing in (34) have been chosen in such a way that the equation is identical with (24) in the strip $0\leqslant\text{Im}p'<\mu$.

Besides the cuts of $V_l^M(p',p)$, the function $a_l(p',p;s)$ will have an additional cut in the strip $0\leqslant\text{Im}p'<2\mu$, coming from the second term in (34). That is, the pole of $F_l(\alpha;p')$ at $p'=i\alpha+\sqrt{s}$ becomes a cut in a_l when we integrate over α. This cut is along the line $p'=i\alpha+\sqrt{s}$ for $\mu\leqslant\alpha<2\mu-\text{Im}\sqrt{s}$. The discontinuity is easily calculated from

$$\begin{aligned}
&a_l(i\alpha'+\sqrt{s}-\epsilon;p;s)-a_l(i\alpha'+\sqrt{s}+\epsilon,p;s)\underset{\epsilon\to0}{\longrightarrow}\\
&[-2\pi/(i\alpha'+\sqrt{s})]\int_\mu^M d\alpha\,\alpha[\Delta_l(\alpha;i\alpha'+\sqrt{s}+\epsilon)\\
&-\Delta_l(\alpha;i\alpha'+\sqrt{s}-\epsilon)]\to[2\pi^3/(i\alpha'+\sqrt{s})]\\
&\times\rho_l(\alpha';i\alpha'+\sqrt{s},\sqrt{s})a_l(\sqrt{s},p;s).
\end{aligned}\tag{35}$$

Since the singularities of $a_l(p',p;s)$ are now known exactly in the strip $0\leqslant\text{Im}p'<2\mu$, we may extend Eqs. (32) and (33) to the new region $2\mu\leqslant\text{Im}p'<3\mu$ and determine all singularities of $\Delta_l(\alpha;p')$ in this region. For example, the cut of $a_l(p',p;s)$ along $p'=i\alpha'+\sqrt{s}$, for $\mu\leqslant\alpha'<2\mu-\text{Im}\sqrt{s}$, produces a cut in $\Delta_l(\alpha;p')$ along $p'=i\alpha+i\alpha'+\sqrt{s}$. Extending (34) to the new strip $2\mu\leqslant\text{Im}p'<3\mu$, we may determine all singularities of a_l in this new region. Equations (32)–(34) together define an iterative procedure from which the cuts and discontinuities in each strip can be computed from the previous strip.

It is now clear that we may apply the arguments of Sec. II to the present case, determining exactly strip by strip all singularities of $a_l(p',p;s)$ in the region $-M<\text{Im}p'<M$. It is also easy to show that

$$\lim_{p'\to0}a_l(p',p;s)=cp'^l\tag{36}$$

and

$$\lim_{|p'|\to\infty}|a_l(p',p;s)|=0.$$

We define the contour C_M to be along the large "rectangle" formed by the lines $\text{Im}p'=\pm M$ and closed at $\pm\text{Re}p'=\infty$. Applying Cauchy's theorem to the function $a_l(p',p;s)/p'^l$, we have

$$\frac{a_l(p',p;s)}{p'^l}=\frac{1}{2\pi i}\oint_{C_M}\frac{dp''a_l(p'',p;s)}{p''^l(p''-p')}-\Sigma_{\text{disk}}{}^M,\tag{37}$$

where $\Sigma_{\text{disk}}{}^M$ represents a sum of integrals taken along the cuts of $a_l(p',p;s)$ within C_M. The procedure above has provided the information to calculate $\Sigma_{\text{disk}}{}^M$ and shown us that a_l has no poles. The integral term is the only unknown in Eq. (37), which is exact. If we take the limit $M\to\infty$, the integral vanished by (36), while $\Sigma_{\text{disk}}{}^M$ approaches a well-defined limit. This establishes the desired result. The final solution can be written in the form

$$\begin{aligned}
a_l(p';p,s)=&r_l(p',p;s)+(-1)^lr_l(p',-p;s)\\
&-(-1)^li\pi(\sqrt{s})r_l(p',-\sqrt{s};s)a_l(\sqrt{s},p;s),
\end{aligned}\tag{38}$$

which, by setting $p'=\sqrt{s}$, becomes

$$a_l(\sqrt{s},p;s)=\frac{r_l(\sqrt{s},p;s)+(-1)^lr_l(\sqrt{s},-p;s)}{1+i\pi(\sqrt{s})(-1)^lr_l(\sqrt{s},-\sqrt{s};s)}.\tag{39}$$

The off-shell solution is then

$$\begin{aligned}
a_l(p',p;s)=&r_l(p',p;s)+(-1)^l\\
&\times r_l(p',-p;s)-(-1)^li\pi(\sqrt{s})r_l(p',-\sqrt{s};s)\\
&\times\frac{r_l(\sqrt{s},p;s)+(-1)^lr_l(\sqrt{s},-p;s)}{1+i\pi(\sqrt{s})(-1)^lr_l(\sqrt{s},-\sqrt{s};s)}.
\end{aligned}\tag{38'}$$

The function $r_l(p',p;s)$ is defined by

$$r_l(p',p;s)\equiv\frac{2\pi i}{p}\left(\frac{ip'}{\mu}\right)^l\times\sum_{m=1}^\infty\int_1^\infty\frac{d\alpha\,d_l{}^m(\alpha;p;s)}{(\alpha+ip/\mu)^l[(\alpha+ip/\mu)^2+p'^2/\mu^2]}\tag{40}$$

in terms of functions $d_l{}^m(\alpha;p;s)$ defined by the following iterative procedure:

$$\begin{aligned}
&d_l{}^1(\alpha;p;s)\equiv\int_1^\alpha d\beta\,\bar\sigma(\beta)P_l\left(1+\frac{\alpha^2-\beta^2}{2(ip/\mu)(\alpha+ip/\mu)}\right),\\
&d_l{}^m(\alpha;p;s)=2\pi^2\theta(\alpha-m)\\
&\qquad\times\int_{m-1}^{\alpha-1}\frac{d\beta\,\bar\rho_l(\alpha,\beta;p)d_l{}^{m-1}(\beta;p;s)}{(\beta+ip/\mu)^2+s/\mu^2}
\end{aligned}\tag{41}$$

for $m\geqslant2$. Here, in order to make our integrals dimensionless, we have defined new quantities $\bar\sigma$ and $\bar\rho_l$ by the relations

$$\begin{aligned}
&\bar\sigma(\beta)\equiv\sigma(\beta\mu),\\
&\bar\rho_l(\alpha,\beta;p)=(1/\mu)\rho_l(\mu\alpha-\mu\beta;i\mu\alpha-p,i\mu\beta-p)\\
&\qquad=\int_1^{\alpha-\beta}d\beta'\bar\sigma(\beta')\\
&\qquad\qquad\times P_l\left(1+\frac{(\alpha-\beta)^2-\beta'^2}{2(\alpha+ip/\mu)(\beta+ip/\mu)}\right).
\end{aligned}\tag{42}$$

The solution for on-shell amplitude $a_l(\sqrt{s},\sqrt{s};s)$ has the particular form

$$a_l(\sqrt{s},\sqrt{s};s)=\frac{\hat f_l(\sqrt{s})+\hat f_l(-\sqrt{s})}{1+i\pi(\sqrt{s})\hat f_l(-\sqrt{s})},\tag{43}$$

with

$$\hat f_l(\sqrt{s})\equiv r_l(\sqrt{s},\sqrt{s};s).$$

It is easily proved that the infinite sum defining $r_l(p',p;s)$ in Eq. (39) converges for all p and p' such that $p/\mu\neq i\alpha$, $(p\pm p')/\mu\neq i\alpha$, where $1\leqslant\alpha<\infty$. The integrals involved all exist due to (29). For the on-shell amplitudes, the expressions defining $\hat{f}_l(\pm\sqrt{s})$ converge for all s except the left-hand cut, $\mathrm{Im}s=0$, $-\infty<s/\mu^2<-\frac{1}{4}$. We defer a proof of this until Sec. V, where we will also allow l to become complex. (We have derived the above result assuming l to be an integer.)

IV. SUPERPOSITIONS OF EXPONENTIAL POTENTIALS

In this section we discuss briefly the modification needed in the work of Sec. III to deal with potentials of the form

$$V(r)=2\pi^2\int_\mu^\infty d\alpha\,\rho(\alpha)e^{-\alpha r}. \tag{44}$$

We shall be interested primarily in comparing the result with previous results found by Martin.[7–9]

The derivation given in Sec. III is valid for the above potential, provided that we require that $|\rho(\alpha)|<C\alpha^{1-\epsilon}$, with $\epsilon>0$. It is convenient to distinguish two cases: If $\rho(\alpha)$ is differentiable, we define $\sigma(\alpha)\equiv d\rho(\alpha)/d\alpha+\rho(\mu)\times\delta(\alpha-\mu)$ and proceed exactly as before; if $\rho(\alpha)$ is not differentiable, we define the function

$$R_l(\alpha;p,p')\equiv\rho(\alpha)P_l\left(\frac{p'^2+p^2+\alpha^2}{pp'}\right)-\frac{1}{pp'}\times\int_\mu^\infty d\beta\,\beta\rho(\beta)P_l'\left(\frac{p'^2+p^2+\beta^2}{2pp'}\right), \tag{45}$$

where $P_l'(\eta)=dP_l(\eta)/d\eta$. The solution is identical in form to the result of Sec. III, except that $\rho_l(\alpha;p',p)$ is replaced everywhere by $R_l(\alpha;p',p)$, and

$$d_l^1(\alpha;p;s)=\bar{\rho}(\alpha)+\frac{1}{(ip/\mu)(\alpha+ip/\mu)}\times\int_1^\alpha d\beta\,\beta\bar{\rho}(\beta)P_l\left[1+\frac{\alpha^2-\beta^2}{2(ip/\mu)(\alpha+ip/\mu)}\right], \tag{46}$$

with $\bar{\rho}(\alpha)\equiv\rho(\mu\alpha)/\mu$.

In particular, the solution for the on-shell amplitude is given by Eq. (42), with

$$\hat{f}_l(\sqrt{s})\equiv r_l(\sqrt{s},\sqrt{s};s)=\frac{-2\pi}{\mu}\left(\frac{i\sqrt{s}}{\mu}\right)^{l-1}\times\sum_{m=1}^\infty\int_1^\infty\frac{d\alpha\,d_l^m(\alpha;\sqrt{s};s)}{[\alpha+i(\sqrt{s})/\mu]^l\alpha[\alpha+i2(\sqrt{s})/\mu]}. \tag{47}$$

[7] A. Martin, Nuovo Cimento **14**, 403 (1959).
[8] A. Martin, Nuovo Cimento **15**, 99 (1960).
[9] A. Martin, Nuovo Cimento, Suppl. **21**, 157 (1961).

In a fashion similar to Eq. (22), the function $\hat{f}_l(\sqrt{s})$ is related to the Jost function by the equation

$$f(k,0)=1-i\pi k\hat{f}_l(k). \tag{48}$$

Considering the special case $l=0$, this becomes

$$f(k,0)=1+2\pi^2\sum_{m=1}^\infty\int_1^\infty\frac{d\alpha\,d_0^m(\alpha;k)}{\alpha(\alpha+i2k/\mu)}, \tag{49}$$

with

$$d_0^1(\alpha;k)=\bar{\rho}(\alpha)$$

and

$$d_0^m(\alpha;k)=2\pi^3\theta(\alpha-m)\int_{m-1}^{\alpha-1}\frac{d\beta\,\bar{\rho}(\alpha-\beta)}{\beta(\beta+i2k/\mu)}d_0^{m-1}(\beta;k). \tag{50}$$

If we now define a function $f_k(\alpha)$ such that

$$f_k(\alpha)\equiv\frac{2\pi^2}{\alpha(\alpha+i2k/\mu)}\sum_{m=1}^\infty d_0^m(\alpha;k), \tag{51}$$

the Jost function can be expressed as

$$f(k,0)=1+\int_1^\infty f_k(\alpha)d\alpha. \tag{52}$$

Alternatively, we may regard $f_k(\alpha)$ as being the solution of the integral equation

$$\alpha(\alpha+i2k/\mu)f_k(\alpha)=2\pi^2\bar{\rho}(\alpha)+\theta(\alpha-2)\times\int_1^{\alpha-1}2\pi^2\bar{\rho}(\alpha-\beta)f_k(\beta)d\beta. \tag{53}$$

Equation (53) is identical with the result which Martin[7] obtained for this problem, although we have derived it from a completely different point of view. In Martin's work, it is assumed that $U^-(k,r)\equiv e^{-ikr}f(k,r)$ is a solution of Schrödinger's equation, and that

$$f(k,r)=1+\int_1^\infty\rho_k(\alpha)e^{-\alpha r}d\alpha, \tag{54}$$

with $f(k,0)=\lim_{r\to0}f(k,r)$. Equation (53) is then obtained by substituting Eq. (54) into the Schrödinger equation and making use of the properties of Laplace transforms. In a later paper,[8] Martin extended his result to the case of higher partial waves, $l\neq0$.

Our solution to the superposition problem, therefore, is not new, although it does possess several advantages. Among these are an explicit form for the off-shell amplitude, and a form for $l\neq0$ which is considerably more compact. The latter is particularly convenient for studying analyticity in the l plane, as we shall show in Sec. V.

However, the most important feature of our result is that we have derived it directly from the integral equation (24) without reference to the associated differential form, i.e., the Schrödinger equation. For this reason, our method is extensible to more general integral equations.

V. EXTENSION TO COMPLEX ANGULAR MOMENTUM

In Sec. III we obtained an explicit representation for the partial-wave amplitude $a_l(\sqrt{s},\sqrt{s};s)\equiv T_l(s)$, which is valid for superpositions of Yukawa potentials. In deriving this result we assumed l to be a positive integer and that $0<\text{Im}(\sqrt{s})<\mu$. We shall now show that our expression for $T_l(s)$ is a meromorphic function of l for all complex l, for all fixed s except the left-hand cut $-\infty<s<-\frac{1}{4}\mu^2$. It is convenient to use the variable $k\equiv\sqrt{s}$ and to write $T_l(s)$ in the form $N_l(s)/D_l(s)$, where we have established that

$$\begin{aligned}N_l(s)&=\hat{f}_l(k)+\hat{f}_l(-k)\,,\\ D_l(s)&=1+i\pi k\hat{f}_l(-k)\end{aligned} \tag{55}$$

in Eq. (43). The function $\hat{f}_l(k)$ is analytic in k in the cut plane $\text{Re}k=0$, $\frac{1}{2}\mu\leq\text{Im}k<\infty$, and is given by (47), which we repeat in the form

$$\hat{f}_l(k)=-(2\pi/\mu)(ik/\mu)^{l-1}\times\sum_{m=1}^{\infty}\int_1^{\infty}\frac{d\alpha\; d_l{}^m(\alpha;k;s)}{(\alpha+ik/\mu)^l\alpha(\alpha+i2k/\mu)}. \tag{56}$$

It is clear from (41), which defines $d_l{}^m(\alpha;k;s)$, that each integral is well defined unless k is on the cut. In fact, since $P_l(z)$ and x^l are entire functions of l, it is easy to show that each term of the sum in (56) is an entire function of l and analytic in k in the cut k plane. The same will be true of $\hat{f}_l(k)$ provided that the sum converges uniformly.

To prove this convergence we write

$$|\hat{f}_l(k)|\leq\frac{2\pi}{\mu}\left|\left(\frac{ik}{\mu}\right)^{l-1}\right|\sum_{n=1}^{\infty}I_n(l,k)\,, \tag{57}$$

where

$$I_n(l,k)=\int_1^{\infty}\left|\frac{d_l{}^n(\alpha;k;s)}{(\alpha+ik/\mu)^l\alpha(\alpha+i2k/\mu)}\right|d\alpha.$$

From Eqs. (41) and (42) it follows that

$$\left|\frac{d_l{}^n(\alpha;k;s)}{(\alpha+ik/\mu)^l}\right|\leq 2\pi^2\theta(\alpha-n)C_l\times\left|\int_1^{\alpha-n+1}d\beta'\,\bar{\sigma}(\beta')\right|I_{n-1}(l,k)\,, \tag{58}$$

where we have defined

$$C_l\equiv\sup_{\alpha,\beta,\beta'}\left|\frac{(\beta+ik/\mu)^l}{(\alpha+ik/\mu)^l}P_l\left[1+\frac{(\alpha-\beta)^2-\beta'^2}{2(\alpha+ik/\mu)(\beta+ik/\mu)}\right]\right| \tag{59}$$

to be the maximum of the absolute value shown in (59) for all α and for all β, β' such that

$$\begin{aligned}1&\leq\beta\leq\alpha-\beta'\,,\\ 1&\leq\beta'\leq\alpha-1\,.\end{aligned} \tag{60}$$

The constant C_l, so defined, is independent of α and n, depending only on l. By making use of the following representation for the P_l function,[10] we can put a bound on C_l:

$$P_l(z)=\frac{1}{\pi}\int_0^{\pi}[z+(z^2-1)^{1/2}\cos t]^l dt. \tag{61}$$

Equation (61) is valid for all l, and all z except the negative real axis. For our purposes here we will always have $\text{Re}z>0$. Consider first $\text{Re}l>0$. Letting $l=l_1+il_2$, it follows that

$$|P_l(z)|\leq e^{|l_2|\pi/2+l_1\ln|z+(z^2-1)^{1/2}|}. \tag{62}$$

Thus

$$C_l\leq e^{\pi|l_2|}\sup_{\alpha,\beta,\beta'}\{e^{l_1\ln R(\alpha,\beta,\beta')}\}\,,$$

with

$$R(\alpha,\beta,\beta')=\frac{\beta+ik/\mu}{\alpha+ik/\mu}[z+(z^2-1)^{1/2}] \tag{63}$$

and

$$z\equiv1+\frac{(\alpha-\beta)^2-\beta'^2}{2(\alpha+ik/\mu)(\beta+ik/\mu)}.$$

Making use of (60), it is easy to establish that C_l is finite for all k not on the cut. In particular, for k restricted to the region $\text{Im}k<\mu$, we find that

$$0<\left|\frac{\beta+ik/\mu}{\alpha+ik/\mu}[z+(z^2-1)^{1/2}]\right|<1$$

as a purely algebraic consequence. We may therefore regard C_l as being of the form $C_l\leq e^{\pi|l_2|+\delta l_1}$, where $\delta<0$ for $\text{Im}k<\mu$. For $\text{Re}l<0$ we may establish a similar bound by a simple modification of the above analysis. The important observation is that $z+(z^2-1)^{1/2}\cos t$ is bounded from below for all values of the parameters,

We have restricted $|\sigma(\alpha)|<K\alpha^{-\epsilon}$, so that

$$\left|\int_1^{\alpha-m+1}d\beta'\,\bar{\sigma}(\beta')\right|<\frac{K}{1-\epsilon}(\alpha-m+1)^{1-\epsilon}. \tag{64}$$

(Here we have chosen $0<\epsilon<1$, which is always possible.) We also have

$$\int_m^{\infty}\frac{d\alpha(\alpha-m+1)^{1-\epsilon}}{\alpha|\alpha+ik/\mu|}<\int_m^{\infty}\frac{d\alpha\;\alpha^{-(1+\epsilon)}}{A}=\frac{1}{A\epsilon m^{\epsilon}}, \tag{65}$$

where $A=2$ if $\text{Im}k<\frac{1}{2}\mu$; $A=|\text{Re}k|/|k|$ if $\text{Im}k>\frac{1}{2}\mu$. Returning to (58) and making use of the above estimates, we finally obtain

$$I_n(l,s)<\frac{1}{n^{\epsilon}}\left(\frac{2\pi^2K}{A\epsilon(1-\epsilon)}\right)e^{\pi|l_2|+\delta l_1}I_{n-1}(l,s). \tag{66}$$

This is sufficient to prove the uniform convergence of

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[10] H. Bateman, *Higher Transcendental Functions* (McGraw-Hill Book Co. Inc., New York, 1953), Vol. 1, p. 155.

(56), provided that $(ik/\mu)^{l-1}I_1(l,s)$ is bounded. An analysis similar to the above then shows that

$$\left|\left(\frac{ik}{\mu}\right)^l\right| I_1(l,s) < \frac{K}{A\epsilon(1-\epsilon)} e^{\pi|l_2|+\delta' l_1}. \qquad (67)$$

We have thus established that $\hat{f}_l(k)$ is analytic in k for k in the cut plane and analytic in l (except possibly at infinity). It follows from (55) that $N_l(s)$ and $D_l(s)$ are analytic in l, $D_l(s)$ is analytic in k for k not on the lower cut $\mathrm{Re}k=0$, $-\infty<\mathrm{Im}k<-\frac{1}{2}\mu$, and $N_l(s)$ is analytic in k for k in the doubly cut plane $\mathrm{Re}k=0$, $\frac{1}{2}\mu\leq|\mathrm{Im}k|<\infty$.

The analyticity properties in s and k are, of course, well known for potentials of this form. What is more interesting is the analyticity properties we have established in the complex l plane. We note in particular that the analytic properties shown by Regge[1] (and subsequently Calogero,[11,12] Martin,[13] etc.) involve only $\mathrm{Re}l\geq-\frac{1}{2}$. The importance of proving analyticity in the left-half l plane is that the Regge representation of the scattering amplitude contains a so-called background integral. This background integration can be pushed farther to the left if $T_l(s)$ is known to be analytic for $\mathrm{Re}l<-\frac{1}{2}$. With this end in mind, Mandelstam[2] proved such analyticity under the condition that the weight function $\sigma(\alpha)$ decreases exponentially as $\alpha\to\infty$. Our restriction (29) on $\sigma(\alpha)$ is much weaker, and the result is manifestly given in Eq. (56).

To prove the Regge representation it is necessary to show that $|T_l(s)|\to 0$ as $|l|\to\infty$, $\mathrm{Re}l\geq-\frac{1}{2}$. This involves proving the limit for both $\mathrm{Re}l\to\infty$ and $|\mathrm{Im}l|\to\infty$. Calogero,[11,12] under essentially equivalent conditions on $\sigma(\alpha)$, established both results for all k in the doubly cut plane. More precisely, however, it has been conjectured that the Born term $T_l^B(s)\equiv V_l(k,k)$ dominates as $|l|\to\infty$, for some domain in k not yet precisely established. This has been shown by Regge[1] for $\mathrm{Re}k$, and for $\mathrm{Re}l\geq-\frac{1}{2}$. Beginning with Eq. (56), and utilizing some of the estimates derived earlier, we will give a simple proof that $T_l(s)\to T_l^B(s)$ as $|l|\to\infty$, for s in the parabolic region $|\mathrm{Im}(\sqrt{s})|<\mu$. We have plotted this in Fig. 1. The result is valid in any direction such that $|\arg(l-l_0)|\leq\frac{1}{2}\pi$, where l_0 is any (finite) complex number. We note that this is in agreement with the result of Cheng and Wu,[14] who showed that $\hat{f}_l(\pm k)$ increases exponentially when $\frac{1}{2}\pi<\arg l<\frac{3}{2}\pi$, for $|l|\to\infty$. Thus we have proven our result in the maximum allowable region.

We begin by defining

$$\hat{f}_l^B(k)\equiv\frac{-2\pi}{\mu}\left(\frac{ik}{\mu}\right)^{l-1}\int_1^\infty \frac{d\alpha\, d_l^1(\alpha;k;s)}{(\alpha+ik/\mu)^l\alpha(\alpha+i2k/\mu)}, \qquad (68)$$

[11] F. Calogero, Nuovo Cimento **28**, 66 (1963).
[12] F. Calogero, Nuovo Cimento **28**, 761 (1963).
[13] A. Martin, Nuovo Cimento **31**, 1229 (1964).
[14] H. Cheng and T. T. Wu, Phys. Rev. **144**, 1232 (1966).

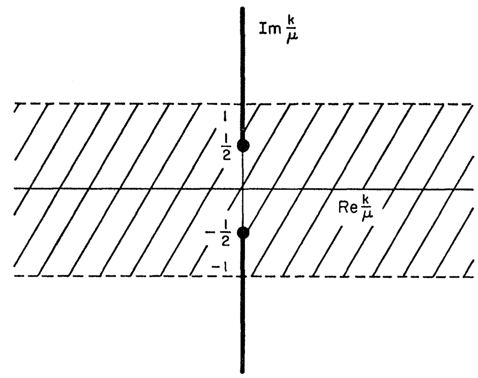

FIG. 1. Domain in k plane where $T_l(s)\to T_l^B(s)$ as $|l|\to\infty$ (shaded area).

so that

$$\frac{|\hat{f}_l(k)-\hat{f}_l^B(k)|}{(2\pi/\mu)|(ik/\mu)^{l-1}|I_1(l,k)}\leq\sum_{n=2}^\infty\frac{I_n(l,k)}{I_1(l,k)}. \qquad (69)$$

Here $I_n(l,k)$ is given as before by (57) and we may make use of the estimate given in Eq. (66). It follows that

$$\lim_{\mathrm{Re}l\to\infty}\frac{I_n(l,s)}{I_{n-1}(l,s)}\leq\lim_{\mathrm{Re}l\to\infty} ce^{\delta\,\mathrm{Re}l}=0 \qquad (70)$$

since we have established that $\delta<0$ for $\mathrm{Im}k<\mu$. From (67) we find that $|(ik/\mu)^l|I_1(l,s)$ also decreases exponentially as $\mathrm{Re}l\to\infty$. Together with (70) this implies

$$\lim_{\mathrm{Re}l\to\infty}|\hat{f}_l(k)-\hat{f}_l^B(k)|=0,$$
$$\lim_{\mathrm{Re}l\to\infty}|\hat{f}_l^B(k)|=0, \qquad (71)$$

for any fixed $\mathrm{Im}l$, provided that $\mathrm{Im}k<\mu$. The latter condition has been chosen primarliy as a convenience to simplify the algebra involved in proving $\delta<0$; it should be possible to extend (71) to a larger domian in k.

We have thus shown that $D_l(s)\to 1$ as $\mathrm{Re}l\to\infty$, for $\mathrm{Im}k>-\mu$. For $|\mathrm{Im}k|<\mu$, it also follows that

$$\lim_{\mathrm{Re}l\to\infty}N_l(s)=\hat{f}_l^B(k)+\hat{f}_l^B(-k). \qquad (72)$$

We must now demonstrate that the right-hand side of (72) is indeed the Born term $N_l^B(s)\equiv V_l(k,k)$. Using the definition of $d_l^1(\alpha;k;s)$ as given in (41), we may rewrite (68) as

$$\hat{f}_l^B(k)=-\frac{2\pi}{\mu}\left(\frac{ik}{\mu}\right)^{l-1}\int_1^\infty d\beta\,\bar{\sigma}(\beta)$$
$$\times\int_\beta^\infty\frac{d\alpha\,P_l[1+(\alpha^2-\beta^2)/(2ik/\mu)(\alpha+ik/\mu)]}{(\alpha+ik/\mu)^l\alpha(\alpha+2ik/\mu)}. \qquad (73)$$

Defining the complex variable

$$\eta=-\left[1+\frac{\alpha^2-\beta^2}{(2ik/\mu)(\alpha+ik/\mu)}\right], \qquad (74)$$

and using $P_l(-\eta)=(-1)^l P_l(\eta)$, we may rewrite (73) in the form

$$\hat{f}_l^B(k)=\frac{-2\pi\mu}{k^2}\oint_1^\infty d\beta\,\bar{\sigma}(\beta)\int_{-1\,c}^{\infty}\frac{d\eta}{(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}}\times\frac{P_l(\eta)}{[\eta+(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}]^{l-1}\{[\eta+(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}]^2-1\}}. \quad (75)$$

The η integration is along the contour defined by (74) as α varies from β to ∞. Considering the analogous expression for $\hat{f}_l^B(-k)$ explicitly, we obtain similarly

$$\hat{f}_l^B(-k)=\frac{2\pi\mu}{k^2}\int_1^\infty d\beta\,\bar{\sigma}(\beta)\oint_{1\,c'}^{\infty}\frac{d\eta}{(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}}\times\frac{P_l(\eta)}{[\eta+(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}]^{l-1}\{[\eta+(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}]^2-1\}}, \quad (76)$$

where the contour now is given by

$$\eta=1+\frac{\alpha^2-\beta^2}{-(2ik/\mu)(\alpha-ik/\mu)}$$

as α varies from β to ∞. We have taken the square-root branch line in both (75) and (76) to be along the positive real axis of its argument. It follows that we may distort the contours as we choose, provided only that we keep $\mathrm{Im}\eta^2\geq 0$. We may then combine (75) and (76) to obtain

$$\hat{f}_l^B(k)+\hat{f}_l^B(-k)=\frac{-2\pi}{\mu}\int_1^\infty d\beta\,\bar{\sigma}(\beta)\int_{-1}^{1}\frac{d\eta}{(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}}\times\frac{P_l(\eta)}{[\eta+(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}]^{l-1}\{[\eta+(\eta^2-1-\beta^2\mu^2/k^2)^{1/2}]^2-1\}}. \quad (77)$$

The second integration in (77) may be performed explicitly, and (77) thus reduces to

$$\hat{f}_l^B(k)+\hat{f}_l^B(-k)=\frac{2\pi\mu}{s}\int_1^\infty d\beta\,\bar{\sigma}(\beta)Q_l\left(1+\frac{\beta^2\mu^2}{2s}\right). \quad (78)$$

Since we have defined $\bar{\sigma}(\beta)\equiv\sigma(\beta\mu)$, the right-hand side of (78) is identical with the expression for $V_l(k,k)$ given in (24) with $p'=p=k$. We thus obtain the desired result:

$$\lim_{\mathrm{Re}l\to\infty} N_l(s)=N_l^B(s)=V_l(k,k),$$
$$\lim_{\mathrm{Re}l\to\infty} T_l(s)=T_l^B(s)=N_l^B(s), \quad (79)$$

valid for region $|\mathrm{Im}k|<\mu$, which we have plotted in Fig. 2.

It remains to prove the dominance of the Born term in the limit $|\mathrm{Im}l|\to\infty$ for fixed Rel. In establishing this, it will prove advantageous to employ the so-called extended unitarity relation

$$\hat{f}_{l^*}(-k^*)=[\hat{f}_l(k)]^*, \quad (80)$$

which follows directly from (56) and the representation (61) for the P_l function. It is then necessary to consider only three distinct cases, all with fixed Rel: (a) Re$k<0$, Im$k<1$, $-\mathrm{Im}l\to\infty$; (b) Re$k>0$, Im$k\geq 0$, $-\mathrm{Im}l\to\infty$; and (c) Re$k<0$, Im$k\leq 0$, Im$l\to\infty$.

Considering first case (a), we refer once again to Eqs.

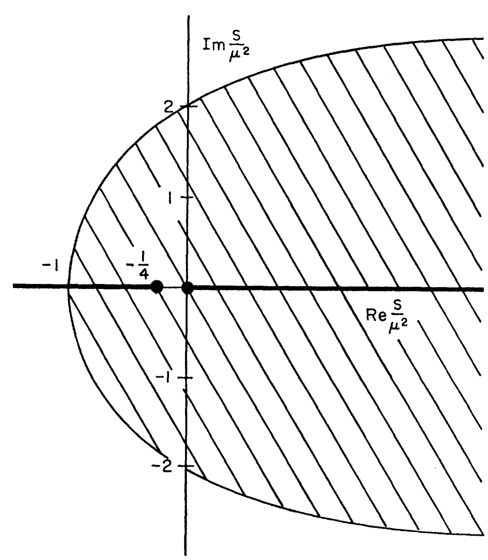

FIG. 2. Domain in s plane where $T_l(s)\to T_l^B(s)$ as $|l|\to\infty$ (shaded area).

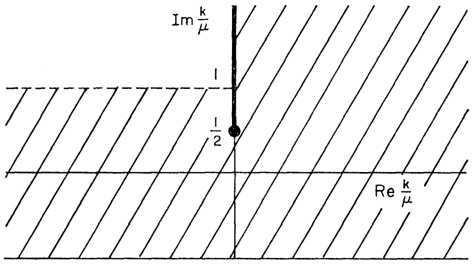

FIG. 3. Domain in k plane where $\hat{f}_l(k) \to \hat{f}_l^B(k)$ as $\mathrm{Im}l \to -\infty$ (shaded area).

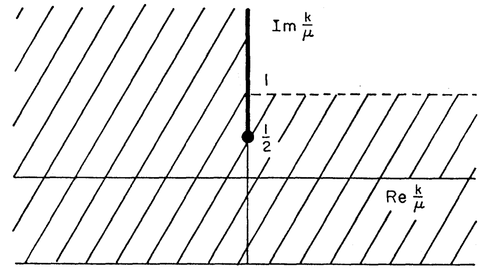

FIG. 4. Domain in k plane where $\hat{f}_l(k) \to \hat{f}_l^B(k)$ as $\mathrm{Im}l \to +\infty$ (shaded area).

(58) and (59). Denoting the argument of the P_l function in (59) by z, we may easily show that

$$\arg[(\beta+ik)/(\alpha+ik)]<0$$

and

$$\arg[z+(z^2-1)^{1/2}\cos t]/z<0. \tag{81}$$

It follows that $C_l \leq ce^{-\theta\,\mathrm{Im}l}$ as $-\mathrm{Im}l \to \infty$, with $\theta<0$. We may employ the remainder of our estimates without change to obtain

$$\lim_{-\mathrm{Im}l\to\infty} \frac{I_n(l,k)}{I_{n-1}(l,k)}=0 \tag{82}$$

and

$$\lim_{-\mathrm{Im}l\to\infty} \left|\left(\frac{ik}{\mu}\right)^{l-1}\right| I_1(l,k)=0.$$

Equation (69) establishes that

$$\lim_{-\mathrm{Im}l\to\infty} \hat{f}_l(k)=\hat{f}_l^B(k), \tag{83}$$

for $\mathrm{Re}k<0$, $\mathrm{Im}k<1$, and $\mathrm{Re}l$ fixed.

We prove cases (b) and (c) by showing that

$$\lim \sum_{m=1}^{\infty} d_l^m(\alpha;k;s)=d_l^1(\alpha;k;s) \tag{84}$$

in either case. To do so we recall that

$$d_l^m(\alpha;k;s)=2\pi^2\theta(\alpha-m)\int_1^{\alpha-m+1} d\beta'\,\bar{\sigma}(\beta')$$
$$\times\int_{m-1}^{\alpha-\beta'} \frac{d\beta}{\beta(\beta+i2k/\mu)} P_l\left(1+\frac{(\alpha-\beta)^2-\beta'^2}{2(\alpha+ik/\mu)(\beta+ik/\mu)}\right)$$
$$\times d_l^{m-1}(\beta;k;s). \tag{85}$$

Denoting the argument of the P_l function in (85) by z, we easily find that $\arg[z+(z^2-1)^{1/2}\cos t]<0$ in case (b). and $\arg[z+(z^2-1)^{1/2}\cos t]>0$ in case (c). From (61) it then follows that

$$\lim|d_l^m(\alpha;k;s)|/|d_l^{m-1}(\alpha;k;s)|=0$$

in either case, establishing (84).

Having proven our three particular cases, it is trivial to obtain the final result. We find that $\hat{f}_l(k) \to \hat{f}_l^B(k)$ as $\mathrm{Im}l \to -\infty$, for all k except the region $\mathrm{Re}k<0$, $\mathrm{Im}k>\mu$. This is shown in Fig. 3. The limit also holds as $\mathrm{Im}l \to +\infty$ for k not in the region $\mathrm{Re}k>0$, $\mathrm{Im}k>\mu$, plotted in Fig. 4. From (55) it then follows that

$$\lim_{|l|\to\infty} T_l(s)=T_l^B(s) \tag{86}$$

for $|\mathrm{Im}k|<\mu$, $|\arg(l-l_0)|\leq\frac{1}{2}\pi$.

We note that (86) is valid for any fixed $\mathrm{Re}l<0$, $|\mathrm{Im}l|\to\infty$, which has not been obtained elsewhere to our knowledge. We also observe that in the region where our result duplicates the results previously quoted, we have demonstrated a particularly simple direct proof. It would be interesting to use our representation for $T_l(s)$ to obtain the other Regge results, but it is less obvious how to proceed. Instead we shall turn to a discussion of other promising areas of application.

VI. ADDITIONAL APPLICATIONS

Although the previous discussion has been restricted to potential scattering, the method developed is not. The same technique may be applied to many integral equations of physical interest. This is because the primary requirement for applicability is that the kernel and inhomogeneous term have simple analytic properties. This is almost invariably the case for scattering equations. In fact, from the point of view of solving such equations, the only important restriction is that the integral equation be one-dimensional. This excludes direct application to equations such as the Bethe-Salpeter equation. However, the method is applicable to the various approximations to the Bethe-Salpeter equation which are reducible to one dimension.

A particular example is the equation proposed by Blankenbecler and Sugar.[15] They consider the Bethe-Salpeter equation for scattering of two unit-mass particles by exchange of a particle of mass μ. By introducing an approximate Green's function they derive an equation of the form

$$T(\mathbf{p},\mathbf{q})=V(\mathbf{p},\mathbf{q})-\int\frac{d^3k}{(k^2+1)^{1/2}}\frac{V(\mathbf{p},\mathbf{k})}{k^2-z}T(\mathbf{k},\mathbf{q}), \tag{87}$$

with

$$V(\mathbf{p},\mathbf{q})=g^2/[(\mathbf{p}-\mathbf{q})^2+\mu^2].$$

[15] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

Here the on-shell condition is $p^2=q^2=z$, where $z\equiv\frac{1}{4}s-1$, and s is the square of the c.m. energy. Equation (87) has exactly the form of the LS equation for a Yukawa potential except for the factor $(k^2+1)^{1/2}$. We may project out partial waves in the usual way to obtain

$$A_l(p',p;z)=V_l(p',p)-\int_0^\infty \frac{dp''p''^2}{p''^2-z}\times\frac{V_l(p',p'')A_l(p'',p;z)}{(p''^2+1)^{1/2}}, \quad (88)$$

with

$$V_l(p',p)=\frac{g^2}{pp'}Q_l\left(\frac{p'^2+p^2+\mu^2}{2pp'}\right).$$

Except for the $(p''^2+1)^{1/2}$ factor, Eq. (88) is the same as (24) with $\sigma(\alpha)=(g^2/2\pi)\delta(\alpha-\mu)$. However, as Blankenbecler and Sugar point out, the square-root factor severely complicates the analytic structure. The second Born approximation of (87) contains a left-hand cut starting at $s=4-4(1+\mu)^2$. This is not present in the correct second Born term as calculated from the original equation. However, an explicit calculation by Blankenbecler and Sugar found the relative difference to be less than 10% at threshold and decreasing above threshold ($s=4$).

If we begin with (88) and proceed as in Sec. III, we again arrive at (34), except that now

$$F_l(\alpha;p')=\int_{-\infty}^{\infty}\frac{dp''p''}{p''^2-z}\frac{(g^2/2\pi)}{(p''^2+1)^{1/2}(p'-p''-i\alpha-i\epsilon)(p'-p''+i\alpha)}P_l\left(\frac{p'^2+p''^2+\mu^2}{2p'p''}\right)+\frac{g^2}{2\alpha}P_l\left[\frac{p'^2+(p'-i\alpha)^2+\mu^2}{2p'(p'-i\alpha)}\right]\frac{(p'-i\alpha)}{[1+(p'-i\alpha)^2]^{1/2}}\frac{A_l(p'-i\alpha,p;z)}{(p'-i\alpha)^2-z}, \quad (89)$$

for $\alpha<\text{Im}p'<\mu+\alpha$. Just as before, (34) and (89) provide us with an iterative scheme for calculating the singularities of $A_l(p',p;z)$. It is evident from (89) that the additional square-root factor will affect our result in two ways. The first is trivial and merely involves putting an extra factor in the discontinuity formulas derived previously. The second is more serious and arises from the branch cut of the square-root function. To insure that the $(p''^2+1)^{1/2}$ term in (88) is well defined for real p'' we take the cut along $p''^2+1\leq 0$; or $\text{Re}p''=0$, $|\text{Im}p''|\geq 1$. It then follows from (89) that $F_l(\alpha;p')$ will have a cut along $p'=i(x+\alpha)$, with $x\geq 1$.

The result is that $A_l(p',p;z)$ has all the cuts found previously in Sec. III, but with slightly modified discontinuities. In addition, it has a cut along the imaginary p' axis for $|\text{Im}p'|\geq 1+\mu$. The discontinuity across this cut is not given explicitly, although it is possible to develop an additional iterative scheme to compute it. We shall show, however, that it is consistent with the approximations already made by Blankenbecler-Sugar to ignore it.

The off-shell solution to (88) can be written in the form

$$A_l(p',p;z)=r_l(p',p;z)+(-1)^l r_l(p',-p;z)-(-1)^l i\pi(\sqrt{z})r_l(p',-\sqrt{z};z)A_l(\sqrt{z},p;z)+\int_{1+\mu}^{\infty}\frac{f_l(p',x)}{p'^2+x}, \quad (90)$$

where the last term is the contribution from the new cut. The function $r_l(p',p;z)$ is given by (40), with

$$d_l^1(\alpha;p;z)=\frac{g^2}{2\pi\mu}P_l\left[1+\frac{\alpha^2-1}{2(ip/\mu)(\alpha+ip/\mu)}\right] \quad (91)$$

and

$$d_l^m(\alpha;p;z)=\frac{\pi g^2}{\mu}\theta(\alpha-m)\int_{m-1}^{\alpha-1}\frac{d\beta\, P_l(1+[(\alpha-\beta)^2-1]/2(\alpha+ip/\mu)(\beta+ip/\mu))d_l^{m-1}(\beta;p;z)}{[1-(\beta u+ip)^2]^{1/2}[(\beta+ip/\mu)^2+z/\mu^2]}.$$

If we now put p' on-shell, we have $p'^2=z=\frac{1}{4}s-1$. From (90) we see that the last term will produce a left-hand cut in s for $s\leq 4-4(1+\mu)^2$. In view of the discussion prior to Eq. (89), we observe that this term cannot be large. It is consistent with Blankenbecler and Sugar to drop it altogether. By doing so we obtain an effective solution to (88) which is a simple modification of our earlier results for potential scattering. In contrast, Eq. (88) is sufficiently different from the LS equation to preclude the application of standard methods. The associated differential form is exceedingly complicated, unlike the extensively studied Schrödinger equation. In such cases our method should be especially useful.

In addition to relativistic two-body equations such as the above, some especially interesting applications can be found in the field of many-particle equations.

The ability of this method to generate off-shell solutions becomes extremely important for this purpose. For example, off-shell two-body solutions such as those obtained in Secs. II and III become the input for equations describing three-particle scattering. The resulting three-body equations may then be solved directly by the same approach. Preliminary results for a particular set of such equations indicate that the method may prove especially valuable in this context. These results and a discussion of additional possibilities will appear in a forthcoming paper with R. F. Peierls.

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Mirror Relations as Nondynamical Tests of Conservation Laws*

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A mirror relation is defined as $L(x_a,y_b;z_c,w_d)=\eta L(z_c,w_d;x_a,y_b)$, with $\eta=+1$ or -1, where $L(x_a,y_b;z_c,w_d)$ is an observable for an arbitrary four-particle reaction $a+b\rightarrow c+d$, and x_a denotes the polarization state of particle a, etc. Invariances under the transformations P (space reflection), T (time reversal), C (charge conjugation), B (detailed balancing), and products of these, are considered. First, the types of reactions are listed which transform into themselves under any one of these transformations. The M matrix of these self-transforming reactions will therefore be restricted by the requirement of invariance under these transformations. This allows a study of the validity of a particular conservation law using one single reaction. The restriction on the M matrices of one kind of self-transforming reactions can be expressed very simply in terms of the requirement that the number of occurrences of each of the three unit vectors used to span the space of momenta be either even or odd throughout each term of the M matrix. Under the transformations T, PT, TB, TC, PTC, TCB, PTB, and $PTCB$, these restrictions are expressed in terms of the number of one of the unit vectors or in terms of the sum of the numbers of all three unit vectors. We call these transformations of the odd type. On the other hand, under transformations P, B, PB, PC, CB, and PCB, the restrictions are expressed in terms of the sum of the numbers of two unit vectors. We call these transformations of the even type. Under transformations of the even type, about half of the observables identically vanish. It is then shown that mirror relations among all observables arise under any of the transformations of the odd type. On the other hand, the only reaction for which mirror relations hold for all observables as a result of any of the transformations of the even type is the reaction $0+\frac{1}{2}\rightarrow 0+\frac{1}{2}$. The factor η for mirror relations for the odd type is shown to be $(-1)^{a_S}$, where a_S is the sum of the numbers of those types of unit vectors appearing in the observable that are involved in characterizing the restriction of that particular transformation on the M matrix of its self-transforming reaction. It is shown that in order to distinguish among invariances under the various transformations of the odd type, observables in the *eee* or *ooo* subclasses are useless, and the observables in at least two other subclasses must be measured in order to pick out unambiguously the transformation under which the reaction is invariant. As a by-product, the number of terms in the M matrix of a self-transforming, but otherwise arbitrary, reaction under any of the various transformations is derived, thus generalizing the results of a previous paper.

I. INTRODUCTION

EXPERIMENTAL tests of the validity of conservation laws and the determination of quantum numbers of particles with respect to such conservation laws have been, and continue to be, in the center of interest of particle physics. Among such tests, some are completely independent of the dynamics of the reaction under investigation and hence have a solid foundation even in instances when our knowledge of the interaction among the particles in the reaction is fragmentary, such as is the case in most of elementary particle physics. Such tests are called nondynamical.

Among such nondynamical tests, there is a type which is particularly simple. In general, nondynamical tests involve linear relations among a number of observables. There are, however, tests which involve only two observables, and in fact are of the type

$$L(x_a,y_b;z_c,w_d)=\eta L(z_c,w_d;x_a,y_b)\,, \qquad (1.1)$$

where η is $+1$ or -1. The notation here is as follows: $L(x_a,y_b;z_c,w_d)$ denotes an observable for a four-particle reaction $a+b\rightarrow c+d$, and the spin states of the particles a, b, c, and d are characterized by x_a, y_b, z_c, and w_d, respectively. Connections of the type (1.1) among observables are called mirror relations.

Mirror relations constitute particularly convenient tests of conservation laws, since they involve only two observables. Furthermore, when used to determine

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