Relations among the Superconvergence Conditions for Forward Elastic Scattering. II. Linear Relations among the Crossed-Channel Helicity Amplitudes*

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The relations among the superconvergence conditions for the forward elastic scattering of two particles with spin are discussed. Because of the existence of identities among the t-channel helicity amplitudes at $t=0$, the corresponding superconvergence conditions are not independent. It is shown that there exists a set of t-channel helicity amplitudes with the following properties: (1) The corresponding superconvergence conditions are independent; (2) the superconvergence conditions for any other t-channel helicity amplitude are linearly related to the superconvergence conditions for this set of amplitudes.

I. INTRODUCTION

It has been shown by Trueman[1] how to obtain superconvergence conditions[2] for arbitrary spin in a general way. The idea is to construct amplitudes free from the s kinematic singularities directly from the t-channel helicity amplitudes. The t-channel helicity amplitudes are kinematically-independent except at values of s and t where higher symmetry exists, for example, the forward elastic scattering. In the case of forward elastic scattering, it is well known that the t-channel helicity amplitudes are related to each other.[3] Therefore, the corresponding superconvergence conditions must also be related to each other.

The relations among the superconvergence conditions for forward elastic scattering are investigated in a series of papers. In the previous paper,[4] we discussed the general properties of the forward amplitudes and investigated the special case where one of the scattering particles is spinless. In this paper, we shall consider the elastic scattering of two particles with spin and discuss the linear relations among the t-channel helicity amplitudes at $t=0$. The linear relations among the derivatives of the t-channel helicity amplitudes at $t=0$ are discussed in the following paper.

A brief review of Ref. 4 is given below. Let us consider the elastic scattering of the particles a (spin J, mass M_a) and b (spin J', mass M_b): $a+b\to a+b$. Their helicities are denoted by α, β, α^*, and β^*, respectively. We use the convention $J\geqslant J'$. The crossed channel is defined to be $\bar{b}+b\to a+\bar{a}$ where $\bar{a}$ means the antiparticle of a and the corresponding helicities are β', β'', α', and α'', respectively. The square of the invariant mass in the direct (crossed) channel is denoted by $s(t)$. The s-channel helicity amplitude $f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,t)$ has the kinematic factor $t^{|\lambda^*-\mu^*|/2}$ where $\lambda^*=\alpha-\beta$ and $\mu^*=\alpha^*-\beta^*$. The t-channel helicity amplitudes $f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,t)$ can be divided into two groups according to whether the numbers $\lambda+\mu$ ($\lambda=\alpha'-\alpha''$, $\mu=\beta'-\beta''$) are even (group A) or odd (group B).

The forward amplitudes in group A satisfy the following identities:

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda+\mu=\text{even}}} f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,0)d_{\alpha''\alpha}{}^J(\tfrac{1}{2}\pi)d_{\beta''\beta}{}^{J'}(\tfrac{1}{2}\pi)\times d_{\alpha'\alpha^*}{}^J(\tfrac{1}{2}\pi)d_{\beta'\beta^*}{}^{J'}(\tfrac{1}{2}\pi)$$
$$=0 \quad \text{if} \quad \lambda^*\neq\mu^*,$$
$$=f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0) \quad \text{if} \quad \lambda^*=\mu^*. \quad (1)$$

The amplitudes in group B have the kinematic factor $t^{1/2}$ and vanish at $t=0$. However, the modified amplitudes $f^{*t}\equiv f^t t^{-1/2}$ do not vanish at $t=0$. The forward amplitudes $f^{*t}(s,0)$ satisfy the following identities:

$$\sum_{\substack{\alpha',\alpha'',\beta',\beta''\\ \lambda+\mu=\text{odd}}} f_{\alpha'\alpha'',\beta'\beta''}{}^{*t}(s,0)d_{\alpha''\alpha}{}^J(\tfrac{1}{2}\pi)d_{\beta''\beta}{}^{J'}(\tfrac{1}{2}\pi)\times d_{\alpha'\alpha^*}{}^J(\tfrac{1}{2}\pi)d_{\beta'\beta^*}{}^{J'}(\tfrac{1}{2}\pi)$$
$$=0 \quad \text{if} \quad |\lambda^*-\mu^*|\neq1,$$
$$=f_{\alpha^*\beta^*,\alpha\beta}{}^{*s}(s,0) \quad \text{if} \quad |\lambda^*-\mu^*|=1, \quad (2)$$

where the modified s-channel amplitudes $f_{\alpha^*\beta^*,\alpha\beta}{}^{*s}(s,0)$ are defined in Ref. 4. Hereafter, we shall call the t-channel helicity amplitudes in group A and the modified t-channel helicity amplitudes in group B the t amplitudes.

The s kinematic factor of the t amplitude is given by[5,6] $[s-(M_a-M_b)^2]^{|\lambda+\mu|/2}[s-(M_a+M_b)^2]^{|\lambda-\mu|/2}$. We shall call the redefined amplitude, which is free from this factor and (in the case of group B) the factor $t^{1/2}$, the invariant amplitude.

In the rest of this paper, we shall use the following simplified notations:

$$d(\alpha\alpha')=d_{\alpha\alpha'}{}^J(\tfrac{1}{2}\pi)\,,$$
$$F(\alpha^*\beta^*\alpha\beta)=f_{\alpha^*\beta^*,\alpha\beta}{}^s(s,0) \quad \text{if} \quad \lambda^*=\mu^*,$$
$$=0 \quad \text{if} \quad \lambda^*\neq\mu^*,$$
$$T(\alpha'\alpha''\beta'\beta'')=f_{\alpha'\alpha'',\beta'\beta''}{}^t(s,0) \quad \text{if} \quad \lambda+\mu=\text{even},$$
$$=0 \quad \text{if} \quad \lambda+\mu=\text{odd},$$
$$F^*(\alpha^*\beta^*\alpha\beta)=f_{\alpha^*\beta^*,\alpha\beta}{}^{*s}(s,0) \quad \text{if} \quad |\lambda^*-\mu^*|=1.$$
$$=0 \quad \text{if} \quad |\lambda^*-\mu^*|\neq1.$$
$$T^*(\alpha'\alpha''\beta'\beta'')=f_{\alpha'\alpha'',\beta'\beta''}{}^{*t}(s,0) \quad \text{if} \quad \lambda+\mu=\text{odd},$$
$$=0 \quad \text{if} \quad \lambda+\mu=\text{even}.$$

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[1] T. L. Trueman, Phys. Rev. Letters **17**, 1198 (1966).

[2] V. de Alfaro, S. Fubini, G. Rossetti, and F. Furlan, Phys. Letters **21**, 576 (1966).

[3] See, e.g., M. L. Goldberger, Comments Nucl. Particle Phys. **1**, 63 (1967).

[4] K. Y. Lin, Phys. Rev. **163**, 1568 (1967).

[5] Y. Hara, Phys. Rev. **136**, B507 (1964).

[6] L. L. C. Wang, Phys. Rev. **142**, 1187 (1966).

Under parity symmetry, we have[7]

$$F(\alpha^*\beta^*\alpha\beta)=F(-\alpha^*-\beta^*-\alpha-\beta)\,,$$
$$F^*(\alpha^*\beta^*\alpha\beta)=-F^*(-\alpha^*-\beta^*-\alpha-\beta)\,,$$
$$T(\alpha'\alpha''\beta'\beta'')=T(-\alpha'-\alpha''-\beta'-\beta'')\,,$$
$$T^*(\alpha'\alpha''\beta'\beta'')=-T^*(-\alpha'-\alpha''-\beta'-\beta'')\,.$$

Time-reversal invariance implies that $F(\alpha^*\beta^*\alpha\beta)=F(\alpha\beta\alpha^*\beta^*)$ and $F^*(\alpha^*\beta^*\alpha\beta)=-F^*(\alpha\beta\alpha^*\beta^*)$. These relations plus the crossing relations imply that $T(\alpha'\alpha''\beta'\beta'')=T(\alpha''\alpha'\beta''\beta')$ and $T^*(\alpha'\alpha''\beta'\beta'')=-T^*(\alpha''\alpha'\beta''\beta')$. The s-channel forward amplitudes F and F^* are kinematically-independent if those amplitudes related by parity symmetry or time-reversal invariance are considered as the same amplitude.

In general, we have several superconvergence conditions for each invariant amplitude. Since the t-amplitudes are not independent at $t=0$, the corresponding superconvergence conditions are related. The special case of $J'=0$ is studied in Ref. 4. In general, there are several t-amplitudes which have the same s kinematic factor; it is shown that only one of them needs to be investigated. However, the superconvergence conditions corresponding to t-amplitudes with different s kinematic factors are independent. In this paper, we generalize this result to the case where $J'\neq 0$. It is shown that there exists a set of t-amplitudes with the following properties (hereafter referred to as $P1$ and $P2$ respectively): (1) They are independent. (2) Any other t-amplitude is linearly related to them with constant coefficients, and the corresponding s kinematic factor is the common factor of all terms with nonvanishing coefficients. The first property means that the superconvergence conditions for this set of amplitudes are independent. The second property means that the superconvergence conditions for any other t-amplitude are linearly related to the superconvergence conditions for this set of amplitudes.[4] Therefore, we have the following conclusion: *In order to obtain all independent superconvergence conditions for the t-amplitudes at $t=0$, it is sufficient to consider a particular set of independent t-amplitudes.*

In Secs. II and III, we shall discuss how to select a set of amplitudes from group A (B) which satisfy these two properties. In general, it is not sufficient to consider just one amplitude among all t-amplitudes having the same s kinematic factor. The mathematical details are given in the appendices.

II. SUPERCONVERGENCE CONDITIONS FOR THE FORWARD AMPLITUDES IN GROUP A

In this section, we shall consider the forward amplitudes in the group A. The total number of independent forward amplitudes in group A is the same as that of the s-channel forward amplitudes $F(\alpha^*\beta^*\alpha\beta)$. This number is

$$[(2J+1)(2J'+1)(2J'+2)-2J'(2J'+1)(2J'+2)3^{-1}+v]4^{-1}\,,$$

where

$$\begin{aligned} v&=2J'+1 && \text{if both particles are fermions,}\\ &=2J'+2 && \text{if both particles are bosons,}\\ &=0 && \text{otherwise.}\end{aligned}$$

The simplest case of $J'=0$ is discussed in Ref. 4. Let us consider the next simplest case, $J'=\frac{1}{2}$. A set of independent t-amplitudes of group A can be chosen in this way: Let us choose one t-amplitude among all amplitudes which have the same kinematic factor. In other words, the amplitudes are simply classified by ($|\lambda|\leqslant 2J$)

$$\mu=0\,,\quad \lambda=0,2,4,\cdots,$$
$$\mu=\pm1\,,\quad \lambda=1,3,5,\cdots.$$

In Appendix I, we shall prove that this set of amplitudes have the properties $P1$ and $P2$.

In the case where $J'>\frac{1}{2}$ is more complicated, in order to obtain a complete set of independent amplitudes, it is not enough to choose just one amplitude among all amplitudes which have the same factor. For example, let us consider the case of $J'=1$. We choose one amplitude from each subgroup of amplitudes which are classified by

$$\mu=0\,,\quad \lambda=0;$$
$$\mu=1\,,\quad \lambda=\pm1,\ \pm3,\ \text{etc.};$$
$$\mu=2\,,\quad \lambda=0,\ \pm2,\ \text{etc.};$$
$$\mu=0\ (\beta'=1,0)\,,\quad \lambda=2,\ 4,\ \text{etc.}$$

These amplitudes satisfy the properties $P1$ and $P2$ (see Appendix I). Notice that we choose two amplitudes which are classified by the same set of λ and μ if $\mu=0$ and $\lambda\geqslant 2$.

In the general case, the amplitudes which satisfy the properties $P1$ and $P2$ can be chosen in a similar way. We divide the t-amplitudes into several subgroups. The amplitudes in each subgroup have the same λ and μ. We use the convention that $\lambda\geqslant 0$ since the subgroup of amplitudes classified by $\lambda=m$, $\mu=n$ is identical to the subgroup of amplitudes classified by $\lambda=-m$, $\mu=-n$. In general, we choose several amplitudes from the subgroup of amplitudes which have the same λ and μ. The number of amplitudes chosen from the subgroup of amplitudes classified by λ and μ is denoted by $N(\lambda,\mu)$.

In general, we have (remember that $\lambda+\mu=$ even and $J\geqslant J'$):

$$\begin{aligned} N&=\tfrac{1}{2}(\lambda-|\mu|)+1 && \text{if}\quad 2J'\geqslant\lambda\geqslant|\mu|\,,\\ &=\tfrac{1}{2}(|\mu|-\lambda) && \text{if}\quad |\mu|>\lambda\,,\\ &=1+\tfrac{1}{2}(2J'-|\mu|) && \text{if}\quad \lambda>2J' \ \text{and}\ \lambda+2J'=\text{even}\,,\\ &=\tfrac{1}{2}(2J'-|\mu|+1) && \text{if}\quad \lambda>2J' \ \text{and}\ \lambda+2J'=\text{odd}\,.\end{aligned}$$

[7] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

TABLE I. The numerical values of $N(\lambda,\mu)$ for the case of $J'=3$.

$\mu \backslash \lambda$	0	1	2	3	4	5	6	$2m^{a}$	$2m+1$
0	1		2		3		4	4	
±1		1		2		3			3
±2	1		1		2		3	3	
±3		1		1		2			2
±4	2		1		1		2	2	
±5		2		1		1			1
±6	3		2		1		1	1	

[a] $m \geqslant 3$.

TABLE II. The numerical values of $N(\lambda,\mu)$ for the case of $J'=\frac{5}{2}$.

$\mu \backslash \lambda$	0	1	2	3	4	5	$2m^{a}$	$2m+1$
0	1		2		3		3	
±1		1		2		3		3
±2	1		1		2		2	
±3		1		1		2		2
±4	2		1		1		1	
±5		2		1		1		1

[a] $m \geqslant 3$.

The numerical values of N for the cases $J'=3$ and $J'=\frac{5}{2}$ are given in Tables I and II, respectively. If N is bigger than 1, the amplitudes are chosen in this way:

$$\begin{aligned}&\beta'(i)=J'-i+1, \quad i=1,2,\cdots,N-1\\&J'-N+1\geqslant\beta'(N)\geqslant -J'+|\mu|+N-1,\end{aligned}\tag{3}$$

if $\lambda>\mu\geqslant 0$; replace β' by β'' in (3) if $\lambda>-\mu>0$; replace β' by α', J' by J, and μ by λ in (3) if $|\mu|>\lambda$.

In Appendix I, we shall prove that the amplitudes chosen in this way satisfy $P1$ and $P2$.

III. SUPERCONVERGENCE CONDITIONS FOR THE FORWARD AMPLITUDES IN GROUP B

In this section, we are concerned with the forward t-amplitudes in group B. The total number of independent amplitudes is $\frac{1}{2}(N-2J'-1)$ if both particles are fermions or bosons; otherwise it is $\frac{1}{2}N$, where $N=(2J+1)(2J'+1)^2-\frac{2}{3}J'(2J'+1)(2J'+2)$.

The simplest case of $J'=0$ is discussed in Ref. 4. In order to obtain all independent superconvergence conditions in this case, it is sufficient to consider just one amplitude among all amplitudes having the same kinematic factor. In the general case where $J'\neq 0$, we have to choose several amplitudes from the subgroup of amplitudes which have the same λ and μ.

A set of amplitudes satisfying $P1$ and $P2$ can be chosen in the following way. [The proof is given in Appendix II and the number of amplitudes chosen from the amplitudes classified by λ and μ is denoted by $N^*(\lambda,\mu)$. The subgroup of amplitudes classified by λ and μ is identical to that classified by $-\lambda$ and $-\mu$.]

$$\begin{aligned}N^*&=|(|\lambda|-|\mu|)|+1 && \text{if } |\mu|\neq 2J' \text{ and } |\lambda|\leqslant 2J',\\&=\tfrac{1}{2}(1+2J'-|\lambda|) && \text{if } |\mu|=2J' \text{ and } |\lambda|\leqslant 2J',\\&=1+2J'-|\mu| && \text{if } 2J>|\lambda|>2J',\\&=\tfrac{1}{2}(2J'-|\mu|)+1 && \text{if } J+J'=\text{half integer and } \lambda=2J,\\&=\tfrac{1}{2}(2J'-|\mu|+1) && \text{if } J+J'=\text{integer and } \lambda=2J;\end{aligned}$$

$$\beta'(n)=\beta'(n+\tfrac{1}{2}N^*)=J'-n+1, \quad n=1,2,\cdots,\tfrac{1}{2}N^*\tag{4}$$

if $2J>\lambda>\mu\geqslant 0$ and $N^*=$even; replace β' by β'' in (4) if $2J>\lambda>-\mu>0$ and $N^*=$even; replace β', J', μ by α', J, λ, respectively, in (4) if $|\mu|\neq 2J'$ and $|\mu|>\lambda\geqslant 0$;

$$\begin{aligned}&\beta'(n)=\beta'(n+\tfrac{1}{2}N^*-\tfrac{1}{2})=J'-n+1,\\&\qquad n=1,2,\cdots,\tfrac{1}{2}N^*-\tfrac{1}{2},\\&J'-\tfrac{1}{2}N^*+\tfrac{1}{2}\geqslant\beta'(N^*)\geqslant -J'+|\mu|+\tfrac{1}{2}N^*-\tfrac{1}{2},\end{aligned}\tag{5}$$

if $2J>\lambda>\mu\geqslant 0$ and N^* odd; replace β' by β'' in (5) if $2J>\lambda>-\mu>0$ and $N^*=$odd;

$$\begin{aligned}&\beta'(n)=J'-n+1, \quad n=1,2,\cdots,N^*-1,\\&J'-N^*+1\geqslant\beta'(N^*)\geqslant -J'+|\mu|+N^*-1,\end{aligned}\tag{6}$$

if $\lambda=2J$ and $\mu\geqslant 0$; replace β' by β'' in (6) if $\lambda=2J$ and $\mu<0$; replace β', J', μ by α', J, λ, respectively, if $|\mu|=2J'$ and $\lambda\geqslant 0$.

The numerical values of N^* for the cases $J'=3$ and $\frac{5}{2}$ are given in Tables III and IV, respectively.

APPENDIX I: RELATIONS AMONG THE SUPERCONVERGENCE CONDITIONS FOR FORWARD AMPLITUDES OF GROUP A

We shall generalize the method used in Ref. 4. We need a set of special solutions for Eq. (1). Let us define the function $F(\alpha^*\beta^*\alpha\beta)$ to be the coefficient of the term

$$x^{J-\alpha}(x^*)^{J-\alpha^*}y^{J'-\beta}(y^*)^{J'-\beta^*}[(J+\alpha)!(J-\alpha)!(J+\alpha^*)!\times(J-\alpha^*)!(J'+\beta)!(J'-\beta)!(J'+\beta^*)!(J'-\beta^*)!]^{-1/2}$$

in the expansion of

$$(1+xx^*)^{a'}(1-xx^*)^{a}(1+yy^*)^{b'}(1-yy^*)^{b}\times[(1+xy)(1+x^*y^*)]^{c}[(1-xy)(1-x^*y^*)]^{d},$$

where $a+b=$even, $a'+a+c+d=2J$, and $b'+b+c+d=2J'$. This function satisfies the conditions: $F(\alpha^*\beta^*\alpha\beta)=F(\alpha\beta\alpha^*\beta^*)=F(-\alpha-\beta-\alpha^*-\beta^*)$, and $F=0$ if $\lambda^*\neq\mu^*$. For each set of admissible values of a, a', b, b', c, and d, the function F generates a special solution of Eq. (1), namely,

$$T(\alpha'\alpha''\beta'\beta'')=\sum_{\alpha,\beta,\alpha^*,\beta^*}F(\alpha^*\beta^*\alpha\beta)d(\alpha'\alpha^*)\times d(\beta'\beta^*)d(\alpha''\alpha)d(\beta''\beta).$$

The numerical value of the $d(\alpha'\alpha)$ is given[8] by the coefficient of the term $(-x)^{J-\alpha}[(J+\alpha)!(J-\alpha)!]^{-1/2}$ in

[8] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957), p. 57.

TABLE III. The numerical values of $N^*(\lambda,\mu)$ for the case of $J'=3$.

$\mu \backslash \lambda$	0	1	2	3	4	5	6	$2m$	$2m+1$	$(2J-1)^a$	$2J^a$	$(2J-1)^b$	$2J^b$
0		2		4		6			7		4	7	
±1	2		2		4		6	6		6			3
±2		2		2		4			5		3	5	
±3	4		2		2		4	4		4			2
±4		4		2		2			3		2	3	
±5	6		4		2		2	2		2			1
±6		3		2		1			1		1	1	

[a] J = half integer.
[b] J = integer.

TABLE IV. The numerical values of $N^*(\lambda,\mu)$ for the case of $J'=\frac{5}{2}$.

$\mu \backslash \lambda$	0	1	2	3	4	5	$2m$	$2m+1$	$(2J-1)^a$	$2J^a$	$(2J-1)^b$	$2J^b$
0		2		4		6		6	6			3
±1	2		2		4		5			3	5	
±2		2		2		4		4	4			2
±3	4		2		2		3			2	3	
±4		4		2		2		2	2			1
±5	3		2		1		1			1	1	

[a] J = integer.
[b] J = half integer.

the expansion of

$$2^{-J}(1-x)^{J-\alpha'}(1+x)^{J+\alpha'}[(J+\alpha')!(J-\alpha')!]^{-1/2}.$$

A straightforward calculation shows that the special solution $T(\alpha'\alpha''\beta'\beta'')$ is given by the coefficient of the term

$$x^{J-\alpha'}(\bar{x})^{J-\alpha''}y^{J'-\beta'}(\bar{y})^{J'-\beta''}[(J+\alpha')!(J-\alpha')! \times(J+\alpha'')!(J-\alpha'')!(J'+\beta')!(J'-\beta')! \times(J'+\beta'')!(J'-\beta'')!]^{-1/2}$$

in the expansion of

$$(1+x\bar{x})^{a'}(x+\bar{x})^{a}(1+y\bar{y})^{b'}(y+\bar{y})^{b} \times[(1+xy)(1+\bar{x}\bar{y})]^{c}[(x+y)(\bar{x}+\bar{y})]^{d}.$$

A. Case of $J'=0$

This case has been discussed in Ref. 4. We shall give a brief review here. The function $f_n=(1+x\bar{x})^{2J-2n}(x+\bar{x})^{2n}$ generates a special solution of Eq. (1) which has the following properties:

$$T_n(\alpha',\alpha'',0,0)=0 \quad \text{for all} \quad \alpha' \text{ and } \alpha'' \text{ if } |\alpha'-\alpha''|>2n,$$
$$\neq 0 \quad \text{for all} \quad \alpha' \text{ and } \alpha'' \text{ if } |\alpha'-\alpha''|=2n.$$

Let us choose one t-amplitude among those amplitudes which have the same s kinematic factor. The amplitudes chosen in this way are denoted by I^m:

$$I^m=T(\alpha_m', \alpha_m'',0,0), \quad m=0, 1, \cdots$$

where $|\alpha_m'-\alpha_m''|=2m$. We shall prove that the amplitudes I^m have the properties $P1$ and $P2$.

Let us assume that the amplitudes I^m are not independent. This assumption means that the I^m are linearly related; namely, we have the identity $\sum_m C_m I^m=0$ where the C_m are constants. This identity is satisfied by any special solution. In fact, each special solution gives a linear relation among the coefficients C_m. Starting from the special solution T_0 which is generated by f_0, we have $C_0=0$. The special solution T_1 which is generated by f_1 implies that C_1 is proportional to C_0. Since we know $C_0=0$, it follows that $C_1=0$. We can continue this procedure to prove that all of the C_m must be zero. In other words, the I^m are independent. Since the total number of the I^m is the same as the number of independent s-channel forward amplitudes, any other t-amplitude is linearly related to the I^m and we have

$$T(\alpha',\alpha'',0,0)=\Sigma_m h(\alpha',\alpha'',m)I^m,$$

where the h are constants. These linear relations are identities and are satisfied by any special solution. Each special solution gives a set of linear relations among the h. The property $P2$ means that $h=0$ if $|\alpha'-\alpha''|>2m$ but $h\neq 0$ if $|\alpha'-\alpha''|=2m$. To see this, let us start from the special solution T_0. This solution implies that $h(\alpha',\alpha'',0)=0$ if and only if $|\alpha'-\alpha''|>0$. The same kind of argument can be used to the special solutions T_1, T_2, etc. Notice that all nonvanishing constants h can be determined in this way.

B. Case of $J'=\frac{1}{2}$

The amplitudes corresponding to different kinematic factors are independent. To see this, let us associate each amplitude with a generating function of a special solution in the following way:

$$(1+x\bar{x})^{2J-2n}(1+y\bar{y})(x+\bar{x})^{2n} \quad \text{for} \quad \lambda=2n, \quad \mu=0$$
$$(1+x\bar{x})^{2J-2n-1}(1+xy)(1+\bar{x}\bar{y})(x+\bar{x})^{2n} \quad \text{for} \quad \lambda=2n+1, \quad \mu=1$$
$$(1+x\bar{x})^{2J-2n-1}(x+y)(\bar{x}+\bar{y})(x+\bar{x})^{2n} \quad \text{for} \quad \lambda=2n+1, \quad -\mu=1.$$

The rest of the discussion is essentially the same as before.

C. Case of $J'>\frac{1}{2}$

We shall prove that the amplitudes chosen according to our rules have the properties $P1$ and $P2$. For convenience, we use the following notations: $[+]=(1+xy)\times(1+\bar{x}\bar{y})$, $[-]=(x+y)(\bar{x}+\bar{y})$, $X=1+x\bar{x}$, $Y=1+y\bar{y}$, and $I^m(\lambda,\mu)$ = the mth amplitude chosen from the subgroup of amplitudes classified by λ and μ.

We associate each of these amplitudes with a generating function of a special solution in this way:

$I(n, \pm n)$ with $X^{2J-n}Y^{2J'-n}[\pm]^n$,
$I(n, \pm n\pm 2)$ with $X^{2J-n}Y^{2J'-n-2}(y+\bar{y})^2[\pm]^n$,
$I^1(n+2, \pm n)$ with $X^{2J-n-2}Y^{2J'-n}(x+\bar{x})^2[\pm]^n$,
$I^2(n+2, \pm n)$ with $X^{2J-n-2}Y^{2J'-n-2}[\pm]^{n+1}[\mp]$,
$I^1(n+4, \pm n)$ with $X^{2J-n-4}Y^{2J'-n}(x+\bar{x})^4[\pm]^n$,
$I^2(n+4, \pm n)$ with $X^{2J-n-4}Y^{2J'-n-2}(x+\bar{x})^2[\pm]^{n+1}[\mp]$,
$I^3(n+4, \pm n)$ with $X^{2J-n-4}Y^{2J'-n-4}[\pm]^{n+2}[\mp]^2$,
etc.

If these amplitudes are not independent, they must satisfy a linear relation

$$\sum_{\lambda,\mu,m} C^m(\lambda,\mu) I^m(\lambda,\mu) = 0.$$

Each special solution implies a linear relation among the constants C. Let us consider first the special solution associated with $I(0,0)$. This solution implies $C(0,0)=0$. The special solution associated with $I(1,1)$ implies that $C(1,1)$ is proportional to $C(0,0)$. Since we know $C(0,0)=0$, it follows $C(1,1)=0$. Similar consideration shows that all C are zero.

The total number of I equals the number of independent s-channel forward amplitudes. Therefore, any other t-amplitude can be written as a linear combination of $I^m(\lambda,\mu)$, say,

$$T(\alpha'\alpha''\beta'\beta'') = \sum_{\lambda,\mu,m} h^m(\lambda,\mu) I^m(\lambda,\mu),$$

where the coefficients h are constants. The property $P2$ means that the constants h vanish if

$$|\lambda+\mu| < |\alpha'-\alpha''+\beta'-\beta''|$$

or if

$$|\lambda-\mu| < |\alpha'-\alpha''-\beta'+\beta''|,$$

and at least one of the h^m with $\lambda=\alpha'-\alpha''$ and $\mu=\beta'-\beta''$ does not vanish. Let us denote the special solution associated with $I^m(\lambda,\mu)$ by $T^{\lambda,\mu,m}(\alpha_1\alpha_2\beta_1\beta_2)$. One observes that the following conditions hold for all λ and μ:

$$T=0 \text{ if } |\lambda+\mu| < |\alpha_1-\alpha_2+\beta_1-\beta_2|,$$
$$\text{or if } |\lambda-\mu| < |\alpha_1-\alpha_2-\beta_1+\beta_2|,$$

and there exists at least one m such that $T\neq 0$ if $\lambda=\alpha_1-\alpha_2$ and $\mu=\beta_1-\beta_2$. These conditions have the following consequence: If there exists one amplitude $T(\alpha'\alpha''\beta'\beta'')$ which does not satisfy $P2$, then those amplitudes $I^m(\lambda,\mu)$ with $|\lambda+\mu| < |\alpha'-\alpha''+\beta'-\beta''|$ or $|\lambda-\mu| < |\alpha'-\alpha''-\beta'+\beta''|$ must be linearly related. On the other hand, we have shown that all $I^m(\lambda,\mu)$ are independent. Therefore, we conclude that the property $P2$ is shared by all amplitudes of group A.

APPENDIX II: RELATIONS AMONG THE SUPERCONVERGENCE CONDITIONS FOR FORWARD AMPLITUDES OF GROUP B

The discussions given here are closely related to those of Appendix I. Again we need a set of special solutions for Eq. (2). Let us define the functions $F_1{}^*(\alpha^*\beta^*\alpha\beta)$ and $F_2{}^*(\alpha^*\beta^*\alpha\beta)$ to be the coefficients of the term

$$x^{J-\alpha}(x^*)^{J-\alpha^*}y^{J'-\beta}(y^*)^{J'-\beta^*}[(J+\alpha)!(J-\alpha)!(J+\alpha^*)! \times(J-\alpha^*)!(J'+\beta)!(J'-\beta)!(J'+\beta^*)!(J'-\beta^*)!]^{-1/2}$$

in the expansion of, respectively, $(x-x^*)f(a',a,b',b,c,d)$ and $(y-y^*)f(a^*,a'',b^*,b'',c',d')$, where

$$f(a',a,b',b,c,d)=(1+xx^*)^{a'}(1+yy^*)^{b'}(1-xx^*)^{a}(1-yy^*)^{b} \times[(1+xy)(1+x^*y^*)]^c[(1-xy)(1-x^*y^*)]^d,$$

$$a+b=\text{even},\quad a''+b''=\text{even},\quad a'+a+c+d=2J-1,$$
$$b'+b+c+d=2J',\quad a^*+a''+c'+d'=2J,$$
$$b^*+b''+c'+d'=2J'-1.$$

These functions satisfy the following conditions: $F^*=0$ if $|\lambda^*-\mu^*|\neq 1$ and $F^*(\alpha^*\beta^*\alpha\beta)=-F^*(\alpha\beta\alpha^*\beta^*)=F^*(-\alpha-\beta-\alpha^*-\beta^*)$. Each function generates a special solution of Eq. (2). The special solutions $T_1{}^*(\alpha'\alpha''\beta'\beta'')$ and $T_2{}^*(\alpha'\alpha''\beta'\beta'')$ are given by the coefficients of the term

$$x^{J-\alpha'}\bar{x}^{J-\alpha''}y^{J'-\beta'}\bar{y}^{J'-\beta''}[(J+\alpha')!(J+\alpha'')! \times(J-\alpha')!(J-\alpha'')!(J'+\beta')!(J'+\beta'')! \times(J'-\beta')!(J'-\beta'')!]^{-1/2}$$

in the expressions of, respectively, $(\bar{x}-x)f^*$ and $(\bar{y}-y)f^*$ where

$$f^*(a',a,b',b,c,d)=(1+x\bar{x})^{a'}(1+y\bar{y})^{b'}(x+\bar{x})^{a} \times(y+\bar{y})^{b}[(1+xy)(1+\bar{x}\bar{y})]^c[(x+y)(\bar{x}+\bar{y})]^d.$$

A. Case of $J'=0$

This case has been discussed in Ref. 4. The special solutions generated by

$$(x-\bar{x})(1+x\bar{x})^{2J-2n-1}(x+\bar{x})^{2n},\quad n=0,1,2,\cdots$$

satisfy the conditions:

$$T^*=0 \quad\text{if}\quad |\lambda|>2n+1,$$
$$T^*\neq 0 \quad\text{if}\quad |\lambda|=2n+1.$$

B. Case of $J'>0$

Any linear combination of different solutions of Eq. (2) is itself a solution. Let us consider the solutions generated by the following functions:

$$F^1=(x-\bar{x})(x+\bar{x})(y+\bar{y})F,$$
$$F^2=(y-\bar{y})(x+\bar{x})^2F,$$
$$F^3=(x-\bar{x})(1+xy)(1+\bar{x}\bar{y})F,$$
$$F^4=(x-\bar{x})(1+x\bar{x})(1+y\bar{y})F,$$

where $F=f^*(a',a,b',b,c,d)$, $a+b=\text{even}$, $a'+a+c+d=2J-2$, $b'+b+c+d=2J'-1$. The following function generates a solution:

$$F^1+F^2+2F^4-2F^3=2F[(x-\bar{x})(x\bar{x}+y\bar{y})+2x\bar{x}(y-\bar{y})].$$

Similar argument leads to the following generating function:

$$[(y-\bar{y})(x\bar{x}+y\bar{y})+2y\bar{y}(x-\bar{x})]f^*(a',a,b',b,c,d),$$

where $a+b=\text{even}$, $a'+a+c+d=2J-1$, $b'+b+c+d=2J'-2$. We use the notations:

$$A\equiv(x-\bar{x})(x\bar{x}+y\bar{y})+2x\bar{x}(y-\bar{y}),$$
$$B\equiv(y-\bar{y})(x\bar{x}+y\bar{y})+2y\bar{y}(x-\bar{x}).$$

We associate each of the amplitudes $I^m(\lambda,\mu)$ with a generating function of a special solution in the following way:

For $|\mu|\neq 2J'$ and $\lambda\leqslant 2J'$,

$I^1(n+1,\pm n)$ with $(x-\bar{x})[\pm]^nX^{2J-n-1}Y^{2J'-n}$,

$I^2(n+1,\pm n)$ with $A[\pm]^nX^{2J-n-2}Y^{2J'-n-1}$,

$I^1(n,\pm n\pm 1)$ with $(y-\bar{y})[\pm]^nX^{2J-n}Y^{2J'-n-1}$,

$I^2(n,\pm n\pm 1)$ with $B[\pm]^nX^{2J-n-1}Y^{2J'-n-2}$,

$I^1(n+3,\pm n)$ with $(x-\bar{x})(x+\bar{x})^2[\pm]^nX^{2J-n-3}Y^{2J'-n}$,

$I^2(n+3,\pm n)$ with $(x-\bar{x})[\pm]^{n+1}[\mp]X^{2J-n-3}Y^{2J'-n-2}$,

$I^3(n+3,\pm n)$ with $A(x+\bar{x})^2[\pm]^nX^{2J-n-4}Y^{2J'-n-1}$,

$I^4(n+3,\pm n)$ with $A[\pm]^{n+1}[\mp]X^{2J-n-4}Y^{2J'-n-3}$,

etc.

For $|\mu|=2J'$ and $\lambda\leqslant 2J'$,

$I(2J'-1,\pm 2J')$ with $(y-\bar{y})[\pm]^{2J'-1}X^{2J-2J'+1}$,

$I^1(2J'-3,\pm 2J')$ with $(y-\bar{y})(y+\bar{y})^2[\pm]^{2J'-3}X^{2J-2J'+3}$,

$I^2(2J'-3,\pm 2J')$ with $(y-\bar{y})[\pm]^{2J'-2}[\mp]X^{2J-2J'+1}$,

etc.

For $\lambda=2J$ and $J+J'=\text{half integer}$,

$I(2J,\pm 2J')$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'-1}[\pm]^{2J'}$,

$I^1(2J,\pm 2J'\mp 2)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'+1}[\pm]^{2J'-2}Y^2$,

$I^2(2J,\pm 2J'\mp 2)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'-1}[\pm]^{2J'-1}[\mp]$,

$I^1(2J,\pm 2J'\mp 4)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'+3}[\pm]^{2J'-4}Y^4$,

$I^2(2J,\pm 2J'\mp 4)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'+1}[\pm]^{2J'-3}[\mp]Y^2$,

$I^3(2J,\pm 2J'\mp 4)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'-1}[\pm]^{2J'-2}[\mp]^2$,

etc.

For $\lambda=2J$ and $J+J'=\text{integer}$,

$I(2J,\pm 2J'\mp 1)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'}[\pm]^{2J'-1}Y$,

$I^1(2J,\pm 2J'\mp 3)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'+2}[\pm]^{2J'-3}Y^3$,

$I^2(2J,\pm 2J'\mp 3)$ with $(x-\bar{x})(x+\bar{x})^{2J-2J'}[\pm]^{2J'-2}[\mp]Y$,

etc.

For $2J>\lambda>2J'$,

$I(2J'+2n+1,\pm 2J')$ with $(x-\bar{x})(x+\bar{x})^{2n}[\pm]^{2J'}X^{2J-2J'-2n-1}$,

$I^1(2J'+2n,\pm 2J'\mp 1)$ with $(x-\bar{x})(x+\bar{x})^{2n}[\pm]^{2J'-1}X^{2J-2J'-2n}Y$,

$I^2(2J'+2n,\pm 2J'\mp 1)$ with $A(x+\bar{x})^{2n}[\pm]^{2J'-1}X^{2J-2J'-2n+1}$,

$I^1(2J'+2n+1,\pm 2J'\mp 2)$ with $(x-\bar{x})(x+\bar{x})^{2n+2}[\pm]^{2J'-2}X^{2J-2J'-2n-1}Y^2$,

$I^2(2J'+2n+1,\pm 2J'\mp 2)$ with $A(x+\bar{x})^{2n+2}[\pm]^{2J'-2}X^{2J-2J'-2n}Y$,

$I^3(2J'+2n+1,\pm 2J'\mp 2)$ with $(x-\bar{x})(x+\bar{x})^{2n}[\pm]^{2J'-1}[\mp]X^{2J-2J'-2n-1}$,

etc.

Using these special solutions, one can use the same argument of Appendix I to prove that the properties $P1$ and $P2$ are satisfied by the amplitudes chosen according to our rules.