Relativistic Model for Electroproduction of Nucleon Resonances*

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We have made a simple model to obtain some estimates of the inelastic form factors for electron excitation of the nucleon resonances. The model is covariant and gauge invariant, and satisfies all the general requirements of the theory. It relies heavily on the lead of previous theoretical work on the form factor for the $\frac{3}{2}^+$, $\frac{3}{2}$ (1238-MeV) level. The basic idea is to take from experiment the knowledge of which nucleon states are resonant. We then look at single-pion electroproduction and project out the relevant multipoles from a covariant, gauge-invariant set of graphs which are thought to play an important role as an excitation mechanism. The multipoles are then multiplied by a final-state enhancement factor which provides a resonance mechanism. We give some theoretical justification for this procedure. If just the electron is detected, one measures the virtual-photon width for formation of the resonance, and our result contains the possibility that the resonance can decay into other channels than just $\pi+N$. We calculate only the inelastic form factors. The individual contributions of the resonance levels are normalized to photoabsorption, where such data exist. We keep π, ω, and N exchange as an excitation mechanism, and treat the over-all contribution of the ω exchange as a single parameter, with which we are able to fit all the existing inelastic-electron-scattering data. We actually find two equally acceptable fits with quite different properties. The ω-nucleon coupling constant we get from this analysis is in reasonable agreement with other determinations of this quantity. Form factors for all the nucleon levels up through 2650 MeV are presented out to momentum transfers of interest in the SLAC experiments.

1. INTRODUCTION

THERE are two reasons why electron scattering is such a powerful tool for studying nuclear structure. The first is that the interaction is known. The electrons interact with the local charge and current density in the target. Since this interaction is relatively weak, of order $\alpha=1/137$, one can make measurements without greatly disturbing the structure of the target. Of course, the same holds for real photon processes, but electrons have the second great advantage that for a fixed energy transfer, one can *vary* the 3-momentum transferred, the only restriction being that the 4-momentum transferred be spacelike. Thus one can map out the Fourier transform of the transition charge and current densities, and this is a rich and unique source of information on nuclear structure. With the advent of very-high-energy electron accelerators, for example, the Cambridge Electron Accelerator (CEA), the Deutsches Elektronen-Synchrotron (DESY), and especially the Stanford Electron Accelerator Center (SLAC), electron excitation has become a practical means for studying the details of the excited states of the nucleon. To indicate the richness of possibilities here we show the "low-lying" spectrum of the nucleon in Fig. 1.[1]

From both a theoretical and experimental standpoint one would like to have some idea of what to expect in these experiments. From a theoretical point of view one would at least like to make some predictions *before* the experiments are carried out. From an experimental point of view, estimates of the transition form factors are useful in planning new experiments and in interpreting, understanding, and correlating the data as they accumulate.

The detailed theoretical understanding of the excited states of the nucleon requires a theory of strong interactions, but reliable, quantitative calculations are exceedingly difficult and in many cases impossible at the present time. We shall therefore make a very simple model to attempt to get some physical insight into what is going on and to get some feeling for what is to be expected in these experiments. Although the model can be, at best, only a crude approximation to the strong-interaction dynamics, it does have the distinct advantage that it keeps all the general properties of the theory, including covariance, gauge invariance, analytic properties, and threshold behaviors.

We first make a general covariant analysis of the transition matrix element for the process

$$\gamma^*+N\rightarrow N+\pi,$$

where γ^* is the Møller potential from the electron. In this we follow the work of Fubini, Nambu, and Wataghin (FNW)[2] and find that there are six independent kinematic invariants which can be chosen to be explicitly gauge invariant [that is, replacing $\epsilon_\mu\rightarrow k_\mu$ gives identically zero] and six independent invariant amplitudes which are functions of three scalar variables. Going to the center-of-momentum frame for the above process, these invariant amplitudes can be expressed, in standard fashion, in terms of the independent electromagnetic transition multipoles: a transverse electric and magnetic multipole and a Coulomb multipole. Thus at this stage, whatever approximation we put in for the

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† National Science Foundation Predoctoral Fellow.
[1] A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, Matts Roos, Paul Soding, W. J. Willis, C. G. Wohl, University of California Radiation Laboratory Report No. UCRL-8030 Rev., 1967 (unpublished).

[2] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. 111, 329 (1958).

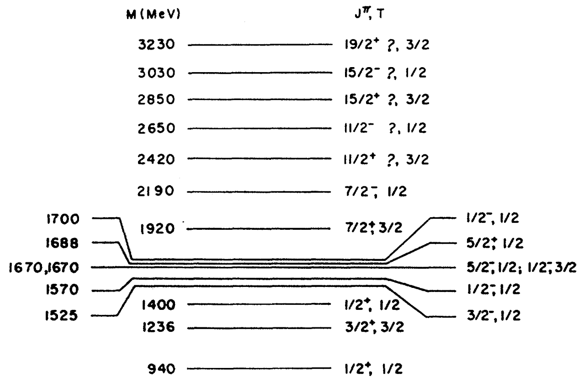

FIG. 1. The known nucleon resonances (Ref. 1).

individual multipoles, we still have an over-all amplitude which is covariant and explicitly gauge invariant. From experiment we take the knowledge of which particular multipoles are resonant. We then choose a covariant, gauge-invariant set of graphs that we believe should play an important role as an excitation mechanism and project out the appropriate multipoles. These multipoles are then multiplied by a final-state enhancement factor; we give some theoretical justification for this procedure. The transition multipole amplitudes then resonate at the appropriate energy. The fact that these resonances may decay inelastically is easily incorporated into this model.

Since the final-state enhancement factor depends only on the total energy in the center-of-momentum frame, the momentum-transfer dependence of the multipole amplitudes enters through a known function, and we can calculate the transition form factors. At present, it is only the form factors that we attempt to calculate. We normalize the over-all contribution of the various resonant states to the values known from a phenomenological analysis of photoproduction. There is thus one scale factor for each state which is taken from experiment, but the model then predicts the relative contributions of the multipoles for any given state at all momentum transfers.

As an excitation mechanism we have kept π, ω, and N exchanges. The first and last must be included together for gauge invariance, and we give some arguments that the second is the most important vector-meson exchange. The over-all contribution of the ω-exchange diagram is treated as a single parameter, and we attempt to fit all the presently known inelastic data. The value that we obtain for this parameter is found to be in reasonable agreement with other estimates of this quantity. Our contributions thus come from three different regions of the exchanged-particle mass spectrum. We have the most peripheral contribution, which is expected to play a dominant role in the excitation of higher spin states at low momentum transfers because of the centrifugal barrier $\sim(kR)^l$; further, we can calculate it exactly. Also included is an intermediate-mass contribution about which we have a great deal of knowledge and a high-mass contribution where our ignorance becomes glaringly evident. Undaunted by this, we make predictions for the form factors of all the known nucleon resonances, and calculate them out to momentum transfers of interest in the SLAC experiments.[3]

Now electron excitation of the $\frac{3}{2}^+$, $\frac{3}{2}$ (1238) resonance has been studied in great detail and with much more sophisticated treatments than that presented here, because starting from the work of Chew and Low[4] we have had some dynamical understanding of this resonance. The work goes back to FNW[2] and important contributions have been made by Dennery,[5] by Zagury,[6] by Simon and Gutbrot,[7] and by Vik.[8] Vik also calculated the one-particle exchange contributions in the regions of the second and third nucleon resonances. The treatment presented here is only a simplified version and summary of these others, but it does contain the essential physics of these approaches, we believe, and we have merely tried to extend these ideas to the higher nucleon resonances, and to higher momentum transfers.

In Sec. 2 we briefly review what can be said, on general grounds, about the electron excitation of nucleon resonances. In Sec. 3, we review the general analysis of single-pion production by electrons. In Sec. 4 we discuss our model for the resonant multipoles. In Sec. 5 we calculate the multipole contributions of the graphs which we use as an excitation mechanism. In Sec. 6 we present the numerical results and a comparison with the existing experiments; Sec. 7 is a discussion and summary.

2. ELECTRON SCATTERING

We first give a very brief review of the theory of electron scattering. We concentrate on the case where only the final electron is detected, as in most of the experiments which have been done so far, and as will be the case in the SLAC experiments.[3] Bjorken and Walecka[9] have given a relativistically covariant analysis of the process of electron excitation of the nucleon and have discussed all that can be said, on general grounds, about the transition form factors. They also show the relation to photoexcitation of the nucleon resonances. We summarize their results here.

The kinematical situation in the one-photon exchange approximation is shown in Fig. 2. The angular

[3] W. Panofsky, D. Coward, H. DeStaebler, J. Litt, L. Mo, R. Taylor, J. Friedman, H. Kendall, L. VanSpeybroek, C. Peck, and J. Pine, Stanford Linear Accelerator Center Group A—Proposal 4B (1966) (unpublished).
[4] G. F. Chew and F. Low, Phys. Rev. **101**, 1570 (1956); **101**, 1579 (1956).
[5] P. Dennery, Phys. Rev. **124**, 2000 (1961).
[6] N. Zagury, Phys. Rev. **145**, 1112 (1966).
[7] F. Gutbrod and D. Simon, DESY Report No. 67/1, 1967 (to be published).
[8] R. C. Vik, Phys. Rev. **163**, 1535 (1967).
[9] J. D. Bjorken and J. D. Walecka, Ann. Phys. (N. Y.) **38**, 35 (1966).

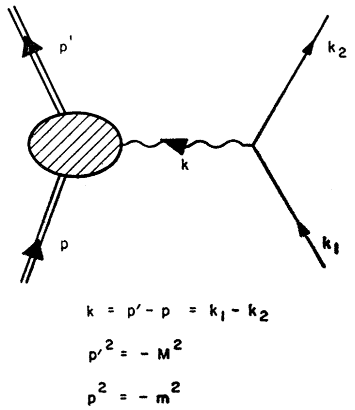

FIG. 2. The kinematics for inelastic electron scattering.

momentum analysis is best carried out in the rest frame of the final isobar, because one then has an eigenstate of angular momentum and parity. The electromagnetic vertex is characterized by four reduced matrix elements, or, equivalently, by the four linear combinations

$$f_\rho=\left(\frac{EE'\Omega^2}{8\pi M^2}\right)^{1/2}\sum_j\left(\frac{2j+1}{2J+1}\right)^{1/2}(j\,\tfrac{1}{2}\,1\rho|j\,1\,J\,\tfrac{1}{2}+\rho)\times\langle\pi_R J\|\mathbf{J}(0)\|k^*\pi j\rangle,\quad(2.1)$$

with $\rho=\pm1,0$ and

$$f_c=(EE'\Omega^2/8\pi M^2)^{1/2}\langle\pi_R J\|J_0(0)\|k^*\pi J\rangle.\quad(2.2)$$

In these expressions E and E' are the initial and final target energies, respectively, M is the isobar mass, Ω is the normalization volume, $J_\mu(0)=(\mathbf{J}(0),iJ_0(0))$ is the electromagnetic current operator taken at the origin, and $J^{\pi R}$ is the angular momentum and parity of the isobar. In the rest frame of the isobar one has

$$k_\mu\equiv(\mathbf{k}^*,ik_0).\quad(2.3)$$

There is still one relation among these four quantities coming from current conservation and it simply eliminates f_0;

$$f_0=(k_0/k^*)f_c.\quad(2.4)$$

The electron-scattering cross section in the laboratory is then shown to be (we set $m_e=0$)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}}=\frac{\alpha^2\cos^2\frac{1}{2}\theta}{4\epsilon^2\sin^4\frac{1}{2}\theta[1+2(\epsilon/m)\sin^2\frac{1}{2}\theta]}\times\{(k^4/k^{*4})|f_c|^2+(k^2/2k^{*2}+(M^2/m^2)\tan^2\tfrac{1}{2}\theta)\times[|f_+|^2+|f_-|^2]\}.\quad(2.5)$$

In this expression ϵ is the initial electron energy, θ is the electron scattering angle, m is the nucleon mass, and $k^2=k_\mu{}^2$ is the invariant 4-momentum transfer. We see that electron scattering measures two independent combinations of form factors, the Coulomb and transverse form factors. These may be separated experimentally by keeping k^2 and the energy loss $k_0=\epsilon-\epsilon'$ fixed and varying θ or by working at $\theta=180°$, where only the transverse contribution remains. The transverse form factor can also be measured at one momentum transfer, namely, $k_\mu{}^2=0$ or

$$k^*_{\text{thresh}}=(M^2-m^2)/2M\quad(2.6)$$

in photoexcitation,

$$\int_{\text{lab; over resonance}}\sigma_\gamma(\omega)d\omega=\frac{4\pi^2\alpha}{M^2-m^2}\frac{M^2}{m}\times[|f_+|^2+|f_-|^2]_{k^2=0}.\quad(2.7)$$

Thus with electron scattering, we can add a whole new dimension onto the photon problem. There is also the possibility of direct Coulomb excitation.

Detailed properties of the form factors $f_c,f_\pm$ are highly model-dependent. However, in the limit $k^*\to0$ (which implies $k_0\to M-m$) the form factors have simple threshold behaviors:

1. Normal parity transitions[10] $\frac{1}{2}^+\to\frac{3}{2}^-,\frac{5}{2}^+\cdots$.

$$f_c\sim(k^*)^{J-1/2},$$
$$f_\pm\sim(k^*)^{J-3/2}.$$

2. Abnormal parity transitions $\frac{1}{2}^+\to\frac{1}{2}^-,\frac{3}{2}^+,\frac{5}{2}^-\cdots$.

$$f_c\sim(k^*)^{J+1/2},$$
$$f_\pm\sim(k^*)^{J-1/2}.$$

One of the interesting questions which we would like our model to shed some light on is whether or not these threshold behaviors are of any use, because only space-like momentum transfers are available experimentally $[k^2\geq0]$, and it is not clear whether the threshold behavior still persists there since this implies a minimum 3-momentum transfer

$$k^*\geqslant k^*_{\text{thresh}}=(M^2-m^2)/2M.\quad(2.8)$$

For the normal parity transitions there is an additional relation between f_c and $f_\pm$ valid near threshold:

$$\frac{|f_+|^2+|f_-|^2}{|f_c|^2}\underset{k^*\to0}{\approx}\left(\frac{J+\frac{1}{2}}{J-\frac{1}{2}}\right)\left(\frac{k_0}{k^*}\right)^2.\quad(2.9)$$

This relation is well known in nuclear physics. In particular, it is the relation which allows one to get photon lifetimes for electric transitions from Coulomb excitation.

3. GENERAL ANALYSIS OF SINGLE-PION ELECTROPRODUCTION

In this section we review what can be said about the process of single-pion electroproduction on general grounds alone.[11] We assume a single-photon exchange

[10] For the special case $\frac{1}{2}^+\to\frac{1}{2}^+$, $f_c\sim(k^*)^2$ and $f_-\sim k^*(f_+=0)$.

[11] We use a metric so that $a_\mu=(\mathbf{a},ia_0)$. Our γ matrices are Hermitian and satisfy $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}$. The Dirac equation is $(i\gamma\cdot p+m)u(p)=0$ and the spinors are normalized to $\bar{u}u=1$. We set $\hbar=c=1$. Note that $e^2/4\pi=\alpha=1/137$ and $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$.

mechanism; the kinematical situation is shown in Fig. 3. The analysis follows closely that of photoproduction of pions by Chew, Goldberger, Low, and Nambu (CGLN)[12] and of electroproduction by Fubini, Nambu, and Wataghin.[2] The results are similar to those obtained recently by Zagury,[6] although the emphasis is somewhat different. In particular, we will keep explicit current conservation (or equivalently gauge invariance) throughout.

We define the quantities

$$\begin{aligned} P&\equiv\tfrac{1}{2}(p_1+p_2)\,,\\ \Delta&\equiv\tfrac{1}{2}(p_2-p_1)=\tfrac{1}{2}(k-q)\,,\\ \nu&\equiv-k\cdot P/m=-q\cdot P/m\,,\\ \nu_1&\equiv-k\cdot q/2m. \end{aligned} \tag{3.1}$$

The c.m. system is defined by $\mathbf{p}_1+\mathbf{k}=\mathbf{p}_2+\mathbf{q}=0$. In this frame we write $k_\mu=(\mathbf{k}^*,ik_0)$ and denote the total energy by W. The nucleon energies are E_1 and E_2 and the meson energy is ω_q. The following relations exist between these quantities:

$$\begin{aligned} 2k_0W&=W^2-m^2-k^2\,,\\ 2\omega_qW&=W^2-m^2+\mu^2\,,\\ \nu&=(1/2m)(W^2-m^2)+k\cdot q/2m\,,\\ \frac{k_0}{E_1+m}&=\frac{W-m}{W+m}-\frac{k^2}{(W+m)(E_1+m)}. \end{aligned} \tag{3.2}$$

For the strong-interaction part of the above process we need the covariant amplitude

$$\left(\frac{2\omega_qE_1E_2\Omega^3}{m^2}\right)^{1/2}\langle qp_2{}^{(-)}|J_\mu\epsilon_\mu|p_1\rangle =\bar{u}(p_2)\Big[\sum_{i=1}^{6}\epsilon_\mu M_i{}^\mu A_i(W,\Delta^2,k^2)\Big]u(p_1)\,, \tag{3.3}$$

where on the left side we have the matrix element of the electromagnetic current operator between exact Heisenberg states. ϵ_μ is the Møller potential

$$\epsilon_\mu=(e/k^2)\bar{u}(k_2)\gamma_\mu u(k_1) \tag{3.4}$$

and Ω is our normalization volume. We have expanded the Heisenberg matrix element, which is now a Lorentz scalar, in terms of a complete set of six kinematic invariants. Following FNW, we choose $[\epsilon_\mu M_\mu{}^{(i)}\equiv M_{(i)}$, $i=A,B,\cdots,F]$

$$\begin{aligned} M_A&=\tfrac{1}{2}i\gamma_5[(\gamma\cdot\epsilon)(\gamma\cdot k)-(\gamma\cdot k)(\gamma\cdot\epsilon)]\,,\\ M_B&=2i\gamma_5[(P\cdot\epsilon)(q\cdot k)-(P\cdot k)(q\cdot\epsilon)]\,,\\ M_C&=\gamma_5[(\gamma\cdot\epsilon)(q\cdot k)-(\gamma\cdot k)(q\cdot\epsilon)]\,,\\ M_D&=2\gamma_5[(\gamma\cdot\epsilon)(P\cdot k)-(\gamma\cdot k)(P\cdot\epsilon)]\\ &\quad-im\gamma_5[(\gamma\cdot\epsilon)(\gamma\cdot k)-(\gamma\cdot k)(\gamma\cdot\epsilon)]\,,\\ M_E&=i\gamma_5[(k\cdot\epsilon)(q\cdot k)-(q\cdot\epsilon)k^2]\,,\\ M_F&=\gamma_5[(\gamma\cdot k)(k\cdot\epsilon)-(\gamma\cdot\epsilon)k^2]. \end{aligned} \tag{3.5}$$

[12] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957); **106**, 1345 (1957); see also, J. S. Ball, *ibid.* **124**, 2014 (1961).

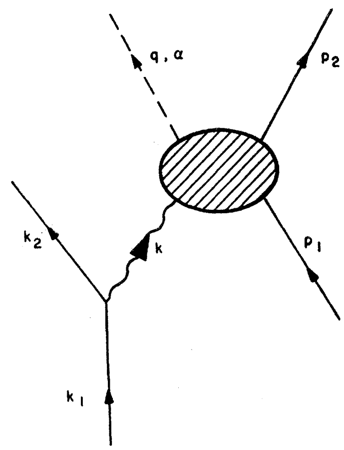

FIG. 3. The kinematics for electroproduction of one pion by single-photon exchange.

These invariants have the distinct advantage that they are explicitly gauge invariant;

$$k_\mu M_i{}^\mu=0\,,\quad i=1,\cdots,6. \tag{3.6}$$

This is the statement of current conservation for the strongly interacting part of the process. The invariant amplitudes are now functions of W, Δ^2, and k^2.

Let us proceed to a further analysis of the invariant matrix element in the c.m. system. Because of current conservation we have

$$\langle qp_2{}^{(-)}|\mathbf{J}\cdot\hat{k}|p_1\rangle=(k_0/k^*)\langle qp_2{}^{(-)}|\rho|p_1\rangle\,, \tag{3.7}$$

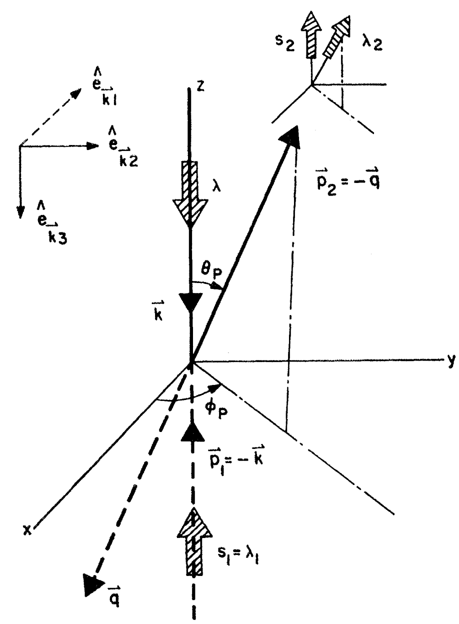

FIG. 4. Our choice of angles and helicity unit vectors in the center-of-momentum frame.

and we only need to analyze the current matrix elements, for the charge matrix elements are then determined. We therefore only need to know the matrix elements for ϵ_μ a complete set of three-dimensional unit vectors, $\hat{e}_{k\lambda}$, where

$$\begin{aligned}\hat{e}_{k\pm1}&\equiv\mp\tfrac{1}{2}\sqrt{2}[\hat{e}_{k1}\pm i\hat{e}_{k2}],\\ \hat{e}_{k0}&\equiv\hat{e}_{k3}\equiv\hat{k}.\end{aligned}\tag{3.8}$$

We chose our coordinate system in the c.m. system, as indicated in Fig. 4. In this case we can make a reduction of the amplitude from Dirac spinors to Pauli spinors, and we find

$$\left(\frac{2\omega_q E_1E_2\Omega^3}{m^2}\right)^{1/2}\langle qp_2^{(-)}|\mathbf{J}\cdot\hat{e}|p_1\rangle=\eta_{s_2}{}^{+}\Big(\sum_{i=1}^{6}m_iG_i\Big)\eta_{s_1},\tag{3.9}$$

where we have defined

$$\begin{aligned}m_1&=i\boldsymbol{\sigma}\cdot\hat{e},\\ m_2&=(\boldsymbol{\sigma}\cdot\hat{q})[\boldsymbol{\sigma}\cdot(\hat{k}\times\hat{e})],\\ m_3&=i(\boldsymbol{\sigma}\cdot\hat{k})(\hat{q}\cdot\hat{e}),\\ m_4&=i(\boldsymbol{\sigma}\cdot\hat{q})(\hat{q}\cdot\hat{e}),\\ m_5&=i(\boldsymbol{\sigma}\cdot\hat{q})(\hat{k}\cdot\hat{e}),\\ m_6&=i(\boldsymbol{\sigma}\cdot\hat{k})(\hat{k}\cdot\hat{e}).\end{aligned}\tag{3.10}$$

This expression is still exact, of course. The relation between the two sets of invariant amplitudes is given by

$$\begin{aligned}G_1&=\left[\frac{(E_1+m)(E_2+m)}{4m^2}\right]^{1/2}(W-m)\left[A+(W-m)D+\frac{2m\nu_1}{W-m}(C-D)+\frac{k^2}{W-m}F\right],\\
G_2&=\frac{qk^*(W+m)}{[4m^2(E_1+m)(E_2+m)]^{1/2}}\left[-A+(W+m)D+\frac{2m\nu_1}{W+m}(C-D)+\frac{k^2}{W+m}F\right],\\
G_3&=qk^*(W+m)\left[\frac{E_2+m}{4m^2(E_1+m)}\right]^{1/2}\left[C-D+(W-m)B-\frac{k^2}{W+m}E\right],\\
G_4&=\mathbf{q}^2(W-m)\left[\frac{E_1+m}{4m^2(E_2+m)}\right]^{1/2}\left[C-D-(W+m)B+\frac{k^2}{W-m}E\right],\\
G_5&=\frac{qk^*}{[4m^2(E_1+m)(E_2+m)]^{1/2}}\{k_0[-A+(W+m)(D-F)+2m\nu_1(B-E)]+2m\nu_1[C-D-(W+m)(B-E)]\},\\
G_6&=k^{*2}\left[\frac{E_2+m}{4m^2(E_1+m)}\right]^{1/2}[-A-2m\nu_1(E-B)-(W+m)F-(W-m)D].\end{aligned}\tag{3.11}$$

The connection with the photoproduction amplitudes $\mathfrak{F}_i$ of CGLN[12] is

$$\begin{aligned}(m/4\pi W)G_i&\equiv\mathfrak{F}_i, && i=1,\cdots,6\\ \mathfrak{F}_i(k^2=0)&=\mathfrak{F}_i{}^{\mathrm{CGLN}}, && i=1,2,3,4.\end{aligned}\tag{3.12}$$

We can further make a multipole analysis of our invariant amplitude in this frame. This is most easily done through a helicity analysis following Jacob and Wick.[13] The invariant amplitude can be expanded in terms of helicity amplitudes as[14]

$$\eta_{\lambda_2}{}^{+}\Big(\sum_{i=1}^{6}m_i\mathfrak{F}_i\Big)\eta_{s_1}=\frac{1}{(4k^*q)^{1/2}}\sum_J(2J+1)\mathfrak{D}_{\lambda_1-\lambda_k,\lambda_2}{}^{J}(-\phi_p-\theta_p\phi_p)^*\times\langle\lambda_2|T^J(W,k^2)|\lambda_1\lambda_k\rangle,\tag{3.13}$$

where λ_1 and λ_2 are the initial and final nucleon helicities, respectively, and λ_k is the virtual photon helicity. The Pauli spinor $\eta_{\lambda_2}{}^{+}$ can be written in terms of our previous Pauli spinor $\eta_{s_2}{}^{+}$ (representing spin up and down along the $-\hat{k}^*$ axis) by

$$\eta_{\lambda_2}{}^{+}=\sum_{s_2}\mathfrak{D}_{\lambda_2,s_2}{}^{1/2}(-\phi_p\theta_p\phi_p)\eta_{s_2}{}^{+}.\tag{3.14}$$

It is convenient to go to amplitudes of definite parity by introducing the linear combinations of helicity states

$$\begin{aligned}|J^{\pm}M\tfrac{1}{2}1\rangle&\equiv\tfrac{1}{2}\sqrt{2}[|JM\tfrac{1}{2}1\rangle\mp|JM-\tfrac{1}{2}-1\rangle],\\ |J^{\pm}M-\tfrac{1}{2}1\rangle&\equiv\tfrac{1}{2}\sqrt{2}[|JM-\tfrac{1}{2}1\rangle\mp|JM\tfrac{1}{2}-1\rangle],\\ |J^{\pm}M\tfrac{1}{2}0\rangle&\equiv\tfrac{1}{2}\sqrt{2}[|JM\tfrac{1}{2}0\rangle\mp|JM-\tfrac{1}{2}0\rangle],\end{aligned}\tag{3.15}$$

for the initial states. These states satisfy

$$\pi|J^{\pm}M\rangle=(-1)^{J\pm1/2}|J^{\pm}M\rangle.\tag{3.16}$$

We do the same for the final π-N states. In this case we can label the states by l since this is the same as specify-

[13] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).
[14] We use the angular momentum notation of A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

ing the parity. Therefore,

$$|l^{\pm}M\tfrac{1}{2}\rangle \equiv \tfrac{1}{2}\sqrt{2}[|JM\tfrac{1}{2}\rangle \pm |JM-\tfrac{1}{2}\rangle]. \tag{3.17}$$

(Note that we use the convention of $\eta_N=+1$, $\eta_\pi=-1$, and $\eta_\gamma=-1$ for the intrinsic parities in accordance with Jacob and Wick.[13]) For the left side we use the familiar notation $l^{\pm}$ for states with $J=l\pm\frac{1}{2}$ and the parity of these states is again

$$\pi|l^{\pm}M\tfrac{1}{2}\rangle=(-1)^{J\pm 1/2}|l^{\pm}M\tfrac{1}{2}\rangle \equiv (-1)^{l\pm 1}|l^{\pm}M\tfrac{1}{2}\rangle. \tag{3.18}$$

The parity amplitudes are then defined by

$$\begin{aligned} T_{1/2}{}^{l\pm} &\equiv (4k^*q)^{-1/2}\langle l^{\pm}|T^J(W,k^2)|\tfrac{1}{2},1^{\pm}\rangle,\\ T_{3/2}{}^{l\pm} &\equiv (4k^*q)^{-1/2}\langle l^{\pm}|T^J(W,k^2)|-\tfrac{1}{2},1^{\pm}\rangle,\\ L^{l\pm} &\equiv (4k^*q)^{-1/2}\langle l^{\pm}|T^J(W,k^2)|\tfrac{1}{2},0^{\pm}\rangle. \end{aligned} \tag{3.19}$$

The more familiar electric, magnetic, and longitudinal or Coulomb multipoles are then obtained from these quantities by the relations

$$\begin{aligned} (l+1)M_{l+} &\equiv -\tfrac{1}{2}i\sqrt{2}[T_{1/2}{}^{l+}+((l+2)/l)^{1/2}T_{3/2}{}^{l+}],\\ (l+1)E_{l+} &\equiv -\tfrac{1}{2}i\sqrt{2}[T_{1/2}{}^{l+}-(l/(l+2))^{1/2}T_{3/2}{}^{l+}],\\ lM_{l-} &\equiv -\tfrac{1}{2}i\sqrt{2}[T_{1/2}{}^{l-}-(l-1/l+1)^{1/2}T_{3/2}{}^{l-}],\\ lE_{l-} &\equiv \tfrac{1}{2}i\sqrt{2}[T_{1/2}{}^{l-}+(l+1/l-1)^{1/2}T_{3/2}{}^{l-}],\\ C_{l\pm} &\equiv (k^*/k_0)iL_{l\pm}\equiv (k^*/k_0)N_{l\pm}. \end{aligned} \tag{3.20}$$

We can now use Eqs. (3.13), (3.14), and the orthogonality of the $\mathfrak{D}_{\lambda\mu}{}^J$ functions to write the multipole analysis of the invariant amplitudes;

$$\begin{aligned} \mathfrak{F}_1 &= \sum_l \{[lM_{l+}+E_{l+}]P_{l+1}'(x)+[(l+1)M_{l-}+E_{l-}]P_{l-1}'(x)\},\\ \mathfrak{F}_2 &= \sum_l \{[(l+1)M_{l+}+lM_{l-}]P_l'(x)\},\\ \mathfrak{F}_3 &= \sum_l \{[E_{l+}-M_{l+}]P_{l+1}''(x)+[E_{l-}+M_{l-}]P_{l-1}''(x)\},\\ \mathfrak{F}_4 &= \sum_l \{[M_{l+}-E_{l+}-M_{l-}-E_{l-}]P_l''(x)\}, \end{aligned} \tag{3.21}$$

$$\mathfrak{F}_7 \equiv \left(\frac{k^*}{k_0}\right)(\mathfrak{F}_5+x\mathfrak{F}_4)=\sum_l \{[C_{l-}-C_{l+}]P_l'(x)\},$$

$$\mathfrak{F}_8 \equiv \left(\frac{k^*}{k_0}\right)(\mathfrak{F}_1+x\mathfrak{F}_3+\mathfrak{F}_6)=\sum_l \{[C_{l+}P_{l+1}'(x)-C_{l-}P_{l-1}'(x)]\},$$

where $x\equiv\cos(\sphericalangle\hat{k}\hat{q})$ and $P_l'(x)\equiv dP_l(x)/dx$. For $k^2\to 0$, the first four equations are those of CGLN.[12]

We note equivalently that if we analyze the Coulomb matrix element directly, we find

$$\begin{aligned} (2\omega_q E_1E_2\Omega^3/m^2)^{1/2}\langle qp_2{}^{(-)}|(-)J_0\epsilon_0|p_1\rangle &= \eta_{s_2}{}^+[m_7G_7+m_8G_8]\eta_{s_1},\\ m_7 &= -i\epsilon_0\boldsymbol{\sigma}\cdot\hat{q},\\ m_8 &= -i\epsilon_0\boldsymbol{\sigma}\cdot\hat{k}, \end{aligned} \tag{3 22}$$

and

$$\begin{aligned} \mathfrak{F}_7 &= \left(\frac{k^*}{k_0}\right)(\mathfrak{F}_5+x\mathfrak{F}_4)=\frac{1}{8\pi W}\left(\frac{E_2-m}{E_1-m}\right)^{1/2}k^*\{(W-m-k_0)[-A+(W+m)(D-F)]\\ &\quad +\omega_q[(W-m)(C-D)+k^2E-(W^2-m^2)B]+(k\cdot q)[C-D-k_0(E-B)-2WB]\},\\ \mathfrak{F}_8 &= \left(\frac{k^*}{k_0}\right)(\mathfrak{F}_1+x\mathfrak{F}_3+\mathfrak{F}_6)=\frac{1}{8\pi W}\left(\frac{E_2+m}{E_1+m}\right)^{1/2}k^*\{(W+m-k_0)[A+(W-m)(D-F)]\\ &\quad +\omega_q[(W+m)(C-D)+(W^2-m^2)B-k^2E]+(k\cdot q)[C-D+k_0(E-B)+2WB]\}. \end{aligned} \tag{3.23}$$

The multipole amplitudes here are functions of k^2 and W Equations (3.21) may be inverted. If we write

$$\mathfrak{F}_l{}^i(W,k^2)\equiv\tfrac{1}{2}\int_{-1}^{1}P_l(x)\mathfrak{F}^i(W,k^2,x)dx, \tag{3.24}$$

then

$$
\begin{aligned}
lE_{l-}&=\mathfrak{F}_l^1-\mathfrak{F}_{l-1}^2+\frac{l+1}{2l+1}[\mathfrak{F}_{l+1}^3-\mathfrak{F}_{l-1}^3]+\frac{l}{2l-1}[\mathfrak{F}_l^4-\mathfrak{F}_{l-2}^4],\\
lM_{l-}&=-\mathfrak{F}_l^1+\mathfrak{F}_{l-1}^2-\frac{1}{2l+1}[\mathfrak{F}_{l+1}^3-\mathfrak{F}_{l-1}^3],\\
(l+1)E_{l+}&=\mathfrak{F}_l^1-\mathfrak{F}_{l+1}^2-\frac{l}{2l+1}[\mathfrak{F}_{l+1}^3-\mathfrak{F}_{l-1}^3]-\frac{l+1}{2l+3}[\mathfrak{F}_{l+2}^4-\mathfrak{F}_l^4],\\
(l+1)M_{l+}&=\mathfrak{F}_l^1-\mathfrak{F}_{l+1}^2+\frac{1}{2l+1}[\mathfrak{F}_{l+1}^3-\mathfrak{F}_{l-1}^3],\\
C_{l+}&=\mathfrak{F}_{l+1}^7+\mathfrak{F}_l^8,\\
C_{l-}&=\mathfrak{F}_{l-1}^7+\mathfrak{F}_l^8.
\end{aligned}
\tag{3.25}
$$

These equations can be used for all l provided that we remember that

$$E_{1-}=M_{0+}\equiv 0. \tag{3.26}$$

Thus, given a set of invariant amplitudes $(A\cdots F)$ we can construct the multipole amplitudes $(M_{l\pm}, E_{l\pm}, C_{l\pm})$. These relations can clearly be inverted to provide the invariant amplitudes if the complete set of multipole amplitudes is known.

For the isospin properties of our amplitudes we know that the general form of the T matrix is

$$T=T^{(+)}\delta_{\alpha 3}+T^{(-)}\tfrac{1}{2}[\tau_\alpha,\tau_3]+T^{(0)}\tau_\alpha, \tag{3.27}$$

where α labels the (Hermitian) final pion state. The amplitude for electroproduction of a definite state of total isotopic spin $\frac{1}{2}$ or $\frac{3}{2}$ from a proton can be related to these amplitudes by

$$
\begin{aligned}
T(\tfrac{3}{2},\text{proton})&=(\surd\tfrac{2}{3})(T^+-T^-),\\
T(\tfrac{1}{2},\text{proton})&=-(\surd\tfrac{1}{3})(T^++2T^-+3T^0).
\end{aligned}
\tag{3.28}
$$

Some further kinematic relations expressing everything in terms of W and k^2 will be useful in Sec. 4 and we summarize them here:

$$
\begin{aligned}
E_1+m&=[(W+m)^2+k^2]/2W,\\
E_2+m&=[(W+m)^2-\mu^2]/2W,\\
q&=(1/2W)[W-(m+\mu)]^{1/2}[W+(m+\mu)]^{1/2}\\
&\quad\times[W-(m-\mu)]^{1/2}[W+(m-\mu)]^{1/2},\\
k^*&=(1/2W)[(W-m)^2+k^2]^{1/2}[(W+m)^2+k^2]^{1/2},\\
\frac{k^*}{E_1+m}&=\left[\frac{(W-m)^2+k^2}{(W+m)^2+k^2}\right]^{1/2},\\
\frac{q}{E_2+m}&=\left[\frac{(W-m)^2-\mu^2}{(W+m)^2-\mu^2}\right]^{1/2},\\
k_0&=(1/2W)[W^2-m^2-k^2].
\end{aligned}
\tag{3.29}
$$

These quantities are related to the laboratory variables by

$$
\begin{aligned}
W^2-m^2&=-k^2-2k\cdot p_1,\\
k^2&=4\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\theta,\\
-k\cdot p_1&=(\epsilon_1-\epsilon_2)m.
\end{aligned}
\tag{3.30}
$$

Finally, from Eqs. (3.9), (3.12), and (3.13), we can write the differential cross section in the laboratory for single-pion electroproduction in the case where only the final electron is detected as

$$
\begin{aligned}
\left(\frac{d^2\sigma}{d\Omega_2 d\epsilon_2}\right)_{\text{lab}}&=\frac{\alpha^2\cos^2\frac{1}{2}\theta}{\epsilon_1{}^2\sin^4\frac{1}{2}\theta}\sum_{J\pi}(J+\tfrac{1}{2})\\
&\times\left\{\frac{k^4}{k^{*4}}|C_{l\pm}|^2+\left(\frac{1}{2}\frac{k^2}{k^{*2}}+\frac{W^2}{m^2}\tan^2\tfrac{1}{2}\theta\right)\right.\\
&\left.\times[|T_{3/2}{}^{l\pm}|^2+|T_{1/2}{}^{l\pm}|^2]\right\}\left(\frac{mq}{W}\right).
\end{aligned}
\tag{3.31}
$$

In terms of the multipole amplitudes these expressions become

$$
\begin{aligned}
(J+\tfrac{1}{2})[|T_{3/2}{}^{l+}|^2+|T_{1/2}{}^{l+}|^2]&=l[(l+1)M_{l+}]^2\\
&\quad+(l+2)[(l+1)E_{l+}]^2,\\
(J+\tfrac{1}{2})[|T_{3/2}{}^{l-}|^2+|T_{1/2}{}^{l-}|^2]&=(l+1)[lM_{l-}]^2\\
&\quad+(l-1)[lE_{l-}]^2.
\end{aligned}
\tag{3.32}
$$

The multipole amplitudes are functions of k^2 and W. In order to integrate over final lab energies, we need

$$\left(\frac{\partial\epsilon_2}{\partial W}\right)_{\theta,\epsilon_1}=\left(\frac{W}{m}\right)\frac{1}{1+2(\epsilon_1/m)\sin^2\frac{1}{2}\theta}. \tag{3.33}$$

4. COVARIANT, GAUGE-INVARIANT MODEL

In this section we discuss a simple model for the multipole amplitudes. Any simple approximation which we make for the multipole amplitudes leaves the expression

$$\left(\frac{2\omega_q E_1 E_2 \Omega^3}{m^2}\right)^{1/2} \langle p_2 q^{(-)} | \epsilon_\mu J_\mu | p_1 \rangle = \bar{u}(p_2)\left(\sum_{i=1}^{6} \epsilon_\mu M_i{}^\mu A_i\right) u(p_1) \quad (4.1)$$

gauge invariant and covariant. Let us suppose that we have solved the problem of π-N scattering and have the solution for the partial-wave amplitudes ($l\pm$) in the form[15]

$$f = e^{i\delta}\sin\delta/q = N(W)/D(W), \quad (4.2)$$

where $N(W)$ has only the left-hand singularities and is real for $W > m+\mu$. If we have really solved the problem correctly, we will find resonances in the appropriate partial-wave amplitudes. We can now define a resonance as the place where $\mathrm{Re}D(W_R)=0$ and in the vicinity of the resonance we can expand

$$D(W) \cong (W-W_R)(d/dW)\,\mathrm{Re}D(W)|_{W=W_R} + i\,\mathrm{Im}D(W_R) = \mathrm{Re}'D(W_R)[W-W_R+\tfrac{1}{2}i\Gamma]. \quad (4.3)$$

Now by unitarity we know

$$\mathrm{Im}\frac{1}{f} = \frac{\mathrm{Im}D(W)}{N(W)} = -q\frac{\sigma_{\mathrm{tot}}}{\sigma_{\mathrm{el}}}, \quad (4.4)$$

where we are allowing for the possibility of inelastic processes. σ_{tot} and σ_{el} are the total and elastic cross sections for the $l\pm$ channel. Therefore we find

$$\frac{\mathrm{Im}D(W_R)}{N(W_R)} = -q_R\frac{\sigma_{\mathrm{tot}}(W_R)}{\sigma_{\mathrm{el}}(W_R)}, \quad (4.5)$$

and we can identify

$$\tfrac{1}{2}\Gamma = \frac{N(W_R)}{-\mathrm{Re}'D(W_R)} q_R \frac{\sigma_{\mathrm{tot}}(W_R)}{\sigma_{\mathrm{el}}(W_R)}. \quad (4.6)$$

If we further define

$$\tfrac{1}{2}\Gamma_{\mathrm{el}} \equiv \frac{N(W_R)}{-\mathrm{Re}'D(W_R)} q_R \quad (4.7)$$

we can write, in the sharp-resonance approximation,

$$f = \frac{e^{i\delta}\sin\delta}{q} \cong \frac{-\Gamma_{\mathrm{el}}/2q_R}{W-W_R+\tfrac{1}{2}i\Gamma}, \quad (4.8)$$

which is just the Breit-Wigner form[15a]

[15] We do not discuss the detailed problems of choosing a normalization for $D(W)$ here. One convenient choice is $D(W)\to 1$ as $W\to\infty$.

[15a] *Note added in proof:* We are dealing here, of course, with just the resonant amplitude. These arguments all go through even in the presence of an elastic background in the channel under consideration, but then it is only the resonant cross sections which we must use in Eq. (4.4). Similarly, it is only the resonant part of the electroproduction amplitude which we are computing.

Let us now take the following as an approximate electroproduction multipole amplitude in the vicinity of a resonance in the $l\pm$ channel:

$$a(W,k^2) \cong a^{BA}(W,k^2)[e^{i\delta}\sin\delta/qN(W)], \quad (4.9)$$

$$a(W,k^2) \cong a^{BA}(W,k^2)/D(W), \quad (4.10)$$

where $a^{BA}(W,k^2)$ stands for the multipole projections of any gauge-invariant set of exchange graphs that are believed to play an important role as an excitation mechanism. This approximation has the following features to recommend it:

(i) It has the correct singularity structure since $a^{BA}(W,k^2)$ has the correct left-hand singularities in W, and $D(W)$ has the physical right-hand cut.

(ii) It has built into it the correct threshold behaviors in both k^* and q.

(iii) In the weak-coupling limit $D(W)\to 1$, and this formula is exact.

(iv) It satisfies the final-state theorem in the region of elastic scattering since there

$$D(W) = |D(W)|e^{-i\delta}. \quad (4.11)$$

(v) It is a solution to the Omnès equation[16] for the multipole amplitude in the elastic case provided only that $a^{BA}(W,k^2)$ is a slowly varying function of W in the region where $\sin\delta\neq 0$. *Note that the approximation is not restricted to the elastic region however.*

(vi) The electroproduction amplitude then resonates at the same place as the scattering amplitude.

Since this result has so many features of the exact theory in it, we will simply take it as a model of the electroproduction amplitude in the vicinity of a resonance. An approximate form for $D(W)$, which relates it entirely to the strong scattering phase shifts in the elastic case is due to Watson[17]:

$$D(W) = \exp\left[-\frac{1}{\pi}\int_{M+\mu}^{\infty}\frac{\delta(W')dW'}{W'-W-i\epsilon}\right] = e^{-i\delta}\exp\left[-\frac{\mathcal{P}}{\pi}\int_{M+\mu}^{\infty}\frac{\delta(W')dW'}{W'-W}\right]. \quad (4.12)$$

In the last form it is clear that $\mathrm{Re}D(W)$ vanishes at resonance, so again, in the sharp-resonance approximation we can write

$$D(W) \cong \mathrm{Re}'D(W_R)[W-W_R+\tfrac{1}{2}i\Gamma]. \quad (4.13)$$

We shall give two derivations of this model at the end of this section. We merely note here that in this model

$$a(W,k^2) \cong a^{BA}(W,k^2)/D(W). \quad (4.14)$$

The entire k^2 dependence is in $a^{BA}(W,k^2)$, so that we can evaluate the electron form factors directly from this quantity. We also note that this expression also predicts

[16] R. Omnès, Nuovo Cimento 8, 316 (1958).

[17] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

the shape of the resonance peak; however, this is nothing but a Breit-Wigner form in the sharp-resonance approximation. A quantity of interest is the inelastic cross section integrated over a peak, since it is this quantity that has a direct interpretation as an inelastic isobar form factor, as we saw in Sec. 2. We have

$$a(W,k^2)\cong\frac{-\frac{1}{2}(\Gamma_{\gamma^*}\Gamma_{\rm el})^{1/2}}{W-W_R+\frac{1}{2}i\Gamma}, \tag{4.15}$$

where we have defined

$$\tfrac{1}{2}\Gamma_{\gamma^*}\equiv\frac{[a^{BA}(W_R,k^2)]^2}{q_R N(W_R)[-\mathrm{Re}'D(W_R)]}, \tag{4.16}$$

and we therefore find

$$\int_{\rm reson} dW\,|a(W,k^2)|^2{}_{\gamma^*+N\to N+\pi}=\tfrac{1}{2}\pi\Gamma_{\gamma^*}\left(\frac{\Gamma_{\rm el}}{\Gamma}\right)=\tfrac{1}{2}\pi\Gamma_{\gamma^*}\left(\frac{\sigma_{\rm el}}{\sigma_{\rm tot}}\right), \tag{4.17}$$

where this is the expression for the process

$$\gamma^*+N\to N+\pi.$$

Clearly, if there are inelastic processes present and we only observe the electron, we want to sum over all of these processes. The result is just to multiply the above by $\sigma_{\rm tot}/\sigma_{\rm el}$ and

$$\int_{\rm reson} dW\,|a(W,k^2)|^2{}_{\gamma^*+N\to\text{anything}}=\tfrac{1}{2}\pi\Gamma_{\gamma^*}(W_R,k^2)=\frac{\pi|a^{BA}(W_R,k^2)|^2}{q_R N(W_R)[-\mathrm{Re}'D(W_R)]}. \tag{4.18}$$

Let us now discuss a derivation of the simple model presented here. We give two derivations, the first being a slight extension of the work of Chew and Low on photoproduction of π^0 in the 3-3 resonance region,[4] and the second essentially an argument contained in Goldberger and Watson.[17]

(A) Suppose we have a relation for the π-N scattering amplitude in a given $l\pm$partial-wave state,

$$\mathrm{Re}f(\omega)=f^{\rm l.h.s.}(\omega)+\frac{\mathcal{P}}{\pi}\int_1^\infty\frac{\mathrm{Im}f(\omega')d\omega'}{\omega'-\omega}, \tag{4.19}$$

where $f^{\rm l.h.s.}(\omega)$ is some given function of ω with only left-hand singularities, and that this relation together with unitarity (we assume elastic unitarity here for simplicity)

$$\mathrm{Im}f(\omega)=q|f(\omega)|^2,\quad \omega>1 \tag{4.20}$$

completely determine $f(\omega)$. Suppose also that we have a similar relation for an electroproduction multipole into the same partial-wave state

$$\mathrm{Re}a(\omega,k^2)=a^{\rm l.h.s.}(\omega,k^2)+\frac{\mathcal{P}}{\pi}\int_1^\infty\frac{\mathrm{Im}a(\omega',k^2)d\omega'}{\omega'-\omega} \tag{4.21}$$

and that this relation, together with the final-state theorem (time reversal plus unitarity)

$$a(\omega,k^2)=|a|e^{i\delta},\quad \omega>1 \tag{4.22}$$

completely determine this quantity.

Let us assume that in the region on the physical cut which is important for our problem [we essentially assume here that $f(\omega)$ is resonant and that it is the amplitude in the vicinity of the resonance which is important] that we can approximate

$$f^{\rm l.h.s.}(\omega)\cong f^{\rm l.h.s.}(\omega_R)\sum_i\frac{\alpha_i}{\omega+\bar{\omega}_i},\qquad \omega>1$$
$$a^{\rm l.h.s.}(\omega,k^2)\cong a^{\rm l.h.s.}(\omega_R,k^2)\sum_i\frac{\alpha_i}{\omega+\bar{\omega}_i},\quad \omega>1 \tag{4.23}$$

where

$$\sum_i\frac{\alpha_i}{\omega_R+\bar{\omega}_i}\equiv1. \tag{4.24}$$

Thus, we simply scale the functions by their values at resonance and assume that they have the same approximate left-hand singularity structure. This modified problem can now be solved exactly, and we find

$$a(\omega,k^2)=\frac{a^{\rm l.h.s.}(\omega_R,k^2)}{f^{\rm l.h.s.}(\omega_R)}f(\omega), \tag{4.25}$$

as can be seen by substituting in Eq. (4.21) for $a(\omega,k^2)$ and then using Eq. (4.19) for $f(\omega)$. This is just our previous result.

(B) For the electroproduction amplitude, we really have an Omnès equation to solve[16];

$$a(\omega,k^2)=a^{\rm l.h.s.}(\omega,k^2)+\frac{1}{\pi}\int_1^\infty\frac{h^*(\omega')a(\omega',k^2)d\omega'}{\omega'-\omega-i\epsilon}, \tag{4.26}$$

where

$$h(\omega)=e^{i\delta(\omega)}\sin\delta(\omega) \tag{4.27}$$

(we again assume elastic unitarity for simplicity), and

$$\delta(\infty)=0,\qquad \delta(0)=-n\pi, \tag{4.28}$$

where n is the number of bound states in the particular channel. This equation was solved exactly by Omnès, and the solution is[16]

$$a(\omega,k^2)=e^{i\delta(\omega)}\left[a^{\rm l.h.s.}(\omega,k^2)\cos\delta(\omega)+e^{\rho(\omega)}\frac{\mathcal{P}}{\pi}\int_1^\infty\frac{a^{\rm l.h.s.}(\xi,k^2)\sin\delta(\xi)e^{-\rho(\xi)}d\xi}{\xi-\omega}\right], \tag{4.29}$$

where

$$\rho(\omega)=\frac{\mathcal{P}}{\pi}\int_1^\infty\frac{\delta(\zeta)d\zeta}{\zeta-\omega}. \tag{4.30}$$

Let us assume that over the region where $\sin\delta(\xi)\neq0$ (that is, essentially over the resonance region) the function $a^{\mathrm{l.h.s.}}(\xi,k^2)$ varies slowly enough so that we can write

$$a^{\mathrm{l.h.s.}}(\xi,k^2)\cong a^{\mathrm{l.h.s.}}(\omega_R,.k)^2 \tag{4.31}$$

Then in the vicinity of the resonance

$$a(\omega,k^2)\cong a^{\mathrm{l.h.s.}}(\omega_R,k^2)\left[e^{i\delta(\omega)}\cos\delta(\omega)+e^{i\delta(\omega)}e^{\rho(\omega)}\frac{\mathcal{P}}{\pi}\int_1^\infty\frac{\sin\delta(\xi)e^{-\rho(\xi)}d\xi}{\xi-x}\right]$$
$$\equiv a^{\mathrm{l.h.s.}}(\omega_R,k^2)\chi(\omega)\,. \tag{4.32}$$

Now we can write

$$\chi(\omega)\equiv\exp\left[\frac{1}{\pi}\int_1^\infty\frac{\delta(y)dy}{y-\omega-i\epsilon}\right]\psi(\omega)\,, \tag{4.33}$$

where

$$\psi(\omega)\equiv\exp\left[-\frac{1}{\pi}\int_1^\infty\frac{\delta(y)dy}{y-\omega-i\epsilon}\right]+\frac{1}{\pi}\int_1^\infty\frac{\sin\delta(\xi)}{\xi-\omega-i\epsilon}\exp\left[-\frac{\mathcal{P}}{\pi}\int_1^\infty\frac{\delta(z)dz}{z-\xi}\right]d\xi\,. \tag{4.34}$$

$\psi(\omega)$ then has the following properties:

(i) $\psi(\omega)$ is analytic in ω with a cut from $\omega=1$ to $\omega=\infty$.

(ii) $\psi(\omega)\to 1$ as $\omega\to\infty$. (The second integral goes to zero as $\omega\to\infty$ and the first term goes to $e^0=1$.)

Therefore, we can write an unsubtracted dispersion relation for the quantity $\psi(\omega)-1$.

$$\psi(\omega)-1=\frac{1}{\pi}\int_1^\infty\frac{\mathrm{disk}\psi(\omega')d\omega'}{\omega'-\omega}\,. \tag{4.35}$$

But from the above

$$\mathrm{disk}\psi(\omega)=\sin\delta(\omega)\exp\left[-\frac{\mathcal{P}}{\pi}\int_1^\infty\frac{\delta(y)dy}{y-\omega}\right]\times[-1+1]\equiv 0\,. \tag{4.36}$$

Therefore, we conclude

$$\psi(\omega)=1\,, \tag{4.37}$$

$$a(\omega,k^2)=a^{\mathrm{l.h.s.}}(\omega_R,k^2)/D(\omega)\,, \tag{4.38}$$

where

$$D(\omega)=\exp\left[-\frac{1}{\pi}\int_1^\infty\frac{\delta(\omega')d\omega'}{\omega'-\omega-i\epsilon}\right], \tag{4.39}$$

which is our previous model result.

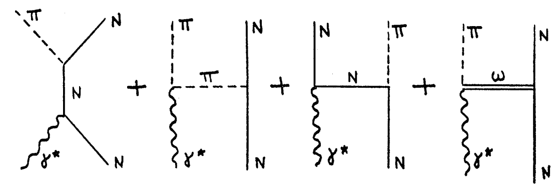

FIG. 5. Exchange graphs used to describe the excitation mechanism.

5. THE EXCITATION MECHANISM

In this section we choose a small gauge-invariant set of graphs which we assume play a dominant role as an excitation mechanism for the nucleon resonances and compute their contribution to $a^{BA}(W_R,k^2)$. As discussed in the Introduction, we limit ourselves to the set shown in Fig. 5. If we assume

$$F_\pi(k^2)=F_1{}^V(k^2)\,, \tag{5.1}$$

then this set of graphs is gauge invariant. In order to get a feel for what is going on, we have simply treated the over-all contribution of the last graph as a parameter and have tried to make a one-parameter fit to all the existing inelastic data. We assume a form factor

$$F_{\omega\pi\gamma}(k^2)=F_2{}^V(k^2)/F_2{}^V(0) \tag{5.2}$$

for the last graph, which is approximately true if the form factor is dominated by ρ exchange. Since $g_{\omega\pi\gamma}$ is known from the ω decay, we are essentially varying the quantity $g_{\omega NN}/g_{\pi NN}$, where we assume a coupling of the ω to the nucleon of the form

$$ig_{\omega NN}\bar{\psi}\gamma_\mu\psi\omega_\mu\,. \tag{5.3}$$

We keep only the charge coupling because of the small value of the nucleon isoscalar magnetic moment. We have only kept ω^0 as a vector-meson exchange because $g_{\rho\pi\gamma}$ is known to be very small and $SU(6)$ plus vector-meson dominance would give $g_{\phi\pi\gamma}\approx 0$.[18] We should, of course, keep N^*, N^{**} exchange, box diagrams, etc., but as a first step we will see just what we can do with this simple model.

We can therefore write our amplitude in the standard fashion as

$$\left(\frac{2\omega_q E_1E_2\Omega^3}{m^2}\right)^{1/2}\langle p_2q^{(-)}|\epsilon_\mu J_\mu|p_1\rangle_{\mathrm{pole}}=-g_{\pi NN}\bar{u}(p_2)[M_\mu{}^{\mathrm{pole}}\epsilon_\mu]u(p_1)\,, \tag{5.4}$$

where we treat these contributions as poles in a dispersion-theory sense in that we use all the renormalized coupling constants; we then have

[18] G. Segrè and J. D. Walecka, Ann. Phys. (N. Y.) **40**, 337 (1966).

$$\tau_\alpha M_\mu{}^{(0)}=\tau_\alpha\left\{\gamma_5\frac{1}{i(\gamma\cdot p_1+\gamma\cdot k)+m}[\tfrac{1}{2}F_1{}^S\gamma_\mu-\tfrac{1}{2}F_2{}^S\sigma_{\mu\nu}k_\nu]+[\tfrac{1}{2}F_1{}^S\gamma_\mu-\tfrac{1}{2}F_2{}^S\sigma_{\mu\nu}k_\nu]\frac{1}{i(\gamma\cdot p_2-\gamma\cdot k)+m}\gamma_5\right\},$$

$$\delta_{\alpha 3}M_\mu{}^{(+)}=\delta_{\alpha 3}\left\{\gamma_5\frac{1}{i(\gamma\cdot p_1+\gamma\cdot k)+m}[\tfrac{1}{2}F_1{}^V\gamma_\mu-\tfrac{1}{2}F_2{}^V\sigma_{\mu\nu}k_\nu]+[\tfrac{1}{2}F_1{}^V\gamma_\mu-\tfrac{1}{2}F_2{}^V\sigma_{\mu\nu}k_\nu]\frac{1}{i(\gamma\cdot p_2-\gamma\cdot k)+m}\gamma_5\right\}$$
$$+\delta_{\alpha 3}\frac{g_{\omega\pi\gamma}g_{\omega NN}}{g_{\pi NN}}\frac{F_{\omega\pi\gamma}(k^2)}{(k-q)^2+m_\omega{}^2}[2\gamma_5(\gamma_\mu(P\cdot k)-\gamma\cdot kP_\mu)-im\gamma_5(\gamma_\mu\gamma\cdot k-\gamma\cdot k\gamma_\mu)],$$

$$\tfrac{1}{2}[\tau_\alpha,\tau_3]M_\mu{}^{(-)}=\tfrac{1}{2}[\tau_\alpha,\tau_3]\left\{\gamma_5\frac{1}{i(\gamma\cdot p_1+\gamma\cdot k)+m}[\tfrac{1}{2}F_1{}^V\gamma_\mu-\tfrac{1}{2}F_2{}^V\sigma_{\mu\nu}k_\nu]\right.$$
$$\left.-[\tfrac{1}{2}F_1{}^V\gamma_\mu-\tfrac{1}{2}F_2{}^V\sigma_{\mu\nu}k_\nu]\frac{1}{i(\gamma\cdot p_2-\gamma\cdot k)+m}\gamma_5-i\gamma_5\frac{(2q_\mu-k_\mu)}{(q-k)^2+\mu^2}F_\pi\right\}. \quad (5.5)$$

The first two terms are clearly gauge invariant, $k_\mu M_\mu{}^{(0+)}=0$. The last term is only gauge invariant if $F_\pi\equiv F_1{}^V$. This is a well-known difficulty, and we shall make this assumption in the interests of simplicity and maintaining current conservation. Our conventions on the nucleon form factors are

$$F_1{}^V(0)=F_1{}^S(0)=1,$$
$$2mF_2{}^S(0)=\lambda_p'+\lambda_n=-0.12, \quad (5.6)$$
$$2mF_2{}^V(0)=\lambda_p'-\lambda_n=3.70.$$

[Note that all the form factors in Eq. (5.5) are functions of k^2.] For the ω^0 contribution, we have assumed an $\omega\pi\gamma$ vertex of the form[18]

$$V_{\omega\pi\gamma}=ieg_{\omega\pi\gamma}\delta_{\alpha 3}\epsilon_{\mu\nu\rho\sigma}k_\mu\epsilon_\nu\omega_\rho q_\sigma \quad (5.7)$$

and from the decay $\omega^0\to\pi^0+\gamma$ we conclude

$$\Gamma_{\omega\to\pi+\gamma}=(\alpha m_\omega{}^3/24)g_{\omega\pi\gamma}{}^2(1-\mu^2/m_\omega{}^2)^3$$
$$\cong 1.3\text{ MeV}. \quad (5.8)$$

Therefore, we have

$$g_{\omega\pi\gamma}{}^2\cong 9/m^2. \quad (5.9)$$

The sign of the coupling is unknown to us. Therefore, we can only identify

$$\beta^2\equiv g_{\omega\pi\gamma}{}^2g_{\omega NN}{}^2/[\tfrac{1}{2}g_{\pi NN}F_2{}^V(0)]^2$$
$$\cong 10(g_{\omega NN}/g_{\pi NN})^2. \quad (5.10)$$

Of course, in adding the amplitudes the sign of β is important. We shall simply treat β as a parameter and try and find a one-parameter fit to all the known inelastic data. Note that we will also assume $F_{\omega\pi\gamma}(k^2)=F_2{}^V(k^2)/F_2{}^V(0)$, as discussed previously.

We can now write the contributions of the pole terms to our invariant amplitudes as

$$A=-\tfrac{1}{2}g_{\pi NN}F_1\left[\frac{1}{(p_1+k)^2+m^2}\pm\frac{1}{(p_2-k)^2+m^2}\right],$$

$$B=\tfrac{1}{2}g_{\pi NN}F_1\left(\frac{1}{q\cdot k}\right)\left[\frac{1}{(p_1+k)^2+m^2}\pm\frac{1}{(p_2-k)^2+m^2}\right],$$

$$C=\tfrac{1}{2}g_{\pi NN}F_2\left[\frac{1}{(p_1+k)^2+m^2}\mp\frac{1}{(p_2-k)^2+m^2}\right],$$

$$D=\tfrac{1}{2}g_{\pi NN}F_2\left\{\left[\frac{1}{(p_1+k)^2+m^2}\pm\frac{1}{(p_2-k)^2+m^2}\right]+\beta\frac{\delta_+}{(k-q)^2+m_\omega{}^2}\right\}, \quad (5.11)$$

$$E=-(g_{\pi NN})F_1\left(\frac{1}{q\cdot k}\right)\left[\frac{\delta_-}{(k-q)^2+\mu^2}\right],$$

$$F=0,$$

TABLE I. Photoproduction amplitudes.

State	Ratio	Walker[a]	Moorhouse *et al.*[b]	Theory $\beta=+4$	$\beta=-6$	$\beta=-8.2$
$\frac{3}{2}^+,\frac{3}{2}$ (1236)	E_{1+}/M_{1+}	-0.04 ± 0.08		-0.14	-0.34	-0.62
$\frac{3}{2}^-,\frac{1}{2}$ (1525)	M_{2-}/E_{2-}	$+0.53\pm0.2$	$+0.34$	$+0.56$	-0.42	-2.3
$\frac{5}{2}^-,\frac{1}{2}$ (1670)	E_{2+}/M_{2+}	$-0.5\ \pm0.5$		$+1.4$	-0.34	-0.28
$\frac{5}{2}^+,\frac{1}{2}$ (1688)	M_{3-}/E_{3-}	$+0.5\ \pm0.3$		$+0.45$	-0.07	-0.24
Ratios of $\lvert f_+\rvert^2+\lvert f_-\rvert^2$						
$\frac{1}{2}^-(1570):\frac{3}{2}^-(1525)$		0.15 ± 0.2	0.07			
$\frac{5}{2}^-(1670):\frac{5}{2}^+(1688)$		0.24 ± 0.3				

[a] Reference 20. [b] Reference 21.

where we use the following convention:

Isospin amplitude	Form factor	Sign	δ_+	δ_-
+	V	upper	1	0
0	S	upper	0	0
−	V	lower	0	1

We note that there is an apparent kinematic singularity $1/k\cdot q$ introduced into two of our invariant amplitudes by our decision to work with explicitly gauge-invariant kinematic invariants. This singularity is really spurious since it disappears when we go to the multipole amplitudes, as it must. That this must happen is clear from Eq. (3.9), which expresses the physical scattering amplitude directly in terms of the transition multipole moments. For our purpose here, it is enough that this apparent kinematic singularity disappears from the multipole amplitudes,[19] since we will work with these quantities directly.

We can now write the contribution of the pole terms to our invariant amplitudes;

$$\begin{aligned}
\mathfrak{F}_1=&\frac{g_{\pi NN}}{16\pi W}[(E_1+m)(E_2+m)]^{1/2}\bigg\{\frac{F_1}{W+m}-F_2\bigg[\frac{W-m}{W+m}\mp1\bigg]\\
&\mp\frac{(W-m)(F_1+2mF_2)}{2k\cdot q+W^2-m^2}+\beta\delta_+F_2\bigg[\frac{W(E_1-m)+W(E_2-m)+\frac{1}{2}m_\omega^2}{k^2-2k\cdot q+m_\omega^2-\mu^2}-\frac{1}{2}\bigg]\bigg\}\,,\\
\mathfrak{F}_2=&\frac{g_{\pi NN}}{16\pi W}\frac{qk^*}{[(E_1+m)(E_2+m)]^{1/2}}\bigg\{\frac{-F_1}{W-m}-F_2\bigg[\frac{W+m}{W-m}\mp1\bigg]\\
&\pm\frac{(W+m)(F_1+2mF_2)}{2k\cdot q+W^2-m^2}+\beta\delta_+F_2\bigg[\frac{W(E_1+m)+W(E_2+m)+\frac{1}{2}m_\omega^2}{k^2-2k\cdot q+m_\omega^2-\mu^2}-\frac{1}{2}\bigg]\bigg\}\,,\\
\mathfrak{F}_3=&\frac{g_{\pi NN}}{16\pi W}2qk^*\bigg[\frac{E_2+m}{E_1+m}\bigg]^{1/2}\bigg\{\mp\frac{F_1+(W+m)F_2}{2k\cdot q+W^2-m^2}+\delta_-\frac{2F_1}{k^2-2k\cdot q}+\tfrac{1}{2}\beta\delta_+F_2\bigg[\frac{-(W+m)}{k^2-2k\cdot q+m_\omega^2-\mu^2}\bigg]\bigg\}\,,\\
\mathfrak{F}_4=&\frac{g_{\pi NN}}{16\pi W}2\mathbf{q}^2\bigg(\frac{E_1+m}{E_2+m}\bigg)^{1/2}\bigg\{\pm\frac{F_1-(W-m)F_2}{2k\cdot q+W^2-m^2}-\delta_-\frac{2F_1}{k^2-2k\cdot q}-\tfrac{1}{2}\beta\delta_+F_2\bigg[\frac{W-m}{k^2-2k\cdot q+m_\omega^2-\mu^2}\bigg]\bigg\}\,,\\
\mathfrak{F}_7=&\frac{g_{\pi NN}}{16\pi W}q\bigg(\frac{E_1+m}{E_2+m}\bigg)^{1/2}\bigg\{\frac{F_1+(W-E_1)F_2}{W-m}-F_2[1\pm1]\\
&\pm\frac{1}{2k\cdot q+W^2-m^2}[\{\tfrac{1}{2}(W^2-m^2)+\tfrac{1}{2}k^2-\mu^2+m(W-2E_2+E_1)\}F_2+(W-m-2E_2)F_1]\\
&-\delta_-\frac{2(W-2E_2+E_1)}{k^2-2k\cdot q}F_1+\beta\delta_+F_2\bigg[\frac{1}{2}+\frac{m(E_1-E_2)-\frac{1}{2}m_\omega^2}{k^2-2k\cdot q+m_\omega^2-\mu^2}\bigg]\bigg\}\,,\\
\mathfrak{F}_8=&\frac{g_{\pi NN}}{16\pi W}k^*\bigg(\frac{E_2+m}{E_1+m}\bigg)^{1/2}\bigg\{-\frac{F_1-(W-E_1)F_2}{W+m}-F_2[1\pm1]\\
&\pm\frac{1}{2k\cdot q+W^2-m^2}[\{\tfrac{1}{2}(W^2-m^2)+\tfrac{1}{2}k^2-\mu^2-m(W-2E_2+E_1)\}F_2-(W+m-2E_2)F_1]\\
&+\delta_-\frac{2(W-2E_2+E_1)}{k^2-2k\cdot q}F_1+\beta\delta_+F_2\bigg[-\frac{1}{2}-\frac{m(E_1-E_2)+\frac{1}{2}m_\omega^2}{k^2-2k\cdot q+m_\omega^2-\mu^2}\bigg]\bigg\}\,.
\end{aligned}\tag{5.12}$$

[Note that the $1/k\cdot q$ term has dropped out of these expressions, as again it must as is evident from Eqs. (3.9) and (3.12).] We need the partial-wave projections of these six amplitudes and therefore we need the following quantities:

$$\begin{aligned}
\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)dx}{2k\cdot q+W^2-m^2}&=\frac{1}{2qk^*}\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)dx}{x+(W^2-m^2-2k_0\omega_q)/2qk^*}=\frac{1}{2qk^*}(-1)^lQ_l(z)\,,\\
\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)dx}{k^2-2k\cdot q}&=\frac{1}{2qk^*}\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)dx}{(k^2+2k_0\omega_q)/2qk^*-x}=\frac{1}{2qk^*}Q_l(\rho)\,,\\
\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)dx}{k^2-2k\cdot q+m_\omega^2-\mu^2}&=\frac{1}{2qk^*}\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)dx}{(k^2+2k_0\omega_q+m_\omega^2-\mu^2)/2qk^*-x}=\frac{1}{2qk^*}Q_l(\rho')\,,
\end{aligned}\tag{5.13}$$

[19] It is easy to convince oneself that this will hold to all orders in perturbation theory.

where

$$z\equiv\frac{W^2-m^2-2k_0\omega_q}{2qk^*}=\frac{k^2+2k_0E_2}{2qk^*}, \tag{5.14}$$

and

$$\begin{aligned}\rho&\equiv(k^2+2k_0\omega_q)/2qk^*,\\ \rho'&\equiv(k^2+2k_0\omega_q+m_\omega^2-\mu^2)/2qk^*,\end{aligned} \tag{5.15}$$

using $q=|\mathbf{q}|$.

Using these results, it follows that

$$\begin{aligned}
\mathfrak{F}_l^1=&\frac{g_{\pi NN}}{32\pi W}\frac{1}{[(E_1-m)(E_2-m)]^{1/2}}\bigg[\frac{2k^*q}{W+m}F_1\delta_{l0}-2k^*q\left(\frac{W-m}{W+m}\mp1\right)F_2\delta_{l0}\\
&\mp(-1)^l(W-m)(F_1+2mF_2)Q_l(z)+\beta\delta_+F_2\{[W(E_1-m)+W(E_2-m)+\tfrac{1}{2}m_\omega^2]Q_l(\rho')-k^*q\delta_{l0}\}\bigg],\\
\mathfrak{F}_l^2=&\frac{g_{\pi NN}}{32\pi W}\frac{1}{[(E_1+m)(E_2+m)]^{1/2}}\bigg[-\frac{2k^*q}{W-m}F_1\delta_{l0}-2k^*q\left(\frac{W+m}{W-m}\mp1\right)F_2\delta_{l0}\\
&\pm(-1)^l(W+m)(F_1+2mF_2)Q_l(z)+\beta\delta_+F_2\{[W(E_1+m)+W(E_2+m)+\tfrac{1}{2}m_\omega^2]Q_l(\rho')-k^*q\delta_{l0}\}\bigg],\\
\mathfrak{F}_l^3=&\frac{g_{\pi NN}}{32\pi W}2\left(\frac{E_2+m}{E_1+m}\right)^{1/2}\{\mp(-1)^l[F_1+(W+m)F_2]Q_l(z)+\delta_-2F_1Q_l(\rho)+\beta\delta_+F_2(-\tfrac{1}{2})(W+m)Q_l(\rho')\},\\
\mathfrak{F}_l^4=&\frac{g_{\pi NN}}{32\pi W}2\left(\frac{E_2-m}{E_1-m}\right)^{1/2}\{\pm(-1)^l[F_1-(W-m)F_2]Q_l(z)-\delta_-2F_1Q_l(\rho)+\beta\delta_+F_2(-\tfrac{1}{2})(W-m)Q_l(\rho')\},\\
\mathfrak{F}_l^7=&\frac{g_{\pi NN}}{32\pi W}\frac{1}{[(E_2+m)(E_1-m)]^{1/2}}\bigg[2k^*q\frac{F_1+(W-E_1)F_2}{W-m}\delta_{l0}-2k^*qF_2(1\pm1)\delta_{l0}\\
&\pm(-1)^l\{[\tfrac{1}{2}(W^2-m^2)+\tfrac{1}{2}k^2-\mu^2+m(W-2E_2+E_1)]F_2+(W-m-2E_2)F_1\}Q_l(z)\\
&-\delta_-2(W-2E_2+E_1)F_1Q_l(\rho)+\beta\delta_+F_2\{k^*q\delta_{l0}+[m(E_1-E_2)-\tfrac{1}{2}m_\omega^2]Q_l(\rho')\}\bigg],\\
\mathfrak{F}_l^8=&\frac{g_{\pi NN}}{32\pi W}\frac{1}{[(E_1+m)(E_2-m)]^{1/2}}\bigg[-2k^*q\frac{F_1-(W-E_1)F_2}{W+m}\delta_{l0}-2k^*qF_2(1\pm1)\delta_{l0}\\
&\pm(-1)^l\{[\tfrac{1}{2}(W^2-m^2)+\tfrac{1}{2}k^2-\mu^2-m(W-2E_2+E_1)]F_2-(W+m-2E_2)F_1\}Q_l(z)\\
&+\delta_-2(W-2E_2+E_1)F_1Q_l(\rho)+\beta\delta_+F_2\{k^*q\delta_{l0}-[m(E_1-E_2)+\tfrac{1}{2}m_\omega^2]Q_l(\rho')\}\bigg].
\end{aligned} \tag{5.16}$$

The multipole projections of our excitation mechanism can now be computed directly from Eqs. (3.25).

6. NUMERICAL RESULTS

In this section we evaluate the inelastic form factors for the various nucleon resonances using our basic approximation, Eqs. (4.10) and (4.14), and the multipole projections of our assumed excitation mechanism as given by Eqs. (5.16) and (3.25). For a particular resonance, we must know the strong-interaction dynamics to get the absolute contribution to electroproduction as we see from Eq. (4.18). In this paper we shall just calculate the k^2 dependence of the form factors for each resonance. We therefore normalize the over-all contribution of each resonance to photoabsorption using Eq. (2.7):

$$\int_{\text{lab; over reson}}\sigma_\gamma(\omega)d\omega=\frac{4\pi^2\alpha}{M^2-m^2}\frac{M^2}{m}\times[|f_+|^2+|f_-|^2]_{k^2=0}. \tag{6.1}$$

The *relative* contributions of the Coulomb and two transverse multipoles for this resonance, and furthermore the electron-scattering cross section, are then determined at all momentum transfers in this model. Such a procedure also works in an energy region where several resonances are important, if the individual contributions are known at one momentum transfer (e.g.

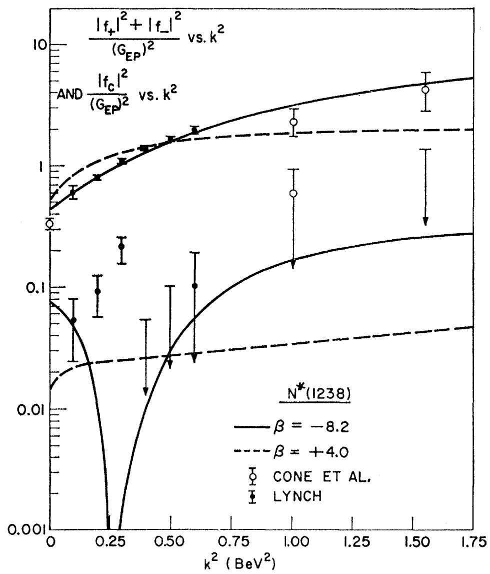

FIG. 6. Comparison of the predicted results (using the two values of the parameter β which best fit the data in Figs. 6–8) to experiment (Refs. 23 and 24). There, the squares of the inelastic transverse and Coulomb form factors are divided by the square of the elastic form factor and plotted. The graph is normalized to the value at photoproduction. [The bottom two curves give the Coulomb form factors.]

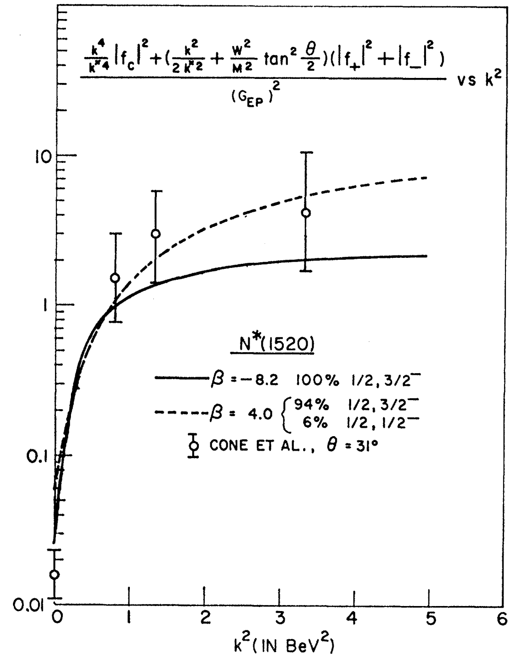

FIG. 7. Inelastic electron-scattering transition probability relative to the square of the elastic form factor, in the $\frac{3}{2}^-$ $N^*(1520)$ region. The background states which are also thought to resonate in this region have been included as indicated (see text) (Refs. 20, 21, and 24).

photoabsorption). We have used the phase-shift analyses of Walker[20] and of Moorhouse *et al.*[21] in the 1512-MeV region and of Walker[20] in the 1688-MeV region in order to normalize the amplitudes for each of the contributing resonances separately. The numerical values we used are shown at the bottom of Table I and on each of the relevant figures.

We notice that all of the inelastic form factors are proportional to the elastic form factors of the nucleon in this model, and therefore it is more convenient to plot the *ratio* to the elastic form factors. This is also the quantity which is most directly of experimental interest. In order to compare with experimental points, however, we need the actual values of the elastic form factors, and we use[22]

$$G_{Ep}=\frac{G_{Mp}}{\mu_p}=\frac{G_{Mn}}{\mu_n}=-\frac{4m^2}{k^2}\frac{G_{En}}{\mu_n}=\frac{1}{[1+k^2/(0.71\ \mathrm{BeV}^2)]^2}. \quad (6.2)$$

Our procedure was to take the over-all contribution of the ω^0 exchange graph as a parameter and to try and fit *all* the existing inelastic data with this single parameter. We were able to find reasonable fits for two different values of β;

$$\beta=+4.0,\qquad \beta=-8.2. \quad (6.3)$$

Figure 6 shows a comparison with the experimental values of Lynch[23] for the $\frac{3}{2}^+$, $\frac{3}{2}$ (1238) and with the values of Cone *et al.*[24] at larger momentum transfers. Lynch was actually able to separate the Coulomb and transverse contributions (the cross section is almost all transverse). Also, Lynch gives the peak height, which he measured very accurately, while for the comparison here, we want the cross section integrated over the resonance, as indicated in Fig. 6. We have used a value for the integrated cross section obtained by subtracting off the s-wave background contribution at photoproduction[9] and then assuming that the background has roughly the same momentum-transfer dependence as the resonance cross section. An estimate of the s-wave background, using our single-particle-exchange diagrams, indicated an error of 10% or less at Lynch's highest momentum-transfer point, resulting from this procedure. The fit to the 3-3 data is very reasonable, and the form factor for the 3-3 resonance is not par-

[20] R. Walker (to be published). Note that our helicity amplitudes are related to those of Walker by

$$i\langle -\mu|T^J(W,0)|\lambda_1\lambda_k\rangle \equiv A_{\mu,\lambda}{}^j \quad \text{for} \quad |\lambda_k|=1,$$

where $\mu=-\lambda_2$ and $\lambda=\lambda_k-\lambda_1$.

[21] Y. C. Chau, Norman Dombey, and R. G. Moorhouse, Phys. Rev. **163**, 1632 (1967).

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[24] A. A. Cone, K. W. Chen, J. R. Dunning, Jr., G. Hartwig, Norman F. Ramsey, J. K. Walker, and Richard Wilson, Phys. Rev. **156**, 1490 (1967); **163**, 1854(E) (1967).

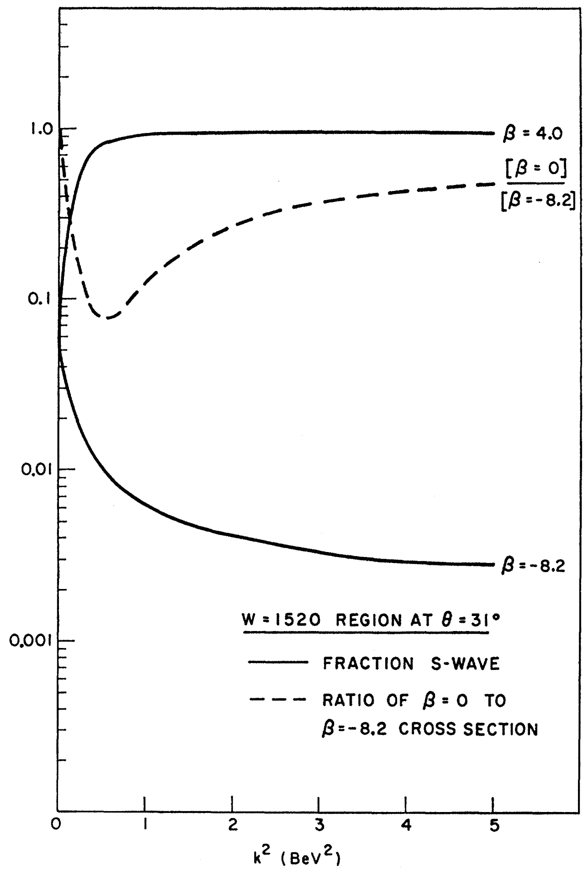

FIG. 8. Relative contributions of the $N^*(1570)$ $\frac{1}{2}^-, \frac{1}{2}$ to the transition probability as plotted in Fig. 7. A value of 6% is used for photoproduction (Refs. 20 and 21). Also shown is the ratio of the result for $\beta=0$ (no ω^0 exchange) to the $\beta=-8.2$ probability graphed in Fig. 7. Note that the $\beta=0$ solution falls to less than 10% of the $\beta=-8.2$ solution which agrees with experiment.

ticularly sensitive to the value of β, as one would expect. We note the very interesting diffraction minimum in the Coulomb form factor in the case $\beta=-8$. This gives a clear distinction between the two fits, but it will be difficult to disentangle experimentally. Notice that the relative magnitude of the Coulomb cross section is given very well in this model.

In Fig. 7 we give a comparison with the data of Cone *et al.*[24] on the 1512 resonance region. Since these authors do not separate the Coulomb and transverse cross section, but measure just the total inelastic cross section at 31°, we plot directly against the measured quantity

$$\left[\frac{k^4}{k^{*4}}|f_c|^2+\left(\frac{k^2}{2k^{*2}}+\frac{W^2}{m^2}\tan^2\tfrac{1}{2}\theta\right)\times(|f_+|^2+|f_-|^2)\right]_{\theta=31^\circ}. \quad (6.4)$$

Note that at photoproduction, $k^2=0$ and one sees only the transverse contribution; however, the Coulomb

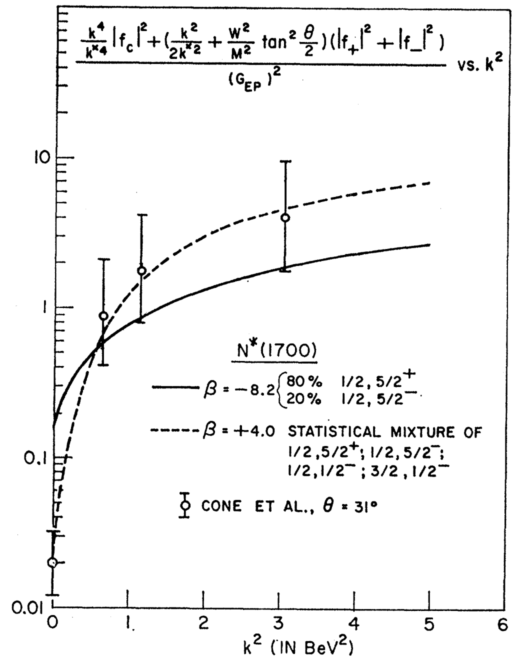

FIG. 9. Same plot as Fig. 7 now made for the $\frac{5}{2}^+$ (1688) resonance region (Ref. 24).

contribution soon becomes very important at these forward angles. The fits with positive and negative β are quite good, but, interestingly enough, they have a

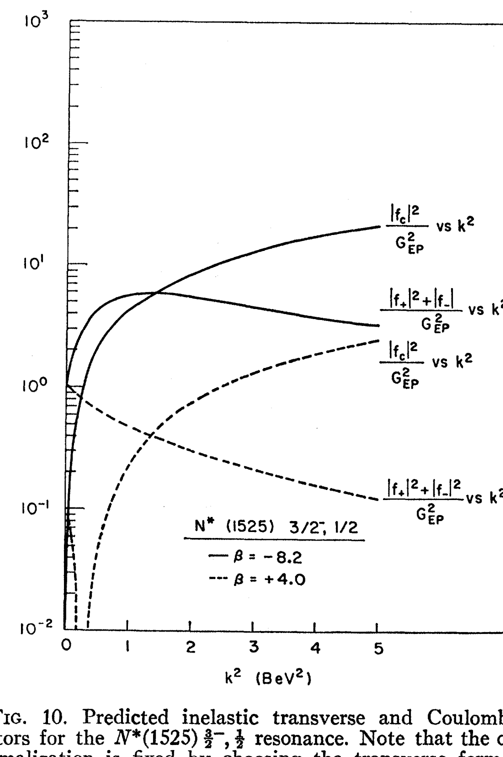

FIG. 10. Predicted inelastic transverse and Coulomb form factors for the $N^*(1525)$ $\frac{3}{2}^-, \frac{1}{2}$ resonance. Note that the over-all normalization is fixed by choosing the transverse form factor equal to unity at photoproduction, and that the ratio to the elastic form factor is given. The two best fit values of β are used.

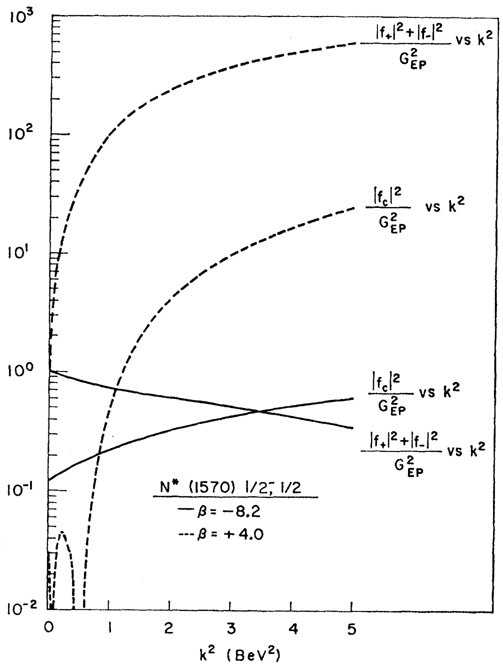

FIG. 11. Same as Fig. 10 except now the $N^*(1570)$ $\frac{1}{2}^-$, $\frac{1}{2}$ is shown.

completely different structure. For $\beta<0$, the s-wave resonance, which is unimportant at photoproduction, remains unimportant, while the high-spin state, the $\frac{3}{2}^-$, $\frac{1}{2}$ (1525), supplies most of the cross section (that is why we do not change our result by taking 100% $\frac{3}{2}^-$, $\frac{1}{2}$ for $\beta=-8.2$ in Fig. 7). On the other hand, for $\beta>0$, the

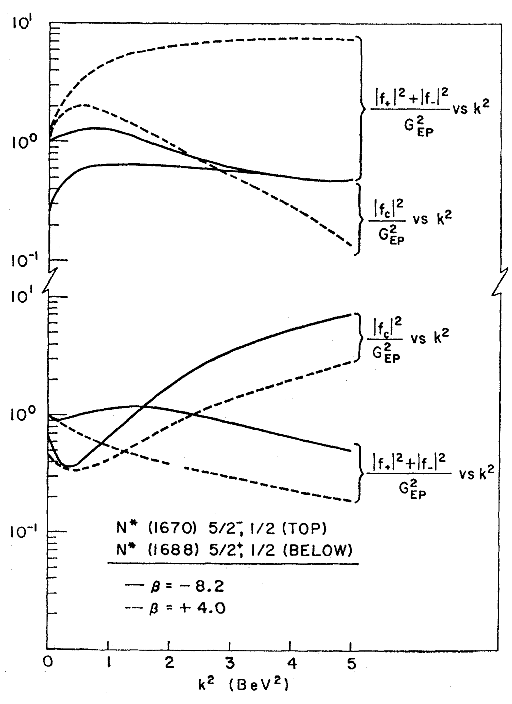

FIG. 12. Same as Fig. 10 except now the $N^*(1670)$ $\frac{5}{2}^-$, $\frac{1}{2}$ and $N^*(1688)$ $\frac{5}{2}^+$, $\frac{1}{2}$ are shown.

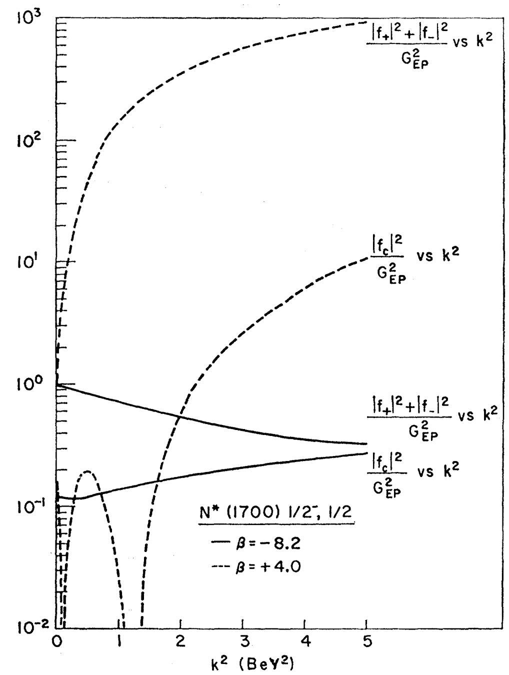

FIG. 13. Same as Fig. 10 except now the $N^*(1700)$ $\frac{1}{2}^-$, $\frac{1}{2}$ is shown.

s-wave states become dominant away from $k^2=0$. That s-wave production could be important in this region is a possibility first noted by Vik.[8] The curve with $\beta=0$ gives a completely unacceptable fit to the data, giving a minimum where the experimental maximum occurs in Fig. 7. An optimist would say, therefore, that we have a very good determination of β, but a realist would only concede that the contribution of intermediate-mass particle exchange is probably important for this resonance. These points are all illustrated in Fig. 8.

A very similar situation holds in the 1688-MeV resonance region, and we show the comparison with the data in Fig. 9. Again, the contribution of the $\frac{1}{2}^-$ resonances which are supposed to exist in this region are completely negligible at all momentum transfers when $\beta=-8.2$, while in the case $\beta=+4.0$ they soon take over and dominate the cross section, particularly the $\frac{1}{2}^-$, $\frac{1}{2}$ (1700) contribution. Since the $\frac{1}{2}^-$ contributions to photoabsorption are not very well known in this region, we simply assumed a statistical mixture

$$[\tfrac{5}{2}^+,\tfrac{1}{2}\,(1688)]:[\tfrac{5}{2}^-,\tfrac{1}{2}\,(1670)]:[\tfrac{1}{2}^-,\tfrac{1}{2}\,(1700)]:[\tfrac{1}{2}^-,\tfrac{3}{2}\,(1670)] =3:3:1:\tfrac{1}{2}$$

at photoproduction in the second case. This is not inconsistent with the phenomenological analyses.[20] Again, just as before, the fit with $\beta=0$ is completely unacceptable. Note that in this case the fit with $\beta=-8.2$ does not fall fast enough at photoproduction.

We can also calculate ratios of the multipole amplitudes at all values of k^2. At the point $k^2=0$ some experimental values which have been determined by the various phenomenological analysis of photoproduction

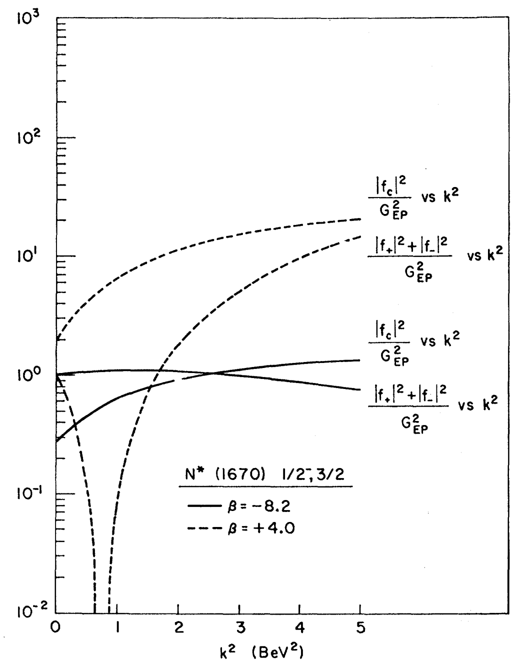

FIG. 14. Same as Fig. 10 except now the $N^*(1670)$ $\frac{1}{2}^-$, $\frac{3}{2}$ is shown.

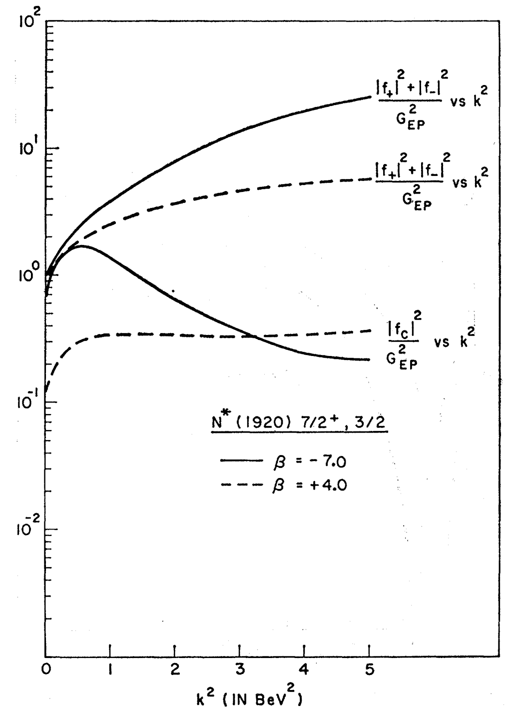

FIG. 15. Same as Fig. 10 except now the $N^*(1920)$ $\frac{7}{2}^+$, $\frac{3}{2}$ is shown.

in the higher resonance regions are available.[20,21] This comparison is shown in Table I. The value $\beta=-6$ gives only a slightly poorer fit to the 1688-MeV form factor while it gives a much more reasonable set of photoproduction values. While the magnitudes are approximately correct, this model does not appear to be a detailed theory of resonant photoproduction in this region.

Using the values of β that give a fit to all the known inelastic data, we have used the model to calculate the form factors of all the known nucleon resonances, and we give some representative results in Figs. 10–18. These curves are all normalized to unity at photoproduction. We note that the levels which are thought to be the Regge recurrences of the $\frac{3}{2}^+$, $\frac{3}{2}$ (1236), namely, the $\frac{7}{2}^+$, $\frac{3}{2}$ (1920) and the $\frac{11}{2}^+$, $\frac{3}{2}$ (2420) have very similar form factors, all the transverse form factors rising relative to the elastic form factor as the momentum transfer is increased. Likewise, the normal parity transition $\frac{3}{2}^-$, $\frac{1}{2}$ (1525); $\frac{5}{2}^+$, $\frac{1}{2}$ (1688); $\frac{7}{2}^-$, $\frac{1}{2}$ (2190); and $\frac{11}{2}^-$, $\frac{1}{2}$ (2650) show similar form factors. The transverse form factors remain about equal to the elastic form factors, while the Coulomb form factors for these levels show a diffraction minimum in all cases, and then eventually surpass the transverse form factors. The minimum come from the fact that the amplitude is the sum over various exchange contributions, and these contributions can cancel each other. Note that for the Coulomb form factors in these normal-parity transitions, the threshold behaviors are completely irrelevant, since we are already past the maximum of these form factors as we go into the physical region.

In Fig. 19 we calculate the form factor of the $\frac{3}{2}^+$, $\frac{3}{2}$ (1236) resonance, using this model, out to momentum transfers of interest in the SLAC experiments. Note the presence of a second diffraction minimum in the Coulomb form factor for the case $\beta=-8.2$. In Figs. 20

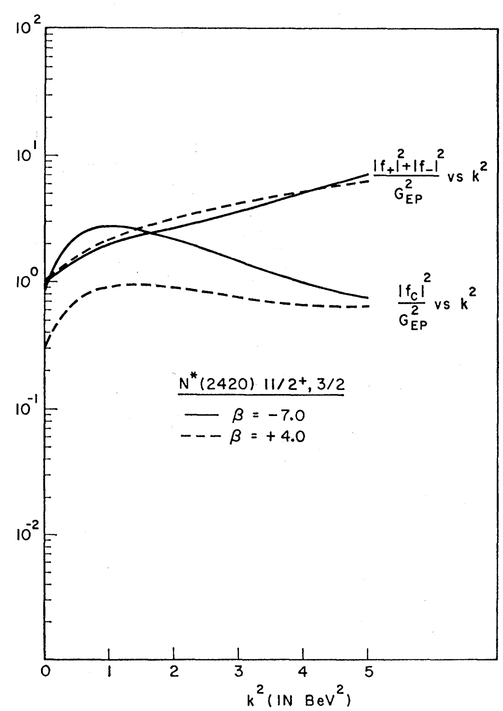

FIG. 16. Same as Fig. 10 except now the $N^*(2420)$ $\frac{11}{2}^+$, $\frac{3}{2}$ is shown.

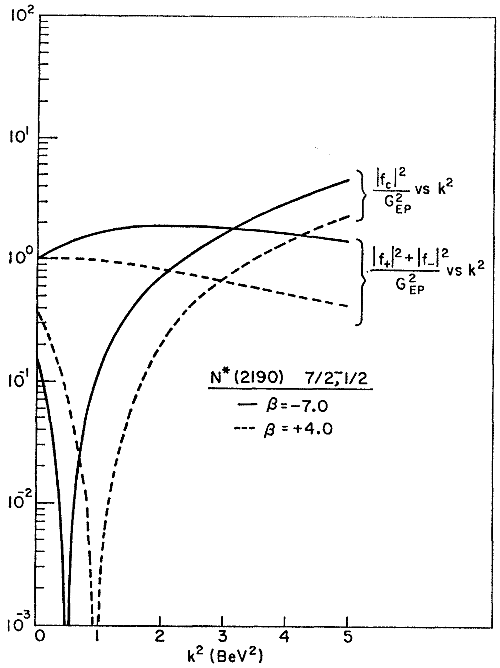

FIG. 17. Same as Fig. 10 except now the $N^*(2190)$ $\frac{7}{2}^-$, $\frac{1}{2}$ is shown.

and 21 we give the form factor of the $\frac{1}{2}^+$, $\frac{1}{2}$ (1400) level out to $k^2=25$ BeV^2 for the two different values of β. Notice the enormous growth of the transverse cross section, particularly in the case $\beta=+4$. We must

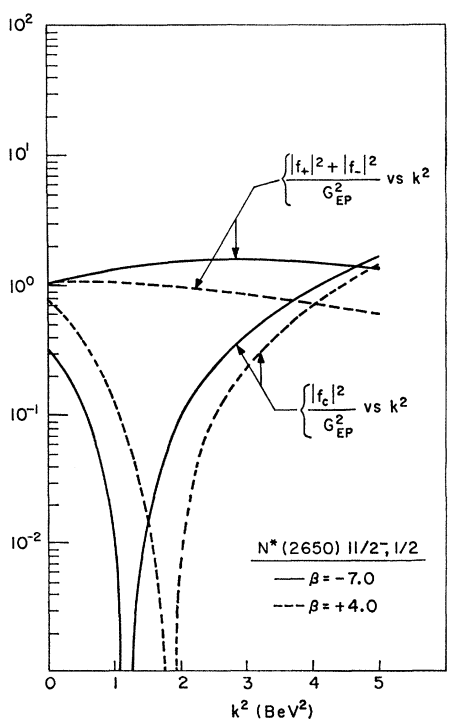

FIG. 18. Same as Fig. 10 except now the $N^*(2650)$ $\frac{11}{2}^-$, $\frac{1}{2}$ is shown.

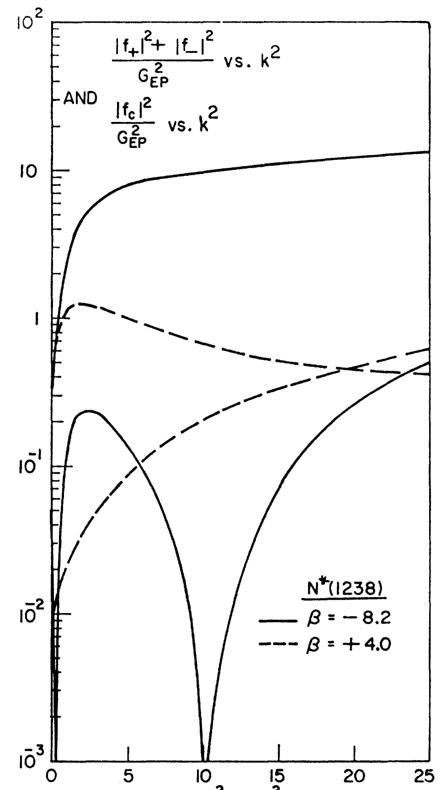

FIG. 19. Same as the preceding figures except now the form factors are calculated out to $k^2=25$ BeV^2. There the $N^*(1236)$ is shown with its transverse form factor normalized to the observed value at photoproduction (as in Fig. 6).

emphasize, however, that our model is necessarily poorest for the low partial-wave resonances.

7. DISCUSSION AND SUMMARY

We have seen that using the constant β as a parameter, where

$$\beta^2=\frac{g_{\omega\pi\gamma}^2 g_{\omega NN}^2}{[\frac{1}{2}g_{\pi NN}F_2^V(0)]^2}\cong 10\left(\frac{g_{\omega NN}}{g_{\pi NN}}\right)^2, \tag{7.1}$$

we can get a reasonable fit to all the known data on electron excitation of nucleon resonances with two values of β:

$$\beta=+4.0$$
$$=-8.2. \tag{7.2}$$

The sign is, of course, crucial. In the first case, the ω^0 exchange serves to enhance the contributions of the $\frac{1}{2}^-$ s-wave resonances and they soon dominate the inelastic form factors, a possibility first suggested by Vik.[8] In the second case, however, the ω^0 exchange serves to cancel the contributions of the s-wave resonances and enhance the contributions of the resonances of higher multipolarity, $\frac{3}{2}^-$, $\frac{5}{2}^+$ etc., so that these latter contributions dominate the cross section. This latter situation would

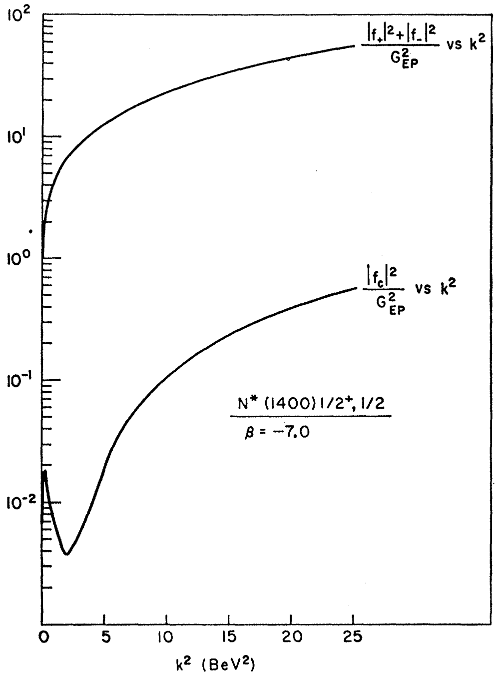

FIG. 20. Same as Fig. 19 except now the $N^*(1400)\ \frac{1}{2}^+,\ \frac{1}{2}$ is shown for $\beta=-7.0$. The transverse form factor is normalized to unity at photoproduction.

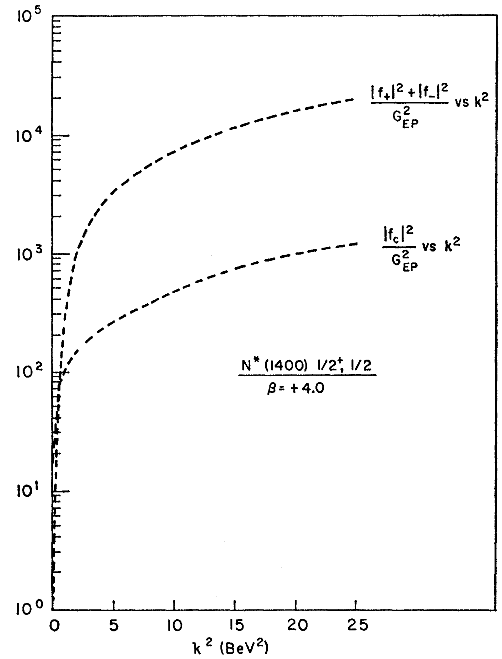

FIG. 21. Same as Fig. 20 except now $\beta=+4.0$ is used. Note the enormous value at $k^2=25$ BeV² compared to photoproduction.

be a little more pleasing, although the over-all fit is a little worse, since it is just in the s waves where we would expect any model such as ours to be the most inadequate.

It is interesting in these calculations to follow the role played by the various multipoles and states as the momentum transfer is increased. The only real way to sort out all the multipole contributions is to do coincidence experiments on the peak at different momentum transfers. This is extremely interesting information which probably cannot be obtained by detecting just the electrons. One may relatively enhance some of the contributions of the background resonances at higher momentum transfers and then sort them out with very good resolution, but again, coincidence experiments will be an invaluable tool here.

We can ask the question as to whether the value of β which we get in our fit is at all reasonable. If we assume an unsubtracted dispersion relation for $F_1^S(k^2)$ and assume that this is dominated by the ω^0 pole, then we have

$$\frac{g_{\omega\gamma}g_{\omega NN}}{m_\omega^2}\left(\frac{1}{1+k^2/m_\omega^2}\right)\cong\tfrac{1}{2}F_1^S(k^2). \qquad (7.3)$$

This is not so unreasonable theoretically since there is some evidence that the φ^0, which also contributes, is only weakly coupled to the nucleon.[25] Although the k^2 dependence is not given correctly, we can use this relation to attempt to get an order-of-magnitude estimate of the coupling constants following the lead of Gell-Mann, Sharp, and Wagner[26]:

$$g_{\omega\gamma}g_{\omega NN}/m_\omega^2\cong\tfrac{1}{2}F_1^S(0)=\tfrac{1}{2}. \qquad (7.4)$$

We can get the coupling constant $g_{\omega\gamma}$ from the decay $\omega^0\rightarrow l^+ + l^-$, which goes through a virtual photon[26,27]

$$\Gamma(\omega^0\rightarrow l^+ + l^-)=\tfrac{4}{3}\pi\alpha^2 m_\omega[g_{\omega\gamma}/m_\omega^2]^2 \times(1+2m_l^2/m_\omega^2)(1-4m_l^2/m_\omega^2)^{1/2}. \qquad (7.5)$$

Using the experimental values[28]

$$2\times10^{-4}\text{ MeV}\leqslant\Gamma(\omega^0\rightarrow e^+ + e^-)\leqslant 6\times10^{-3}\text{ MeV}, \qquad (7.6)$$

we conclude that

$$0.03\leqslant|g_{\omega\gamma}/m_\omega^2|\leqslant 0.18, \qquad (7.7)$$

from which we have

$$17\geqslant|g_{\omega NN}|\geqslant 3. \qquad (7.8)$$

Combining this with $g_{\pi N}^2/4\pi=14.6$, we would conclude from the Gell-Mann–Sharp–Wagner model that

$$4\geqslant|\beta|\geqslant 1. \qquad (7.9)$$

Abarbanel, Callan, and Sharp[29] give a much more

[25] H. Sugawara and F. von Hippel, Phys. Rev. **145**, 1331 (1966).

[26] M. Gell-Mann, D. Sharp, and W. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[27] G. Patsakos, G. Segrè, and J. D. Walecka, Phys. Letters **23**, 141 (1966).

[28] S. Ting, in Proceedings of the 1967 Symposium on Electron and Photon Interactions of High Energies, Stanford Linear Accelerator Center, Stanford, Calif. (to be published).

[29] H. D. I. Abarbanel, C. G. Callan, Jr., and D. H. Sharp, Phys. Rev. **143**, 1225 (1966).

TABLE II. Values of $|\beta|$ from Donnachie *et al.*[a]

$g_{\omega NN}^2/4\pi$	$\beta^2=10(g_{\omega NN}/g_{\pi NN})^2$	$\lvert\beta\rvert$		Ref.
42±13	29±9	5.4±0.8	Fit to low-energy π^0 photoproduction with ω^0 exchange	a
36±10	25±7	5 ±0.7	Fit to low-energy π^0 photoproduction with ω^0 and B^0 exchange	a
36	25	5	Regge-pole fit to high-energy π^0 photoproduction	a, b
21.5	15	3.9	Fit to nucleon-nucleon scattering	a, c
16.7	11.5	3.4	Fit to nucleon-nucleon scattering	a, d
2.77	1.9	1.4	Fit to nucleon-nucleon scattering	e

[a] Reference 20.
[b] M. P. Locker and H. Rollnik, Phys. Letters 22, 696 (1966).
[c] R. A. Bryan and B. L. Scott, Phys. Rev. 135, B434 (1964).
[d] A. Scotti and D. Y. Wong, Phys. Rev. Letters 10, 142 (1963).
[e] A. Scotti and D. Y. Wong, Phys. Rev. 138, B145 (1965).

detailed analysis and conclude that

$$|\beta|=2.7. \tag{7.10}$$

They give some reasons for preferring the plus sign.

There is a recent analysis of low-energy (<500 MeV) photoproduction of π^0 mesons based on including ω^0 and B^0 exchange done by Berends, Donnachie, and Weaver.[30] These authors give a value for $g_{\omega NN}^2/4\pi$, and we show their results in Table II together with some other determinations which these authors quote. The completely independent determination of β which we have made from a fit to the inelastic form factors is in remarkably good agreement with these results.[31]

The model which we have made here is a very naive one, but we believe that it does give one some physical insight into the behavior of the inelastic form factors. There are many extensions and applications of the approach presented here. It is an interesting question whether one can in principle give an exact formulation of the problem of calculating the inelastic form factors from first principles in this fashion. Also, we would like to put in enough strong-interatcion dynamics so that $N(W)$ and $D(W)$ are known. In principle, one wants to be able to predict which levels resonate, and just where they resonate.[32] A more modest problem is to try to evaluate $D(W)$ directly from Eq. (4.12) and the known phase shifts (including inelasticities). This would give one not only relative contributions of the various resonances, but also their shape in W. One should, also, really go back and use the exact solution to the Omnès equation rather than the approximate form which we have utilized here. More complicated excitation mechanisms must be included. $N^*, N^{**}, \cdots, \rho, \varphi, B, \cdots$ exchanges should be added, although always in an explicitly gauge-invariant way, and perhaps as Regge poles. There are also box-diagram excitation mechanisms, processes that go through many-particle intermediate states, etc. Several applications suggest themselves, for example, when one has models of the inelastic form factors for all k^2; the contributions of these states to the n-p mass difference[33] and to the electron-scattering sum rules[34] should be estimated.

The region of high k^2 probes a region of these inelastic form factors where our ignorance is very great, but only by attempting to make some estimates of what will go on there will theoretical progress be made.

[30] F. A. Berends, A. Donnachie, and D. L. Weaver, CERN Report Th. 815, 1967 (to be published).

[31] These values of $(g_{\omega\pi\gamma}g_{\omega NN})^2$ are an order of magnitude larger than the vector-meson coupling constants used by Vik.

[32] See, e.g., P. Carruthers, in *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colo., 1965), for a discussion of this problem and for further references in this area.

[33] W. N. Cottingham, Ann. Phys. (N. Y.) 25, 424 (1963).

[34] J. D. Bjorken, *Lectures at International School of Physics "Enrico Fermi" Course XLI, Varenna, Italy, 1967* (Academic Press Inc., New York, to be published).