Some of the commutation relations employed in the derivation of the projected equations are

$$[\Gamma_{--},\mathcal{J}]=\tfrac{2}{3}i(\gamma^\mu p_\mu-\gamma_5\Gamma_5\mathcal{J}), \quad (B9)$$

$$[\Gamma_{--},\mathcal{G}]=[\Gamma_{+-},\mathcal{G}]=0, \quad (B10)$$

$$[\mathcal{G},\Gamma_{++}+\Gamma_{-+}]=0, \quad (B11)$$

$$[\Gamma_{++},\gamma^\mu p_\mu]\Psi_{+-}=\tfrac{2}{3}\Gamma_5\gamma_4(\boldsymbol{\sigma}\cdot\mathbf{p}-\mathbf{K}\cdot\mathbf{p})\Psi_{+-}+\gamma^\mu p_\mu\Psi_{+-},$$

$$[\Gamma_{++},\gamma^\mu p_\mu]\Psi_{--}=-\tfrac{2}{3}\Gamma_5\gamma_4(2\boldsymbol{\sigma}\cdot\mathbf{K}+\mathbf{K}\cdot\mathbf{p})\Psi_{--},$$

$$[\Gamma_{-+},\gamma^\mu p_\mu]\Psi_{+-}=-\tfrac{2}{3}\Gamma_5\gamma_4(\boldsymbol{\sigma}\cdot\mathbf{p}-\mathbf{K}\cdot\mathbf{p})\Psi_{+-},$$

$$[\Gamma_{-+},\gamma^\mu p_\mu]\Psi_{--}=\tfrac{2}{3}\Gamma_5\gamma_4(2\boldsymbol{\sigma}\cdot\mathbf{p}+\mathbf{K}\cdot\mathbf{p})\Psi_{--}+\gamma^\mu p_\mu\Psi_{--}.$$

We further note the important, but obvious, relations

$$\Gamma_{--}\mathcal{G}\Psi=\mathcal{G}\Psi_{--},\quad \Gamma_{+-}\mathcal{G}\Psi=\mathcal{G}\Psi_{+-},$$

$$\Gamma_{-+}\mathcal{G}\Psi=\Gamma_{-+}\mathcal{G}(\Psi_{++}+\Psi_{-+}),$$

$$\Gamma_{++}\mathcal{G}\Psi=\Gamma_{++}\mathcal{G}(\Psi_{++}+\Psi_{-+}),$$

and

$$\Gamma_{--}\mathcal{J}\Psi=2i\Gamma_{--}\gamma^\mu p_\mu(\Psi_{++}+\Psi_{-+}),$$

$$\Gamma_{+-}\mathcal{J}\Psi=-i\Gamma_{+-}\gamma^\mu p_\mu(\Psi_{++}+\Psi_{-+}),$$

$$\Gamma_{-+}\mathcal{J}\Psi=\Gamma_{-+}\mathcal{J}(\Psi_{--}+\Psi_{+-}),$$

$$\Gamma_{++}\mathcal{J}\Psi=\Gamma_{++}\mathcal{J}(\Psi_{--}+\Psi_{+-}).$$

The above operator relations yield the projected wave equations (4.11–4.13).

(d) In addition to (4.16) we shall also give the action of the operators $\gamma^\mu p_\mu$, $\mathcal{J}$, $\gamma^\mu p_\mu-i\mathcal{J}$ and $2\gamma^\mu p_\mu+i\mathcal{J}$ on $\Psi_{++}+\Psi_{-+}$. Thus we have

$$\begin{aligned}\gamma^\mu p_\mu(\Psi_{++}+\Psi_{-+})&=2i\lambda D[-imcp^2(4p^2-m^2c^2)\\&\quad+2(m^2c^2-p^2)\Phi\gamma^\mu p_\mu+p^2(2p^2-m^2c^2)\mathcal{G}]\Psi_{--}\\&\quad-i\lambda D(m^2c^2-p^2)\\&\quad\times(imcp^2+2\Phi\gamma^\mu p_\mu)\Psi_{+-}, \quad (B12)\end{aligned}$$

$$\begin{aligned}i\mathcal{J}(\Psi_{++}+\Psi_{-+})&=2i\lambda D[2imcp^2(4p^2-m^2c^2)\\&\quad+2(m^2c^2-p^2)\Phi\gamma^\mu p_\mu-2p^4\mathcal{G}]\Psi_{--}-i\lambda D(m^2c^2-p^2)\\&\quad\times(imcp^2+2\Phi\gamma^\mu p_\mu)\Psi_{+-}. \quad (B13)\end{aligned}$$

Hence

$$\begin{aligned}(\gamma^\mu p_\mu-i\mathcal{J})(\Psi_{++}+\Psi_{-+})&=2i\lambda D(4p^2-m^2c^2)\\&\quad\times(p^2\mathcal{G}-3imcp^2)\Psi_{--}, \quad (B14)\end{aligned}$$

$$\begin{aligned}(2\gamma^\mu p_\mu+i\mathcal{J})(\Psi_{++}+\Psi_{-+})&\\=4i\lambda D(m^2c^2-p^2)(3\Phi\gamma^\mu p_\mu-p^2\mathcal{G})\Psi_{--}&\\-3i\lambda D(m^2c^2-p^2)(imcp^2+2\Phi\gamma^\mu p_\mu)\Psi_{+-}. \quad (B15)\end{aligned}$$

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Radiative Corrections to the Fermi Part of Strangeness-Conserving β Decay

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In the $V-A$ theory of weak interactions with coupling constant G, whether or not it is mediated by an intermediate vector meson, it is shown to order αG that the contribution of the vector hadron current to the Fermi part of the amplitude for β decay involving hadrons in the same isomultiplet is independent of the details of the strong interactions. This contribution to the radiative corrections to β decay has been re-evaluated, with results that agree in form with those obtained previously with the use of zero-order perturbation theory in the strong interactions. The contribution of the axial-vector current is evaluated approximately for various models of the strong interactions. These results are compared with the predictions of universality, and generally there is good agreement with the Cabibbo angle determined from K_{e3} decay. The factors which influence the cutoff dependence of the theory are discussed, and among other things, it is pointed out that with the neglect of certain "small" quantities, the ratio of the rates for μ decay and β decay is cutoff-independent in the theory with an intermediate vector meson. The effect of Schwinger terms is studied, as well as the relationship between the cancellation occurring in the vector contribution to β decay and the cancellation in a Yang-Mills theory which occurs as a consequence of the Ward identity.

I. INTRODUCTION

In order to test accurately the conserved-vector-current hypothesis (CVC),[1] as well as to choose between the various versions of universality,[1,2] one must compute the electromagnetic corrections to μ decay and to the Fermi part of β decay. The electromagnetic corrections to the decay of a muon have been

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[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); S. Gershtein and J. Zeldovich, Zh. Eksperim. i Teor. Fiz. **29**, 698 (1955) [English transl.: Soviet Phys.—JETP **2**, 576 (1957)].
[2] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

calculated to order α,[3–7] and in the $V-A$ current-current theory of weak interactions the results are not only free of ultraviolet divergences, but they also agree extremely well with the electron spectrum and polarization data.[8] Hadron decay is considerably more complicated because of the strong interactions. With considerable foresight, in 1955, Behrends, Finkelstein, and Sirlin[7] computed the corrections to neutron decay for a general four-fermion coupling to order α in electromagnetism, but to zeroth order in the strong interactions. Later their results were specialized to the $V-A$ theory by Kinoshita and Sirlin[4] and by Berman and Sirlin.[5] The results of these calculations were logarithmically divergent and could be estimated only by introducing a cutoff and assigning it some hopefully reasonable value. Although it has been suggested that the cutoff in some crude way accounted for the strong interactions,[5,9] and that the divergence could be eliminated by including them completely, it has been argued recently that this divergence would persist even if the strong interactions were handled exactly.[10]

In a letter,[11] which we shall refer to as I, it was shown that as a consequence of the standard commutation relations between the electromagnetic and isospin currents, the part of the corrections to the β-decay amplitude coming from the vector hadron current is independent of the details of the strong interactions; i.e., if there were no axial current, the electromagnetic corrections to β decay would simply universally renormalize the weak coupling constant by the same divergent term which was calculated in perturbation theory.

Our purpose in this paper is to complete the discussion presented in I; first, by filling in the many details which were omitted there, accounting for the infrared divergence properly, and demonstrating the relation of the cancellation of vector-current effects to the Ward identity; second, by including a calculation of the axial hadron current contribution, discussing, in particular, the circumstances under which, as in μ decay, a finite result could be obtained by a cancellation of the divergences due to the vector and axial-vector currents; and third, to compare the results of our calculation with the hypothesis of universality.

In the next section we outline the problem and point out the salient features of the calculation, writing down explicitly the several contributions to the matrix element. In Sec. III the general expression for the decay amplitude discussed in Sec. II is shown to be gauge invariant. After that, all the calculations are done in the Feynman gauge. The formulas for the Fermi part of the decay amplitude[12] are developed further in Sec. IV to get them into a form where they can be more easily evaluated. Section V includes the actual calculations of the electromagnetic corrections to the Fermi part of the amplitude, with the contributions of the vector and axial-vector currents considered separately in subsections A and B, respectively. In Sec. VI these results are combined to obtain the complete result for the corrected decay rate, with the bremsstrahlung rate included to cancel the infrared divergence. Finally, in Sec. VII we show that the theory with an intermediate vector meson[13] can be included in our general results, provided that it is minimally coupled to the electromagnetic field, and in Sec. VIII we summarize the main points of the paper and compare our results with universality. There are four appendices. In Appendix A, the tensors used in the calculation are defined, and a number of relations which they satisfy are derived. Appendix B contains a list of several integrals which occur in the text, as well as a sketch of how they are evaluated. The relationship between the universality discussed in this paper and that characteristic of a Yang-Mills gauge theory[14] is discussed in Appendix C. Appendix D is devoted to explaining how operator Schwinger terms[15] in the equal-time commutation relations of the currents affect the details of our calculations in Secs. II and IV.

II. OUTLINE OF PROBLEM

The part of the weak Hamiltonian density $\mathcal{H}_w$ which is responsible for hadron β decay can be written as

$$\mathcal{H}_w=(G/\sqrt{2})t_\sigma\bar{\psi}_e\gamma_\lambda(1+\gamma_5)\psi_\nu+\text{H.c.}, \qquad (2.1)$$

where the charge-rising component of the total hadron current is denoted by t_σ, including both vector and axial-vector parts. That is,

$$t_\lambda=V_\lambda+A_\lambda. \qquad (2.2)$$

The ψ's in Eq. (2.1) are the Heisenberg fields of the indicated particles and $G\cong 10^{-5}M_p^{-2}$.

The β-decay amplitude is the matrix element of $\mathcal{H}_w$ in Eq. (2.1) between the initial and final states. The momentum of the initial hadron will be labeled by p

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and the momenta of the final hadron, electron, and antineutrino will be indicated by p', e, and $\bar\nu$, respectively. The spin indices will be suppressed except where their appearance is explicitly required to make the discussion clear. The matrix element for the β decay is thus $\langle p'e\bar\nu|\mathcal{H}_w(0)|p\rangle$, and as we will show, to order α it has the form[16,17]

$$\langle p'e\bar\nu|\mathcal{H}_w|p\rangle=(G/\sqrt{2})\bar u(e)\{[(1+v)\gamma_\lambda +s(p+p')_\lambda]\langle p'|V_\lambda|p\rangle_0+[(1+a)\gamma_\lambda +\mathcal{P}(p+p')_\lambda]\langle p'|A_\lambda|p\rangle_0\}(1+\gamma_5)v(\bar\nu)\,, \quad (2.3)$$

where $\langle p'|V_\lambda|p\rangle_0$ and $\langle p'|A_\lambda|p\rangle_0$ are the matrix elements of the vector and axial-vector hadron currents in (2.2) computed in the absence of electromagnetic corrections. Clearly, v, s, a, and $\mathcal{P}$ are proportional to α, and the zero-order decay amplitude is obtained by setting these symbols equal to zero in (2.3). That is,

$$\langle p'e\bar\nu|\mathcal{H}_w|p\rangle_0=(G/\sqrt{2})\bar u\gamma_\lambda(1+\gamma_5)v\langle p'|t_\lambda|p\rangle_0\,. \quad (2.4)$$

The Fermi part of the amplitude in (2.3) comes from the terms involving v and s, and in principle these can be separated experimentally from the Gamow-Teller part (hereafter referred to as G-T) involving a and $\mathcal{P}$. In fact, in any spin-zero to spin-zero transition, such as the β decay of O^{14} or of a pion, only the Fermi part contributes. Although our primary objective is to calculate v and s, for the sake of completeness we will retain the G-T part during the general formulation. However, no attempt is made to calculate a and $\mathcal{P}$.

Figure 1 shows the various kinds of Feynman diagrams which give electromagnetic corrections to the matrix element in Eq. (2.3). Figure 1(a) contains those electromagnetic corrections in which the virtual photon is emitted and absorbed through its interaction with the electric current of the hadrons. Their contribution to the decay amplitude can be expressed conveniently in terms of the tensor $T_{\alpha\mu\nu}$ defined in Eq. (A7) and discussed in Appendix A. Namely, for all theories in which the Hamiltonian density of the electromagnetic interaction is simply $-ej\cdot A$, we have

$$\langle p'e\bar\nu|\mathcal{H}_w|p\rangle_a=\frac{i\alpha G}{8\pi^3\sqrt{2}}\bar u\gamma_\lambda(1+\gamma_5)v \times\int dk\, D_{\mu\nu}(k)T_{\lambda\mu\nu}(k,\,p-p',\,p',\,p)\,, \quad (2.5)$$

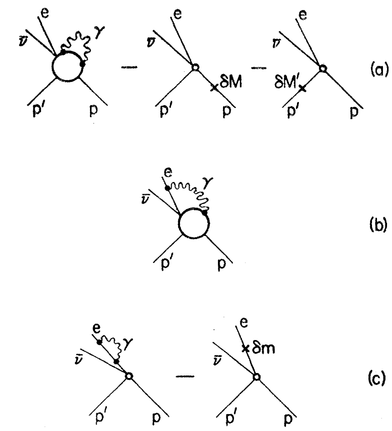

FIG. 1. The order α electromagnetic corrections to the β decay $p\rightarrow p'+e+\bar\nu$.

where $D_{\mu\nu}(k)$ is the Feynman propagator of the photon, and the subscript a refers to the corrections of Fig. 1(a). For theories in which the expression for the electric current involves gradients, as in pion electrodynamics, the electromagnetic interaction Hamiltonian and Lagrangian densities are no longer simply negatives of each other; there are then additional "sea-gull" terms which require Eq. (2.5) to be modified slightly. This possibility causes no essential problem, but to keep the text as simple as possible, we assume here that such terms are absent and relegate the discussion of this complication to Appendix D.

Figures 1(b) and 1(c) refer to the electromagnetic corrections which involve the electric current of the emitted electron. In terms of the tensor $T_{\mu\nu}$ defined in Eq. (A3), the matrix element of Fig. 1(b) is

$$\langle p'e\bar\nu|\mathcal{H}_w|p\rangle_b=\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar u\int dk\, D_{\mu\nu}(k)\gamma_\mu \times\frac{m-\boldsymbol{e}+\boldsymbol{k}}{k^2-2e\cdot k}\gamma_\lambda T_{\lambda\nu}(k,p',p)(1+\gamma_5)v\,, \quad (2.6)$$

where here, and throughout the remainder of the paper, the denominator of Feynman propagators should be understood to contain an additional term $-i\epsilon$, $\epsilon\rightarrow 0_+$. Figure 1(c), properly interpreted, means the correction obtained by multiplying the zeroth-order matrix element in Eq. (2.4) by $Z_2^{1/2}-1=\frac{1}{2}(Z_2-1)$ to order α, where Z_2 is the electron wave-function renormalization constant.[18] That is, to order α,

$$\langle p'e\bar\nu|\mathcal{H}_w|p\rangle_c=(G/\sqrt{2})\tfrac{1}{2}(Z_2-1) \times\bar u\gamma_\lambda(1+\gamma_5)v\langle p'|t_\lambda|p\rangle\,. \quad (2.7)$$

This renormalization factor can be written conveniently

[16] In order to manipulate the matrix element into the form of Eq. (2.3), we can make use of the fact that only one form factor (the change form factor) contributes to $\langle p'|V_\lambda|p\rangle$ to zero order in $(p-p')$. When convenient, we may interchange any two of the three equivalent (to first order in α) forms $\alpha\langle p'|V_\lambda|p\rangle=\alpha K(p+p')_\lambda\delta_{ss'}=2\alpha Kp_\lambda\delta_{ss'}$, where K is an isospin Clebsch-Gordan coefficient.

[17] Our notation and normalization can be summarized as follows: m = electron mass; M = mass of decaying hadron; $\langle p'|p\rangle=(2\pi)^3 2E_p\delta_{ss'}$; $(\boldsymbol{e}+m)u=(-\boldsymbol{e}+m)v=0$, $\bar uu=-\bar vv=2m\delta_{ss'}$; $A\cdot B=\sum_1^4 A_\mu B_\mu=\mathbf{A}\cdot\mathbf{B}-A_0B_0$, $A_4=iA_0$; $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$, $\gamma_\mu^\dagger=-\gamma_\mu$, $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=-2\delta_{\mu\nu}$. $|\mathbf{e}|$ is the magnitude of the electron three momentum, and β is its velocity.

[18] See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., Inc., New York, 1965), pp. 303–309; J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955), pp. 186–187.

as

$$\tfrac{1}{2}(Z_2-1)=-\frac{i\alpha}{16\pi^3m^2}e_\lambda\int dk\, D_{\mu\nu}(k)\frac{\partial}{\partial k_\lambda}\bar{u}(e)\gamma_\mu \times(m+\boldsymbol{e}-\boldsymbol{k})^{-1}\gamma_\nu u(e)\,. \quad (2.8)$$

The complete matrix element is the sum of the expressions in Eqs. (2.4)–(2.7). In the next section we show explicitly that this sum is gauge invariant. After that we proceed with the calculations in the Feynman gauge defined by

$$D_{\mu\nu}(k)=\delta_{\mu\nu}/k^2\,. \quad (2.9)$$

Before obscuring the argument with calculational details, most of which have to do with the proper handling of infrared divergent terms, let us outline the main features of the problems, seeing in particular how the details of $V_{\mu\nu}$ and $V_{\alpha\mu\nu}$, which are the parts of $T_{\mu\nu}$ and $T_{\alpha\mu\nu}$ which contain the vector current, cancel out completely leaving a universal correction to the amplitude; whereas $A_{\mu\nu}$, the axial-current part of $T_{\mu\nu}$, contributes to the Fermi amplitude in a way which depends upon the model of the strong interactions.

Even this partial success of our efforts is possible because we are asking only for the corrections to order α, and because β decay has the special feature that the momentum carried off by either lepton is of order α compared to the hadron mass.

Thus to obtain the corrections to order α, it is almost correct simply to set $p=p'$ and ignore all terms proportional to αm or $\alpha|e_\lambda|$. This is not quite right, because the presence of the denominator $k^2-2e\cdot k$ in (2.6) results in the contributions to some of the integrals coming from the integration region $|k|\lesssim m$ to be effectively proportional to $|e_\lambda|^{-1}$ or m^{-1}. However, $V_{\mu\nu}$ is known near $k=0$, and the corrections to this simple-minded approach can be calculated. If we leave these details for Sec. IV and proceed with the simple argument, we note that when the lepton momenta are set equal to zero in Eqs. (2.5)–(2.7), the order α part of the decay amplitude, expressed in the Feynman gauge becomes

$$\frac{G}{\sqrt{2}}\bar{u}\left[\frac{i\alpha}{4\pi^3}\int\frac{dk}{k^2}(\tfrac{1}{2}\gamma_\lambda T_{\lambda\mu\mu}+\gamma_\mu\boldsymbol{k}\gamma_\lambda k^{-2}T_{\lambda\mu}) +\tfrac{1}{2}(Z_2-1)\gamma_\lambda\langle p'|t_\lambda|p\rangle\right](1+\gamma_5)v\,. \quad (2.10)$$

But

$$\gamma_\mu\boldsymbol{k}\gamma_\lambda T_{\lambda\mu}=-\boldsymbol{k}\gamma_\mu\gamma_\lambda T_{\lambda\mu}-2k_\mu\gamma_\lambda T_{\lambda\mu}\,, \quad (2.11)$$

and according to Eq. (A4) of Appendix A, the second term of (2.11) supplies a universal correction, like the $\frac{1}{2}(Z_2-1)$ term, which is proportional to the zero-order matrix element. The part of the corrections which depends upon the strong interactions is therefore

$$\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\int\frac{dk}{k^2}(\tfrac{1}{2}\gamma_\lambda T_{\lambda\mu\mu}-\boldsymbol{k}k^{-2}\gamma_\mu\gamma_\lambda T_{\lambda\mu})(1+\gamma_5)v\,. \quad (2.12)$$

The second term of this expression may be transformed using

$$\boldsymbol{k}\gamma_\mu\gamma_\lambda T_{\lambda\mu}=-\boldsymbol{k}T_{\mu\mu}+(k_\lambda\gamma_\mu-k_\mu\gamma_\lambda)T_{\lambda\mu} +\epsilon_{\alpha\nu\mu\lambda}k_\nu\gamma_\alpha\gamma_5T_{\lambda\mu}\,. \quad (2.13)$$

Only the $V_{\lambda\mu}$ part of the $T_{\lambda\mu}$ in the first two terms of (2.13), and the $A_{\lambda\mu}$ part of the third term, contribute to the Fermi decay amplitude. Further, the second term contributes only to higher order in α [see (A4) and (A5)] and can be ignored. Hence, the Fermi part of the expression in (2.12) is

$$\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\int\frac{dk}{k^2}(\tfrac{1}{2}\gamma_\lambda V_{\lambda\mu\mu}+\boldsymbol{k}k^{-2}V_{\mu\mu} -\epsilon_{\alpha\nu\mu\lambda}k_\nu\gamma_\alpha k^{-2}A_{\lambda\mu})(1+\gamma_5)v\,. \quad (2.14)$$

In I, the $V_{\lambda\mu\nu}$ was computed using a Fubini-Furlan[19] type sum rule, minimal electromagnetic coupling, and PT invariance. A considerably simpler, and apparently more general procedure is to compute $(\partial/\partial q_\lambda)(q_\beta V_{\beta\mu\nu})$ as discussed in Appendix A. Equation (A17) can then be derived, which when substituted into (2.14), and an integration by parts performed, leads to an exact cancellation between the first two terms of (2.14). (Actually, the integration by parts has to be done more carefully because the integral is formally divergent. A correct treatment is given in Sec. IV.)

The remaining term in (2.14) depends upon the tensor $A_{\lambda\mu}$; i.e., it depends upon the contribution of the axial-vector hadron current to the Fermi part of the decay amplitude. This correction is discussed in detail in Sec. V. In passing, however, we note that it is only this part of the Fermi amplitude which depends upon the details of strong interactions, and remark that the divergent part of this contribution depends only upon the equal-time commutation relations between the spatial components of t_λ and A_μ. To first order in G and α, the divergence from this term will just cancel the divergence of the universal terms leaving a finite result, as in μ decay, if the isospin currents are constructed out of quark-like fermion fields which carry an average charge $\bar{Q}=-\frac{1}{2}$. This point will be discussed in more detail in Sec. V B.

III. GAUGE INVARIANCE

In this section we show that the general form for the decay amplitude is gauge invariant. More precisely, we show that if

$$D_{\mu\nu}(k)\rightarrow k_\mu k_\nu f(k^2)\,, \quad (3.1)$$

the sum of the right-hand sides of (2.5)–(2.7) is transformed to zero.

If we make use of the fact that the electromagnetic mass shifts of the initial and final hadrons are gauge invariant, we can ignore the contribution of the second term on the right of (A10) which appears when $D_{\mu\nu}$

[19] S. Fubini and G. Furlan, Physics 1, 229 (1965).

undergoes the transformation (3.1). With this understanding we can employ (A4), (A10), and the identity

$$\bar{u}(e)\mathbf{k}(m-\mathbf{e}+\mathbf{k})(k^2-2e\cdot k)^{-2}=-\bar{u}(e)\,, \qquad (3.2)$$

to obtain the results of applying the transformation (3.1) to the right-hand sides of (2.5) and (2.6). Indicating the result of this transformation by an arrow, we obtain

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_a \longrightarrow (i\alpha G/8\pi^3\sqrt{2}) \times \bar{u}\gamma_\lambda(1+\gamma_5)v\langle p'|t_\lambda|p\rangle \int dk\ f(k^2) \qquad (3.3)$$

and

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_b \longrightarrow (-i\alpha G/4\pi^3\sqrt{2}) \times \bar{u}\gamma_\lambda(1+\gamma_5)v\langle p'|t_\lambda|p\rangle \int dk\ f(k^2)\,. \qquad (3.4)$$

To see what happens to the electron wave-function renormalization term when the transformation (3.1) is applied, observe the identity

$$e_\lambda k_\mu k_\nu(\partial/\partial k_\lambda)\bar{u}\gamma_\mu(m+\mathbf{e}-\mathbf{k})^{-1}\gamma_\nu = \bar{u}\mathbf{k}(m+\mathbf{e}-\mathbf{k})^{-1}\mathbf{e}(m+\mathbf{e}-\mathbf{k})^{-1}\mathbf{k}u\,. \qquad (3.5)$$

Using first Eq. (3.2) and its adjoint, and then the Dirac equation to eliminate $\mathbf{e}$ in favor of $-m$, we easily obtain

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_c \longrightarrow (i\alpha G/8\pi^3\sqrt{2}) \times \bar{u}\gamma_\lambda(1+\gamma_5)v\langle p'|t_\lambda|p\rangle \int dk\ f(k^2)\,. \qquad (3.6)$$

Comparison of Eqs. (3.3), (3.4), and (3.6) allows the immediate conclusion that the total decay amplitude is transformed to zero under the transformation (3.1). In the remainder of this paper, we will restrict ourselves to the Feynman gauge defined by Eq. (2.9).

IV. TOWARD EVALUATING THE CORRECTIONS

Our purpose in this section is to rewrite the formulas of Sec. II so that the Fermi part of the decay amplitude can be evaluated. Having verified that the total decay amplitude is gauge invariant, we will proceed in the Feynman gauge given by Eq. (2.9).

As written, the integrals in Sec. II are infrared and ultraviolet divergent. The infrared divergence can be handled systematically by introducing a photon mass λ, that is, by replacing k^2 by $k^2+\lambda^2$ in the denominator of $D_{\mu\nu}(k)$. Because of gauge invariance (see Sec. III), we can ignore the term $k_\nu k_\mu/\lambda^2$ which appears in the numerator of the propagator for a massive vector meson. The dependence of the total decay rate upon λ will of course be cancelled, in the limit $\lambda\to 0$, by corresponding terms in the rate for soft photon emission.

The ultraviolet infinity is not cancelled by anything we know how to calculate (see, however, Secs. VB and VII), and some kind of cutoff procedure is therefore necessary. We shall introduce a covariant cutoff by multiplying the photon propagator by $\Lambda^2/(k^2+\Lambda^2)$. We emphasize, however, that this procedure is no more physical than any other, and that the finite part of our result, as well as its explicit dependence upon the cutoff Λ, depends upon this choice.

The foregoing remarks can be summarized by setting

$$D_{\mu\nu}(k)=\frac{\delta_{\mu\nu}}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}, \qquad (4.1)$$

in the decay amplitudes given in Sec. II, and by keeping only those terms in the integrals which do not vanish in the limits $\lambda\to 0$, $\Lambda\to\infty$.

A. Corrections of Fig. 1(a)

With $D_{\mu\nu}$ given by (4.1), the expression in Eq. (2.5) for the matrix element of Fig. 1(a) is

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_a=\frac{i\alpha G}{8\pi^3\sqrt{2}}\bar{u}\gamma_\lambda(1+\gamma_5)v \times\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}T_{\lambda\mu\mu}(k,\,p-p',\,p',\,p)\,. \qquad (4.2)$$

The Fermi part of this amplitude comes only from the contribution of the vector hadron current to $T_{\lambda\mu\mu}$; that is, from $V_{\lambda\mu\mu}$ as defined in Appendix A. Because of the explicit factor of α in Eq. (4.2), we only need $V_{\lambda\mu\mu}$ to zeroth order, for which Eq. (A17) is a handy formula. Hence, using (A17), we can write (4.2) as

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_a=-\frac{i\alpha G}{8\pi^3\sqrt{2}}\bar{u}\gamma_\lambda(1+\gamma_5)v \times\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{\partial}{\partial k_\lambda}V_{\mu\mu}(k,p',p)+(\mathrm{G-T})\,. \qquad (4.3)$$

In order to obtain the cancellation referred to in Sec. II, we integrate by parts the right side of (4.3) and obtain

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_a=-\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\int\frac{dk}{(k^2+\lambda^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2} \times\left(1+\frac{k^2}{k^2+\Lambda^2}\right)\mathbf{k}V_{\mu\mu}(1+\gamma_5)v+(\mathrm{G-T})\,, \qquad (4.4)$$

the resulting term with λ in the numerator has been ignored.

The term proportional to $k^2/(k^2+\lambda^2)$ in (4.4) arises because of the particular way we chose to introduce the ultraviolet cutoff. It gives a finite contribution to the total matrix element, independent of Λ^2 in the limit $\Lambda^2\to\infty$. Although the Λ^2 dependence of the first term in (4.4) will be cancelled exactly by part of the matrix element in Fig. 1(b) given by Eq. (2.6), as outlined in Sec. II, this dependence must be known in order to calculate the finite term.

Noting that

$$\frac{\Lambda^2 k^2}{(k^2+\Lambda^2)^2}=\Lambda^2\frac{\partial}{\partial\Lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}, \quad (4.5)$$

Eq. (4.4) can be rewritten as

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_a=(1+\Lambda^2\partial/\partial\Lambda^2)\times A(\lambda^2,\Lambda^2,p',p)+(\mathrm{G-T}), \quad (4.6)$$

where

$$A=-\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\int\frac{dk}{(k^2+\lambda^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2}\not{k}V_{\mu\mu}(1+\gamma_5)v. \quad (4.7)$$

As will be discussed in the next section, the derivative with respect to Λ^2 in (4.6) is finite in the limit $\Lambda^2\to\infty$ and gives a result which depends only upon the part of A which diverges like $\ln\Lambda^2$.

B. Corrections of Fig. 1(b)

Substituting (4.1) into (2.6) gives

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_b=\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\times\gamma_\mu\frac{\mathbf{m}-\mathbf{e}+\mathbf{k}}{k^2-2e\cdot k}\gamma_\lambda T_{\lambda\mu}(k,p',p)(1+\gamma_5)v. \quad (4.8)$$

It is convenient to separate this into four terms obtained by using the identity

$$\bar{u}\gamma_\mu(m-\mathbf{e}+\mathbf{k})\gamma_\lambda T_{\lambda\mu}=\bar{u}[\mathbf{k}T_{\mu\mu}+2e_\mu\gamma_\lambda T_{\lambda\mu}-(\gamma_\mu k_\lambda+\gamma_\lambda k_\mu)T_{\lambda\mu}+\epsilon_{\alpha\nu\lambda\mu}\gamma_\alpha\gamma_5 k_\nu T_{\lambda\mu}]. \quad (4.9)$$

The first three terms on the right-hand side contribute to the Fermi part of the matrix element only through the $V_{\mu\nu}$ part of $T_{\mu\nu}$, and to zero order in α, the third term can be simplified by using Eqs. (A4) and (A7); namely,

$$(\gamma_\mu k_\lambda+\gamma_\lambda k_\mu)V_{\lambda\mu}=2\gamma_\lambda\langle p'|V_\lambda|p\rangle+O(\alpha), \quad (4.10)$$

noting that $q\sim O(\alpha)$. The last term on the right of (4.9) contributes to the Fermi part of the decay amplitude only through the axial hadron current, that is, through $A_{\lambda\mu}$. Hence, writing explicitly only those terms which contribute to the Fermi amplitude, and omitting cutoffs where they are not required for convergence, we obtain by substituting (4.9) into (4.8)

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_b=\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\bigg[\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{\mathbf{k}}{k^2-2e\cdot k}V_{\mu\mu}+2e_\mu\gamma_\lambda\int\frac{dk}{k^2+\lambda^2}\frac{1}{k^2-2e\cdot k}V_{\lambda\mu}-2\int\frac{dk}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\times\frac{1}{k^2-2e\cdot k}\gamma_\lambda\langle p'|V_\lambda|p\rangle_0+\epsilon_{\alpha\nu\lambda\mu}\gamma_\alpha\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\times\frac{1}{k^2-2e\cdot k}k_\nu A_{\lambda\mu}\bigg](1+\gamma_5)v+(\mathrm{G-T}). \quad (4.11)$$

C. Corrections of Fig. 1(c)

Specializing to the Feynman gauge and adding the cutoffs to the expression (2.8), the electron wave-function renormalization of Eq. (2.7) becomes

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_c=-\frac{i\alpha G}{8\pi^3\sqrt{2}}\bar{u}\gamma_\lambda(1+\gamma_5)v\langle p'|t_\lambda|p\rangle\frac{e_\alpha}{2m^2}\times\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{\partial}{\partial k_\alpha}\bar{u}\gamma_\mu(m+\mathbf{e}-\mathbf{k})^{-1}\gamma_\mu u. \quad (4.12)$$

In exact analogy to our treatment of the hadron current corrections in Eqs. (4.4) through (4.7), Eq. (4.12) can be rewritten as

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_c=\bar{u}\gamma_\lambda(1+\gamma_5)v\langle p'|V_\lambda|p\rangle\times(1+\Lambda^2\partial/\partial\Lambda^2)B(\Lambda^2,\lambda^2,e)+(\mathrm{G-T}), \quad (4.13)$$

where

$$B=\frac{i\alpha G}{2\pi^3 m^2\sqrt{2}}\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{(e\cdot k)(m^2-e\cdot k)}{k^2-2e\cdot k}. \quad (4.14)$$

V. EVALUATION OF THE CORRECTIONS

In this section we evaluate to order α the total electromagnetic correction to the Fermi part of the decay amplitude obtained by adding the matrix elements in Eqs. (4.6), (4.11), and (4.13). As we have emphasized, except for the contribution of the term involving $A_{\lambda\mu}$ in the (4.11), the result of this addition is independent of the details of the strong interactions. Consequently, this section is divided into subsections A and B, which discuss separately the contributions of the vector and axial-vector hadron currents.

A. Corrections from the Vector Hadron Current

Consider first the corrections of Fig. 1(a), which are given in Eq. (4.6). The term proportional to the 1 in (4.6) we shall keep to combine with the first term on the right of Eq. (4.11). The term proportional to the derivative with respect to Λ^2 is finite in the limit $\Lambda^2\to\infty$ (assuming, as we do, that A diverges no worse than like $\ln\Lambda^2$), and unlike most of the other finite terms, does not depend upon the region of integration for small k^2, but rather on the coefficient of $\ln\Lambda^2$ in the integral for A in Eq. (4.7), which in turn is determined from the large-k^2 ($k^2\gg\Lambda^2$) region of integration. It is amusing that of all the terms which combine to give the total decay amplitude only this one depends in any detail on the structure of $V_{\mu\mu}$. A large class of models will have the asymptotic behavior derived by Bjorken[10] assuming quark commutation rules[20]; namely, for fixed $\mathbf{k}$,

$$V_{\mu\mu}\underset{k_0\to\infty}{\longrightarrow}-2k_0^{-1}\langle p'|V_0|p\rangle. \quad (5.1)$$

[20] M. Gell-Mann, Phys. Rev. **125**, 1062 (1962); Physics **1**, 63 (1964).

For large Λ^2 the dependence of A upon this cutoff will be unchanged if any convenient form for $V_{\mu\mu}$ with the behavior (5.1) is used; for example,

$$\text{“}V_{\mu\mu}\text{”}=-2k_\mu\langle p'|V_\mu|p\rangle(k^2-2p\cdot k)^{-1}. \quad (5.2)$$

Using this expression for $V_{\mu\mu}$ in Eq. (4.7), and performing one of the integrals discussed in Appendix B, leads to[16]

$$\text{“}A\text{”}=-(\alpha G/8\pi\sqrt{2})\bar{u}\gamma_\lambda(1+\gamma_5)v \times\langle p'|V_\lambda|p\rangle\ln(\Lambda^2/M^2), \quad (5.3)$$

from which we can obtain

$$\Lambda^2\frac{\partial}{\partial\Lambda^2}A=-(\alpha G/8\pi\sqrt{2})\bar{u}\gamma_\lambda(1+\gamma_5)v\langle p'|V_\lambda|p\rangle. \quad (5.4)$$

The integral for B in Eq. (4.14), which occurs in the correction (4.13) for the electron wave-function renormalization, is also evaluated in Appendix B. The result is

$$B=-(\alpha G/2\pi\sqrt{2})[\tfrac{7}{8}+\tfrac{1}{2}\ln(\Lambda/m)-\ln(m/\lambda)], \quad (5.5)$$

so that

$$\left(1+\Lambda^2\frac{\partial}{\partial\Lambda^2}\right)B=-(\alpha G/2\pi\sqrt{2}) \times[9/8+\tfrac{1}{2}\ln(\Lambda/m)-\ln(m/\lambda)], \quad (5.6)$$

which is, of course, just the standard expression[8] for $\frac{1}{2}(Z_2-1)$ multiplied by $G/\sqrt{2}$.

The analogy between the evaulation of these two corrections is not accidental. If the external hadrons were spin-$\frac{1}{2}$, minimally coupled fermions, to zeroth order in the strong interactions, the two contributions would be identical. The assumption (5.1) says that the asymptotic form of $V_{\mu\mu}$ is the same for hadrons as it is for the corresponding tensor constructed out of electron and neutrino fields.

Now let us add everything up. Using (5.4) and (5.6), the sum of the matrix elements in Eqs. (4.6), (4.11), and (4.13) is

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_{a+b+c}=-\frac{\alpha G}{2\pi\sqrt{2}}[11/8+\tfrac{1}{2}\ln(\Lambda/m)-\ln(m/\lambda)]\bar{u}\gamma_\lambda(1+\gamma_5)v\langle p'|V_\lambda|p\rangle_0$$
$$+\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\Bigg[\int\frac{dk}{(k^2+\lambda^2)^2}\frac{2e\cdot k}{k^2-2e\cdot k}kV_{\mu\mu}+2e_\mu\gamma_\lambda\int\frac{dk}{k^2+\lambda^2}\frac{1}{k^2-2e\cdot k}V_{\lambda\mu}$$
$$+2\int\frac{dk}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}\gamma_\lambda\langle p'|V_\lambda|p\rangle_0+\epsilon_{\alpha\nu\lambda\mu}\gamma_\alpha\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}k_\nu A_{\lambda\mu}\Bigg](1+\gamma_5)v+(\text{G}-\text{T}), \quad (5.7)$$

where we have ignored cutoffs where they are not required. The sum of the three matrix elements is indicated by the subscript.

Consider the two integrals in (5.7) which involve $V_{\mu\mu}$ and $V_{\lambda\mu}$. In terms of the Born approximation $V_{\lambda\mu}{}^B$ discussed in Appendix A, we can write

$$V_{\lambda\mu}=V_{\lambda\mu}{}^B+(V_{\lambda\mu}-V_{\lambda\mu}{}^B) \quad (5.8)$$

with a minimum of assumptions. The difference in the second term on the right-hand side of (5.8) is not singular when $k\rightarrow 0$, and the infrared cutoff λ^2 can be set equal to zero in the corresponding integrand. It is then easy to see that this difference term would contribute to (5.7) in the order $\alpha|e|\ln|e|=O(\alpha^2\ln\alpha)$ and higher orders. Terms of this order are outside the limits of our approximation and have already been ignored in obtaining Eq. (4.6). We therefore replace $V_{\lambda\mu}$ and $V_{\mu\mu}$ in (5.7) by their Born approximations given in (A24) and write these two integrals as

$$\int\frac{dk}{(k^2+\lambda^2)^2}\frac{2e\cdot k}{k^2-2e\cdot k}\frac{\Lambda^2}{k^2+\Lambda^2}kV_{\mu\mu}=-2p_\mu\langle p'|V_\mu|p\rangle_0 \times\int\frac{dk}{(k^2+\lambda^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{2e\cdot k}{k^2-2e\cdot k}\frac{k}{k^2-2p\cdot k} \quad (5.9)$$

and

$$e_\mu\gamma_\lambda\int\frac{dk}{k^2+\lambda^2}\frac{1}{k^2-2e\cdot k}V_{\lambda\mu}=-2(e\cdot p)\gamma_\lambda\langle p'|V_\lambda|p\rangle_0 \times\int\frac{dk}{k^2+\lambda^2}\frac{1}{k^2-2e\cdot k}\frac{1}{k^2-2p\cdot k}. \quad (5.10)$$

These integrals, as well as the remaining explicit integrals in (5.7) are computed in Appendix B. Substituting (5.9) and (5.10) into (5.7), and performing the integrals, leads to an explicit expression for the complete Fermi part of the right-hand side of (5.7), except for the term involving $A_{\lambda\mu}$. If the matrix element is put into the form of Eq. (2.3), the result of this calculation can be expressed by giving the results for v and s. The term of (5.7) involving $A_{\lambda\mu}$ contributes only to v, and it will be evaluated in Sec. VB. If, for the moment, we denote its contribution to v by v_A, then in the rest system of the initial hadron ($\mathbf{p}=0$), our results are

$$v=(\alpha/2\pi)[\tfrac{3}{2}\ln(\Lambda/m)+2(1-\beta^{-1}\tanh^{-1}\beta)\ln(m/\lambda) -11/8+\beta^{-1}L[2\beta/(1+\beta)] -\beta^{-1}(\tanh^{-1}\beta)^2+\beta^{-1}\tanh^{-1}\beta]+v_A, \quad (5.11)$$

where β is the electron velocity, and

$$s=-\frac{\alpha}{4\pi M}\left(\frac{1-\beta^2}{\beta^2}\right)^{1/2}\tanh^{-1}\beta. \quad (5.12)$$

Here $L(x)$ is one of the Spence functions, defined by[21]

$$L(x)\equiv\int_0^x \frac{dt}{t}\ln(1-t). \quad (5.13)$$

B. Corrections from the Axial-Vector Hadron Current

The term v_A in Eq. (5.11) comes from the axial-vector current. This contributes to the Fermi part of the amplitude in Fig. 1(b), since the isoscalar part of the electromagnetic current can combine with the axial current to form an effective vector, G parity $+1$ contribution, given by the term involving $\tilde{A}_{\lambda\mu}$ in (5.7). The electron momentum and infrared cutoff (or photon mass) can be neglected in this term since it contains no infrared divergence. This is because the forward spin-averaged matrix element of an axial current vanishes, and this is all that contributes to $\tilde{A}_{\lambda\mu}$ for $k\rightarrow 0$. Thus, the relevant term in (5.7) is

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_{\text{axial}}=\frac{i\alpha G}{4\pi^3\sqrt{2}}\epsilon_{\alpha\nu\lambda\mu}\bar{u}\gamma_\alpha(1+\gamma_5)v \times\int\frac{dk}{k^4}\frac{\Lambda^2}{k^2+\Lambda^2}k_\nu\tilde{A}_{\lambda\mu}. \quad (5.14)$$

We consider first the divergent contribution to (5.14). This means that we need to know the asymptotic behavior of $A_{\lambda\mu}-A_{\mu\lambda}\simeq 2\tilde{A}_{\lambda\mu}$ for $k^2\rightarrow\infty$. Using the method of Bjorken[10] to determine this limit, we find it depends upon the part of the commutator of $A_\lambda(0)$ and $j_\mu(\mathbf{x},0)$, which is antisymmetric in the tensor indices. This is a model-dependent quantity. Let us assume, as is true for most models, that the currents are constructed out of spin-$\frac{1}{2}$ fields ψ which have canonical anticommutation relations. Then,[20]

$$A_\lambda=\psi^\dagger\gamma_0\gamma_\lambda\gamma_5T^\dagger\psi, \quad (5.15a)$$

$$j_\mu=\psi^\dagger\gamma_0\gamma_\mu Q\psi, \quad (5.15b)$$

where $T^\dagger$ is the isospin raising matrix and Q is the matrix for the electric charge. Thus,

$$[A_\lambda(0),j_\mu(\mathbf{x},0)]-(\lambda\leftrightarrow\mu) =-2i\epsilon_{4\nu\lambda\mu}\delta^3(\mathbf{x})\psi^\dagger\gamma_0\gamma_\nu\{T^\dagger,Q\}\psi. \quad (5.16)$$

If the fields $\psi(x)$ contains one isodoublet, but no other isomultiplets (except isoscalars or isodoublets with the same hypercharge), then the anticommutator $\{T^\dagger,Q\}$ can be written as $2\bar{Q}T^\dagger$, where $\bar{Q}$ is the average charge of the isodoublet of elementary fields. One then finds

$$\tilde{A}_{\lambda\mu}\xrightarrow[k_0\rightarrow\infty]{}-(i/k_0)2\bar{Q}\epsilon_{4\nu\lambda\mu}\langle p'|V_\nu|p\rangle_0+O(1/k_0^2). \quad (5.17)$$

This may be covariantly generalized to give

$$\tilde{A}_{\lambda\mu}=2\bar{Q}\epsilon_{\alpha\nu\lambda\mu}k_\alpha\langle p'|V_\nu|p\rangle_0\frac{1}{k^2}+O(1/k^2) \quad (5.18a)$$

or

$$\tilde{A}_{\lambda\mu}=2\bar{Q}\epsilon_{\alpha\nu\lambda\mu}k_\alpha\langle p'|V_\nu|p\rangle_0\frac{1}{k^2+M_A^2}+O(1/k^2), \quad (5.18b)$$

where the second form, which is the same as the first for $k^2\rightarrow\infty$, is written so that we can evaluate the divergent term without introducing a spurious infrared divergence. For the moment, M_A is simply some mass—presumably roughly equal to the nucleon mass—which crudely marks the value of k^2 above which the asymptotic expression dominates. If (5.18b) is substituted into (5.14), the integral performed, and the result expressed as a value for v_A, we obtain

$$v_A=(3\alpha/2\pi)\bar{Q}\ln(\Lambda/M_A)+\text{finite terms}, \quad (5.19)$$

where the first term includes all of the contribution from (5.18b), and the remaining finite terms have yet to be estimated. Combining the result in (5.19) with the other divergent term in (5.11) coming from the vector hadron current, we see that the coefficient of $\ln\Lambda$ in the total Fermi matrix element is proportional to $1+2\bar{Q}$.

Now $\bar{Q}$ is a model-dependent constant. If the fundamental fields were simply nucleons, as assumed in the perturbation theory approach of previous calculations, then $\bar{Q}=\frac{1}{2}$, which reproduces exactly the result given by Berman and Sirlin for $V-A$ theory. In the Gell-Mann and Zweig quark model[20,22] one finds $\bar{Q}=\frac{1}{6}$. It is also possible to construct models such that $\bar{Q}=-\frac{1}{2}$, so that the divergent term vanishes and one obtains a finite radiative correction for β decay. This is the case for the conventional description of μ decay and explains in our language the finite correction that has long been known in that case. Several models for the hadron fields which have the property have recently been suggested by other authors.[23] However, since other problems with the current-current theory of weak interactions (for example its nonunitarity) indicate that one must eventually modify the theory at least at high energies, we do not consider the vanishing of this divergent term to be of any great significance. In a correct theory it probably would not occur, or as in the intermediate boson theory discussed in Sec. VII, it would occur also in μ decay and indicate only an infinite but universal renormalization of G.

As an interesting aside on the value of $\bar{Q}$ we can estimate, rather crudely, an upper bound on the value $|\bar{Q}|$ from data on photoproduction of the A_1 meson, or more accurately, from the lack of it. The estimate is made by assuming that the A_1 meson dominates the

[21] K. Mitchell, Phil. Mag. 40, 351 (1949).

[22] M. Gell-Mann, Phys. Letters 8, 214 (1964); G. Zweig, CERN Report No. TH412, 1964 (unpublished).

[23] K. Johnson, F. E. Low, and H. Suura, Phys. Rev. Letters 18, 1224 (1967); N. Cabibbo, L. Maiana, and G. Preparata, Phys. Letters 25B, 29 (1967); 25B, 132 (1967).

axial current so that the diagram of Fig. 2 provides the divergent contribution of (5.19). From data for photoproduction of three pions we can obtain an approximate A_1 photoproduction cross section by assuming that, for energies sufficiently above threshold, and for small momentum transfer, the A_1 cross section is proportional to the ω cross section, and that both tend to constant for large photon energies E_γ. This is very possibly a poor assumption since the major contribution to the ω photoproduction at high E_γ is from diffraction production. This process does not contribute to A_1 photoproduction because of the A_1 has the wrong parity. This means that we are possibly overestimating the A_1 cross section, which in turn will mean that, although still an upper bound, our value is not a very restrictive one. From the invariant mass distributions for three-pion photoproduction given by Maor *et al.*,[24] and by the Erbe *et al.*,[25] we find

$$\sigma_{A_1}\leq\tfrac{1}{3}\sigma_\omega \qquad (5.20)$$

as a generous estimate of the number of A_1 mesons produced. We also assume that even though the fact that A_1 and the photon are both virtual particles in the process of Fig. 2, no further form factors are required.

Under these assumptions the amplitude for Fig. 2 is given by

$$F=g_{Ae\nu}eg\delta_{\mu\nu}\delta_{\alpha\rho}k^{-2}(k^2+M_A{}^2)^{-1}i\epsilon_{\lambda\sigma\nu\alpha}k_\lambda p_\sigma \times\bar{u}\gamma_\mu(-kk^{-2})\gamma_\rho(1+\gamma_5)v\,, \qquad (5.21)$$

neglecting the electron mass, where

$$A(\gamma+p\rightarrow n+A_1{}^\dagger)=g\epsilon_A{}^\alpha\epsilon_\gamma{}^\mu i\epsilon_{\lambda\sigma\mu\alpha}k_\rho p_\sigma\,. \qquad (5.22)$$

From the assumptions discussed above, we estimate $g^2<5.1\times10^3$ BeV^{-6}.

The constant $g_{Ae\nu}$ has been shown by Weinberg[26] to be equal to $g_{\rho e\nu}$ under the assumption that the A_1 and the ρ dominate the axial and vector currents, respectively. Combining all these things we can obtain a limit on $\bar{Q}$, which turns out to be $\bar{Q}<10$. This is obviously an overestimate, which is not very surprising when one considers the many approximations made in obtaining it. However, further data on A_1 photoproduction could lead to a considerably lower limit.

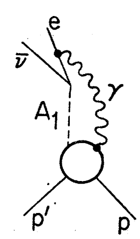

FIG. 2. The A_1 pole in the contribution of the axial current to the Fermi matrix element.

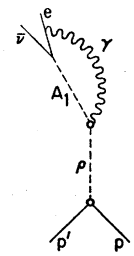

FIG. 3. A possible mechanism to account for the diagram of Fig. 2 at high virtual momenta of the photon and A_1.

To return to the problem of radiative corrections, we have still to evaluate the "finite contribution" of (5.19). Unfortunately, the best we can do is just an estimate. The problem is that we have no real understanding of the source of the infinite contribution, so we cannot rigorously separate it out and then evaluate the remaining contribution by a dispersion approach. However, this is exactly the procedure which we wish to use. We do this for β decay of a single nucleon since this is the number we want, a similar procedure could be applied for any decay. Defining $\nu=-k\cdot(p+p')/2M$, we write

$$A_{\lambda\mu}=\epsilon_{\alpha\nu\lambda\mu}k_\alpha\langle p'|V_\nu|p\rangle A(k^2,\nu)+\cdots\,, \qquad (5.23)$$

and note that to zero order in $q=p-p'$ it is only the part of $A_{\lambda\mu}$ coming from the invariant $A(k^2,\nu)$ which contributes to the integral in (5.14). Then subtracting out a term to account explicitly for the asymptotic behavior as given by Eq. (5.18b), we let

$$A(k^2,\nu)=\frac{2\bar{Q}}{k^2+M_A{}^2}+a(k^2,\nu)\,. \qquad (5.24)$$

This separation occurs, for example, if the asymptotic term arises from the process shown in the graph of Fig. 3. As suggested by this graph, as well as by the previous discussion, we will take M_A to be the mass of the A_1.

Following Cottingham[27] we write a dispersion relation for $a(k^2,\nu)$ in ν for fixed, spacelike k^2, and follow Harari[28] to argue that it should satisfy an unsubtracted dispersion relations, since the leading Regge trajectory in the t channel is that of the ρ meson, and this means that for $\nu\rightarrow\infty$, Im$a\sim\nu^{\alpha(0)-1}$ with $\alpha(0)\approx0.5$. Since only the isoscalar photon contributes, intermediate states with isospin $\frac{1}{2}$ are all that enter the dispersion relation. Thus, the $N_{33}{}^*$ resonance does not contribute, and we are led to hope that a reasonable approximation is obtained by retaining just the Born term. This we have done, and using the form factors $F_A=F_{1\rho}=F_{2\rho}=(1+\eta k^2)^{-2}$, with $\eta=1.37$ BeV^{-2},[29] we obtain 6×10^{-4} as the finite contribution to v_A in Eq. (5.19). Hence,

$$v_A=(3\alpha/2\pi)\bar{Q}\ln(\Lambda/M_{A_1})-6\times10^{-4} \qquad (5.25)$$

[24] Cambridge Bubble Chamber Group, in *Proceedings of the Second Topical Conference on Resonance Particles* (Ohio University, Athens, Ohio, 1965), p. 476.
[25] Aachen-Berlin-Bonn-Hamburg-Heidelberg-München Bubble Chamber Collaboration, Nuovo Cimento **46A**, 795 (1966).
[26] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[27] W. N. Cottingham, Ann. Phys. (N. Y.) **25**, 424 (1963).
[28] H. Harari, Phys. Rev. Letters **17**, 1303 (1966).
[29] *Proceedings of the Argonne International Conference on Weak Interactions*, Argonne National Laboratory Report No. ANL-7130 (unpublished); CERN Heavy Liquid Bubble Chamber Group (unpublished).

is the value of v_A we use in the discussion of Sec. VIII. Fortunately, since it is not a reliable estimate of the finite part of v_A, the 6×10^{-4} gives a very small contribution to cosθ (Cabibbo angle), being almost buried in the uncertainties quoted for the experimental numbers.

VI. CALCULATION OF THE DECAY RATE

A. Rate for Nonradiative Decay

The general form of the decay amplitude is given in Eq. (2.3), and the value of v and s, which describe the Fermi part of this amplitude, are given in Eqs. (5.11), (5.12), and (5.25). We now compute in outline the contribution to the total decay rate coming from the Fermi part of the decay amplitude.

With our normalizations,[17] the decay rate is given by

$$d\Gamma=(2\pi)^{-5}|\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle|^2 \times\delta^4(p'+e+\bar{\nu}-p)\frac{d^3p'd^3ed^3\bar{\nu}}{2E2E'2E_e2E_\nu}. \quad (6.1)$$

We take only the Fermi part of the decay amplitude, as expressed in terms of v and s in Eq. (2.3), substitute this into (6.1), integrate over $d^3p'd^3\bar{\nu}$ and $d\Omega_e$, and the sum over the spins of the electron and nuetrino. Keeping only contributions up to order α, the result of this is

$$d\Gamma=d\Gamma_0(1+2v+4MmE_e^{-1}s), \quad (6.2)$$

where $d\Gamma_0$ is the rate without electromagnetic corrections. If, for the initial hadron, the isospin and its third component are I and I_z, respectively, and if recoil corrections are ignored, then

$$d\Gamma_0=(2\pi^3)^{-1}(I+I_z+1)(I-I_z) \times G^2(M-M'-E_e)^2|e|E_e dE_e. \quad (6.3)$$

B. Rate for Bremsstrahlung

As the bremsstrahlung rate is also infrared divergent, it is necessary to again introduce the same small photon mass λ and keep only terms which do not vanish in the limit $\lambda\to0$. The total decay rate is obtained by adding the bremsstrahlung rate to the rate for the nonradiative decay in (6.2).

Suppressing the spin dependence, the bremsstrahlung rate is

$$d\Gamma_B=(2\pi)^{-8}|\langle p'e\bar{\nu}k|\mathcal{H}_w|p\rangle^2 \times\delta^4(p'+e+\bar{\nu}+k-p)\frac{d^3p'd^3ed^3\bar{\nu}d^3k}{2E2E'2E_e2E_\nu2E_k}. \quad (6.4)$$

As we have stressed, the momentum emitted to the leptons and photon is of order α. Thus, the extra factor of $d^3k/2E_k$ in (6.4) compared to (6.1) means that the bremsstrahlung phase space is of order α^2 compared to the phase space for the nonradiative decay. As a consequence, we need only keep the hadron pole terms in the part of the matrix element which comes from the coupling of the photon to the current of the hadrons, since these are of order α^{-1} and contribute to total decay rate in the same order as do the lowest order radiative corrections to the rate without bremsstrahlung.

Keeping only the hadron pole terms, as well as, of course, the part of the amplitude due to radiation from the emitted electron, the matrix element in (6.4) is

$$\langle p'e\bar{\nu}k|\mathcal{H}_w|p\rangle=iG(2\pi\alpha)^{1/2}\bar{u}(e)\left(\frac{2\epsilon\cdot e+\gamma\cdot\epsilon k}{-\lambda^2-2e\cdot k}+\frac{2\epsilon\cdot p}{-\lambda^2+2k\cdot p}\right)\gamma_\lambda(1+\gamma_5)v(\bar{\nu})\langle p'|t_\lambda|p\rangle_\theta, \quad (6.5)$$

where ϵ_μ is the polarization vector of the photon. When (6.5) is substituted into (6.4), we can ignore the λ^2 in the denominator, since its presence introduces an addition to the rate which vanishes in the limit $\lambda\to0$. Realizing this, it is then easy to see that (6.5) is gauge invariant (in the limit of vanishing λ).

Since we are concerned here with the rate coming from the Fermi part of the amplitude, we will restrict t_λ in (6.5) by replacing it by its vector part V_λ. We then specialize to the rest frame of the initial hadron, ignore recoil corrections (since they are of higher order in α), perform all polarization sums and integrations, except that over the electron energy, to obtain the bremsstrahlung rate in terms of the uncorrected nonradiative decay rate[4] [see Eq. (6.3)]:

$$d\Gamma_B=d\Gamma_0\frac{\alpha}{\pi}\left[2\left(\frac{1}{\beta}\tanh^{-1}\beta-1\right)\ln\frac{M-M'-E_e}{\lambda}+3\left(1-\frac{1}{\beta}\tanh^{-1}\beta\right)+\frac{2}{3}\left(\frac{1}{\beta}\tanh^{-1}\beta-1\right)\frac{M-M'-E_e}{E_e}+\frac{1}{12\beta}\tanh^{-1}\beta\left(\frac{M-M'-E_e}{E_e}\right)^2+C(\beta)\right], \quad (6.6)$$

where

$$C(\beta)=2\ln2(\beta^{-1}\tanh^{-1}\beta-1)+1+(2\beta)^{-1}\times\tanh^{-1}\beta[2+\ln\tfrac{1}{4}(1-\beta)^2]+\beta^{-1}[L(\beta)-L(-\beta)]+(2\beta)^{-1}\left[L\left(\frac{1-\beta}{2}\right)-L\left(\frac{1+\beta}{2}\right)\right]. \quad (6.7)$$

The Spence function $L(x)$ is defined in Eq. (5.13).

C. Total Decay Rate

The total decay rate can be obtained by combining Eqs. (5.11), (5.12), and (5.25) with Eqs. (6.2) and (6.6).

The result is

$$d\Gamma_{\text{tot}}=d\Gamma_0\Big\{1-1.2\times10^{-3}+\frac{\alpha}{\pi}\Big[\tfrac{3}{2}(1+2\bar{Q})\ln\frac{\Lambda}{M}+\tfrac{3}{2}\ln\frac{M}{m}+3\bar{Q}\ln\frac{M}{M_{A_1}}-\tfrac{3}{8}+2(\beta^{-1}\tanh^{-1}\beta-1)$$
$$\times\left(\frac{M-M'-E_e}{3E_e}-\tfrac{3}{2}+\ln\frac{2(M-M'-E_e)}{m}\right)+2(\beta)^{-1}\tanh^{-1}\beta\left[2(1+\beta^2)+\frac{(M-M'-E_e)^2}{6E_e^2}-2\ln\frac{2}{1-\beta}\right]$$
$$+\beta^{-1}\left(L(\beta)-L(-\beta)+L\left(\frac{2\beta}{1+\beta}\right)+\tfrac{1}{2}L\left(\frac{1-\beta}{2}\right)-\tfrac{1}{2}L\left(\frac{1+\beta}{2}\right)\right)\Big]\Big\}. \quad (6.8)$$

This result can be compared with Eq. (7) of Ref. 5. Although complete account has been taken here of the strong interactions to the extent that they affect the contribution to (6.8) coming from the vector hadron current, it is seen that this part of the rate agrees exactly with that obtained by neglecting the strong interactions. In our expressions we have ignored terms of order M^2/Λ^2. These have been retained in Ref. 5.

VII. EFFECTS OF AN INTERMEDIATE VECTOR MESON

A popular modification of the local current-current $V-A$ theory of weak interactions is based on the introduction of a charged intermediate vector meson coupled to t_λ.[13] We show in this section that the principal conclusions of our work still hold when the weak interactions are mediated by such a meson, provided it is minimally coupled to the electromagnetic field. Namely, we show that:

(1) To order α, the Fermi part of the β-decay amplitude coming from the vector hadron current is independent of the details of the strong interactions. As for the case of a theory with a local interaction, one gets the correct answer for these electromagnetic corrections by ignoring the strong interactions in the calculation.

(2) The order α correction to the Fermi part of the β-decay amplitude coming from the axial hadron current is the same as in the local theory, except that the cutoff is replaced by the intermediate vector meson mass M_W.

By referring to the work of Shaffer,[13] we indicate at the end of this section how these conclusions imply that for a theory with an intermediate meson, the comparison of the experimental O^{14} and μ-decay rates with universality is the same as in the local theory with cutoff in the β-decay amplitude set equal to M_W.

In the theory with an intermediate vector meson, the Fermi part of the β-decay matrix element to zero order in α is

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_0=\frac{GM_W^2}{\sqrt{2}}\bar{u}(e)\gamma_\lambda(1+\gamma_5)v(\bar{\nu})\times\frac{\delta_{\lambda\mu}+q_\lambda q_\mu/M_W^2}{q^2+M_W^2}\langle p'|V_\mu|p\rangle, \quad (7.1)$$

where $q=p-p'$. There are eight kinds of diagrams which indicate the corrections to this amplitude to order α. Those which can depend upon the strong interactions are shown in Fig. 4.

In Figs. 4(a) and 4(b) we can ignore the q dependence in the vector meson propagator connected to the e-$\bar{\nu}$ pair, since these terms are higher order in q, and therefore in α, than those retained. The matrix element in Fig. 4(a) is then exactly the same as that given in Eq. (4.6) for the local theory.

Assuming, as we shall, that the W meson is minimally coupled to the electromagnetic field, the matrix element in Fig. 4(b) becomes

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_b=\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}(e)\gamma_\lambda(1+\gamma_5)v(\bar{\nu})\times\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}[(2q-k)_\mu\delta_{\lambda\alpha}-q_\alpha\delta_{\mu\lambda}-(q-k)_\lambda\delta_{\mu\alpha}]\times\frac{\delta_{\alpha\nu}+(q-k)_\alpha(q-k)_\nu/M_W^2}{(q-k)^2+M_W^2}V_{\nu\mu}(k,p',p). \quad (7.2)$$

Using the gauge conditions (A4) and (A7) for $V_{\nu\mu}$, and eliminating terms which are clearly of order $\alpha|q|$ or

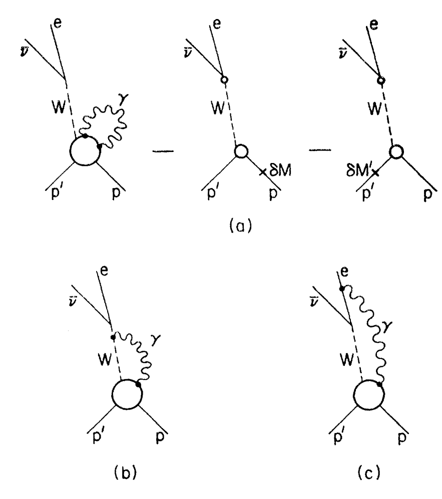

FIG. 4. The part of the electromagnetic corrections which can depend on the strong interactions in a theory with an intermediate meson.

higher, this expression reduces to

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_b=\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2+M_W{}^2}\times(\mathbf{k}V_{\mu\mu}-\gamma_\mu\langle p'|V_\mu|p\rangle)(1+\gamma_5)v(\bar{\nu}). \quad (7.3)$$

In analogy with the graph in Fig. 1(b), the correction to the Fermi part of the amplitude shown in Fig. 4(c) includes a contribution coming from the axial current. Making the decomposition in Eq. (4.9), the Fermi part of the amplitude in Fig. 4(c) is

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_c=\frac{i\alpha GM_W{}^2}{4\pi^3\sqrt{2}}\bar{u}(e)\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}\times[(\mathbf{k}\delta_{\mu\lambda}+2e_\mu\gamma_\lambda-\gamma_\mu k_\lambda-\gamma_\lambda k_\mu)V_{\nu\mu}+\epsilon_{\alpha\beta\lambda\mu}\gamma_\alpha k_\beta A_{\nu\mu}]\times\frac{\delta_{\lambda\nu}+(q-k)_\lambda(q-k)_\nu/M_W{}^2}{(q-k)^2+M_W{}^2}(1+\gamma_5)v(\bar{\nu}). \quad (7.4)$$

Using the relations (A4) and (A7) for $V_{\nu\mu}$, and eliminating terms of order $\alpha|q|$ and $\alpha|e|$, this becomes

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_c=\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{M_W{}^2}{k^2+M_W{}^2}\times\frac{1}{k^2-2e\cdot k}[\mathbf{k}V_{\mu\mu}+2e_\mu\gamma_\lambda V_{\lambda\mu}-(2+k^2M_W{}^{-2})\gamma_\mu\times\langle p'|V_\mu|p\rangle+\epsilon_{\alpha\beta\lambda\mu}\gamma_\alpha k_\beta A_{\lambda\mu}](1+\gamma_5)v. \quad (7.5)$$

If we add the right-hand sides of (7.3) and (7.5), throwing away terms of order $\alpha|e|$ and eliminating cutoffs where they are not required, we obtain for the sum of the matrix elements in Figs. 3(b) and 3(c)

$$\langle p'e\bar{\nu}|\mathcal{H}_w|p\rangle_{b+c}=\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\left[\int\frac{dk}{k^2+\lambda^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{\mathbf{k}}{k^2-2e\cdot k}V_{\mu\mu}+2e_\mu\gamma_\lambda\int\frac{dk}{k^2+\lambda^2}\frac{M_W{}^2}{k^2+M_W{}^2}\frac{1}{k^2-2e\cdot k}V_{\lambda\mu}-2\int\frac{dk}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}\gamma_\mu\langle p'|V_\mu|p\rangle+\epsilon_{\alpha\beta\lambda\mu}\gamma_\alpha\int\frac{dk}{k^2}\frac{M_W{}^2}{k^2+M_W{}^2}\frac{k_\beta}{k^2-2e\cdot k}A_{\lambda\mu}\right](1+\gamma_5)v. \quad (7.6)$$

Comparing (7.6) with (4.11), we see that, except for the factors of $M_W{}^2(k^2+M_W{}^2)^{-1}$ in the integrands of the second and fourth terms on the right-hand side of (7.6) the sum of the corrections in Figs. 3(b) and 3(c) for the theory with an intermediate meson is the same as the correction in Fig. 1(b) for the theory with a point interaction. Remembering that the corrections of Figs. 3(a) and 1(a) are exactly the same, the arguments of Sec. V B can be taken over directly to conclude assertion (1) in the beginning of this section. The second assertion also follows immediately by comparing the last terms on the right of Eqs. (4.11) and (7.6), and recognizing that, for the respective theories, it is only in these terms that the axial current contributes to the Fermi part of the decay amplitude.

In the next section we discuss to what extent the experimental μ-decay and O^{14}-decay rates agree with the hypothesis of universality. As one of the possible theories, we want to include the case discussed here, in which the weak interactions are mediated by a charged vector meson minimally coupled to the electromagnetic field.

For the theory with a minimally coupled intermediate meson, the β-decay and μ-decay rates have been calculated by Shaffer[13] to order α by ignoring the effects of the strong interactions. Neglecting terms of order $M_\mu{}^2/M_W{}^2$, $\alpha M^2/M_W{}^2$, and $\alpha M_W{}^2/\Lambda^2$, the results of Shaffer can be written as

$$\Gamma_\mu{}^W=\{1+(\alpha/\pi)[\tfrac{3}{2}\ln(\Lambda/M_W)+a(\Lambda,M_W)]\}\Gamma_\mu{}^L \quad (7.7a)$$

and

$$\Gamma_\beta{}^W=\{1+(\alpha/\pi)[-\tfrac{3}{2}\ln(\Lambda/M_W)+a(\Lambda,M_W)]\}\Gamma_\beta{}^L(\Lambda), \quad (7.7b)$$

where the superscripts W and L refer, respectively, to the rates for the theories with and without an intermediate vector meson. The argument of $\Gamma_\beta{}^L(\Lambda)$ indicates the Λ dependence of this rate explicitly, and $a(\Lambda,M_W)$ is a divergent (quadratically in Λ) expression which, as we shall see, cancels out in the ratio of the μ-decay and β-decay rates.

The Λ dependence of $\Gamma_\beta{}^L(\Lambda)$ is only in a logarithm, and to order α

$$\Gamma_\beta{}^L(\Lambda)=[1+(3\alpha/\pi)\ln(\Lambda/M_W)]\Gamma_\beta{}^L(M_W). \quad (7.8)$$

Again to order α, the result of substituting this expression into (7.7b) can be written as

$$\Gamma_\beta{}^W=\{1+(\alpha/\pi)[\tfrac{3}{2}\ln(\Lambda/M_W)+a(\Lambda,M_W)]\}\Gamma_\beta{}^L(M_W). \quad (7.9)$$

Equation (7.9) is the decay rate as computed by ignoring the strong interactions. However, we have seen that in the Fermi part of the amplitude, only the contribution coming from the axial current is modified by the strong interactions; and further, that this contribution is the same as in the local theory with Λ replaced by M_W. Hence, Eq. (7.9) can be made to include the strong interactions by including them in $\Gamma_\beta{}^L(M_W)$. Comparing (7.7a) and (7.9), it is then clear that the ratio of μ decay to β decay is the same in the theory with an intermediate vector meson as it is in the local theory, if the cutoff Λ in the β-decay amplitude is replaced by M_W.

VIII. RESULTS AND CONCLUSIONS

In the calculations of the preceding sections we have shown that:

(1) To first order in α, the part of the radiative corrections to pure Fermi β decays which arise from the vector weak current are given correctly by calculations using zero-order perturbation theory in the strong interactions.

(2) The contribution of the axial current to these radiative corrections is model-dependent. It contains a logarithmically divergent term, the coefficient of which depends on the charge structure of the effective fields carrying the axial weak current. In addition, there is a finite correction which has been estimated, using a dispersion approach, to be small.

(3) To order α, the full radiative corrections to the Fermi part of the amplitude in the $V-A$ theory contain a divergent term proportional to $(1+2\bar{Q})\ln\Lambda$, where $\bar{Q}$ is the average charge of the isotopic doublet of fields carrying the weak axial current (see Sec. V B). Therefore, the radiative corrections are finite in any theory for which these fields have an average charge $\bar{Q}=-\frac{1}{2}$.

(4) Introducing an intermediate vector meson of mass M_W and neglecting terms of order M_μ^2/M_W^2, $\alpha M^2/M_W^2$, and $\alpha M_W^2/\Lambda^2$, one finds a universal, divergent renormalization to G in addition to finite corrections. The finite corrections, and thus the comparison with universality, are exactly those obtained in a theory with an intermediate meson, except that for β decay the cutoff is replaced by M_W.

We now proceed to evaluate the results numerically in order to compare the value of the Cabibbo angle[2] obtained from K decay with that from O^{14} decay and thus, hopefully, to investigate the universality of the theory. For K decay we quote the value given by Sirlin[30] of

$$\cos\theta_K = 0.975 \pm 0.001\,. \qquad (8.1)$$

The error stated here is purely the experimental error. Some correction to the calculated rate have been neglected on the basis of estimates that they are small and of opposite sign, but their neglect means that the quoted accuracy is rather optimistic.

For O^{14} decay we use our calculation to obtain the radiative corrections for single nucleon β decay. Since strong interaction effects have been included, it is no longer realistic to choose a cutoff $\Lambda=M$. Therefore, despite the fact that our correction from the vector hadron current has the same form as was obtained using perturbation theory, even this correction will be different from that which has been quoted previously. Introducing the Cabibbo angle, we write

$$P_0 \cos^2\theta_\beta(1+\Delta) = \text{experimental rate} = E\,,$$

[30] A. Sirlin, Phys. Rev. Letters **16**, 872 (1966).

where P_0 represents the zeroth order in α-decay rate for O^{14} prior to Cabibbo, including all nuclear structure corrections, and $P_0\Delta$ represents the radiative corrections. Then

$$\cos\theta_\beta = (EP_0^{-1})^{1/2}(1-\tfrac{1}{2}\Delta)\,.$$

Writing Δ_0 for the radiative corrections used by Sirlin[30] and Δ_1 for our value, we find

$$\cos\theta_\beta = (0.978 \pm 0.001)[1-\tfrac{1}{2}(\Delta_1-\Delta_0)]\,, \qquad (8.2)$$

where 0.978 is the value of $\cos\theta_\beta$ quoted by Sirlin,[30] and the 0.001 represents the experimental errors. We now summarize the values of $\cos\theta_\beta$ obtained in various models (i.e., choices of $\bar{Q}$) by computing $\Delta_1-\Delta_0$ as discussed in Sec. V.

A theory for which $\bar{Q}=-\frac{1}{2}$ gives

$$\cos\theta_\beta(\bar{Q}=-\tfrac{1}{2}) = 0.981\,.$$

If we have correctly estimated the finite axial corrections, we would conclude optimistically that the errors in this value are ± 0.002, in which case the $\bar{Q}=-\frac{1}{2}$ models do not give very good agreement with $\cos\theta_K$ given in Eq. (8.1). The Gell-Mann and Zweig quark model has $\bar{Q}=\frac{1}{6}$, and the value of $\cos\theta_\beta$ depends on a cutoff Λ, which can be interpreted as the mass of the W meson as discussed in Sec. VII. Using the unitarity limit of 300 BeV for this cutoff, we find

$$\cos\theta_\beta(\bar{Q}=\tfrac{1}{6},\ \Lambda=300\ \text{BeV}) = 0.968\,,$$

where again we optimistically estimate an error of ± 0.002. Better agreement with the K-decay result is obtained by choosing a lower cutoff. For example,

$$\cos\theta_\beta(\bar{Q}=\tfrac{1}{6},\ \Lambda=30\ \text{BeV}) = 0.973$$

and

$$\cos\theta_\beta(\bar{Q}=\tfrac{1}{6},\ \Lambda=10\ \text{BeV}) = 0.976\,.$$

Thus, a cutoff or W mass in the range 10–30 BeV gives good agreement with universality.[31]

Of course it should be realized that the dependence on the cutoff is very weak, and that therefore a small modification of some of the other corrections could alter considerably the value of the cutoff, or W mass, rerequired for universality. Theories for which $|\bar{Q}|<\frac{1}{6}$ do not lead to very different results from those implied by $\bar{Q}=\frac{1}{6}$. Larger values of $\bar{Q}$ would have an appreciable effect, tending to decrease the value of $\cos\theta_\beta$.

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[31] A W-meson mass of this order has also been suggested by S. L. Glashow, H. J. Schnitzer, and S. Weinberg, Phys. Rev. Letters **19**, 205 (1967).

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APPENDIX A

The purpose of this appendix is to define the tensor operators which occur in the calculation, to discuss some of their properties, and to generally complete the mathematical details omitted in the discussion of Secs. II and IV.

The charge-raising, strangeness-conserving part of the hadron weak current is called $t_\mu(x)$, and this is a sum of a vector and an axial vector,

$$t_\mu(x)=V_\mu(x)+A_\mu(x)\,. \tag{A1}$$

Following the conserved vector-current hypothesis, we assume that V_μ is the plus component of the isotopic spin current. In the absence of Schwinger terms (see Appendix D), the parts of t_μ are assumed to have the following equal-time commutation relations with the electromagnetic current j_μ:

$$[j_0(\mathbf{x},0),V_\mu(0)]=[j_\mu(0),V_0(x,0)]=V_\mu(0)\delta^3(\mathbf{x}) \tag{A2a}$$

and

$$[j_0(\mathbf{x},0),A_\mu(0)]=[j_\mu(0),A_0(\mathbf{x},0)]=A_\mu(0)\delta^3(\mathbf{x})\,. \tag{A2b}$$

Suppressing the spin dependences, if $|p\rangle$ and $|p'\rangle$ are the initial and final hadron states, respectively, the second-rank tensor which occurs in the expression for for graph of Fig. 1(b) is

$$T_{\mu\nu}(k,p',p)=i\int dx\; e^{-ik\cdot x}\langle p'|T(t_\mu(0)j_\nu(x))|p\rangle\,. \tag{A3}$$

It follows exactly from (A2) and electric current conservation that

$$k_\nu T_{\mu\nu}=\langle p'|t_\mu|p\rangle \tag{A4}$$

and

$$(k_\mu-q_\mu)T_{\mu\nu}=\langle p'|t_\nu|p\rangle-M_\nu\,, \tag{A5}$$

where $q=p-p'$, and M_ν is defined in terms of the divergence D of the total weak current, $D=\partial_\mu t_\mu$, as

$$M_\nu(k,p',p)=\int dx\; e^{-ik\cdot x}\langle p'|T(D(0)j_\nu(x))|p\rangle\,. \tag{A6}$$

We have seen in Sec. V A that terms like $q_\mu T_{\mu\nu}$ can contribute to order α [see (5.9) and (5.10)]. However, we need only keep the Born approximation of such terms, and in the Born approximations, the vector part of $q_\mu T_{\mu\nu}$ is canceled by the part of M_ν involving the divergence of the vector current. Hence, for the vector part of $T_{\mu\nu}$ [see (A15)], we have

$$k_\mu V_{\mu\nu}=\langle p'|V_\nu|p\rangle+O(\alpha,q)\,. \tag{A7}$$

The third-rank tensor which enters into the expression for the graph of Fig. 1(a) is

$$T_{\lambda\mu\nu}(k,q,p',p)=\int dxdy\; e^{-iq\cdot x+ik\cdot y}\times\langle p'|T(t_\lambda(x)j_\mu(y)j_\nu(0))|p\rangle-\delta M_{\lambda\mu\nu}\,, \tag{A8}$$

where the expression abbreviated by $\delta M_{\lambda\mu\nu}$ cancels the pole in the first term at $q=p-p'$ and is equal to the part of the first term which, when (A8) is substituted into (2.5), only contributes to the electromagnetic mass renormalizations of the initial and final hadrons.

From the definition (A8) it is fairly easy to check that the commutation rules (A2) give for $q=p-p'$

$$k_\mu T_{\lambda\mu\nu}=T_{\lambda\nu}-k_\mu\delta M_{\lambda\mu\nu} \tag{A9}$$

and hence, by (A4) that

$$k_\mu k_\nu T_{\lambda\mu\nu}=\langle p'|t_\lambda|p\rangle-k_\mu k_\nu\delta M_{\lambda\mu\nu}\,. \tag{A10}$$

In a similar manner we obtain for arbitrary q

$$q_\lambda T_{\lambda\mu\nu}=T_{\nu\mu}(-k,\,p',\,p)+i\int dy\; e^{i(k-q)\cdot y}\langle p'|T(t_\mu(y)j_\nu(0))|p\rangle+M_{\mu\nu}-q_\lambda\delta M_{\lambda\mu\nu}\,, \tag{A11}$$

where

$$M_{\mu\nu}=-i\int dxdy\; e^{-iq\cdot x+ik\cdot y}\langle p'|T(D(x)j_\mu(y)j_\nu(0))|p\rangle\,, \tag{A12}$$

and, as in (A6) D is the divergence of the total weak current. Differentiating (A11) with respect to q_α and setting $q=p-p'$, we obtain a relation analogous to the Ward identity,[32]

$$T_{\alpha\mu\nu}(k,\,p-p',\,p',\,p)=-(p-p')_\lambda\frac{\partial}{\partial q_\alpha}T_{\lambda\mu\nu}|_{q=p-p'}-\frac{\partial}{\partial k_\alpha}T_{\mu\nu}(k,p',p)+\frac{\partial}{\partial q_\alpha}(M_{\mu\nu}-q_\lambda\delta M_{\lambda\mu\nu})|_{q=p-p'}\,. \tag{A13}$$

The first term on the right side of this equation is proportional to $p-p'$ and hence is of order α. Thus, to zero order in α

$$T_{\lambda\mu\nu}(k,\,p-p',\,p',\,p)=-\frac{\partial}{\partial k_\lambda}T_{\mu\nu}(k,p',p)+\frac{\partial}{\partial q_\lambda}(M_{\mu\nu}-q_\alpha\delta M_{\alpha\mu\nu})|_{q=p-p'}\,. \tag{A14}$$

As in Eq. (A1) the second- and third-rank tensors can be broken up into their parts coming from the vector and axial-vector currents. That is

$$T_{\mu\nu}=V_{\mu\nu}+A_{\mu\nu} \tag{A15}$$

and

$$T_{\lambda\mu\nu}=V_{\lambda\mu\nu}+A_{\lambda\mu\nu}\,. \tag{A16}$$

[32] Y. Takahashi, Nuovo Cimento **6**, 371 (1957).

Since all the relations (A3) through (A14) are linear in the hadron current, similar relations hold separately for their vector and axial-vector parts. We will show that for the vector part $V_{\lambda\mu\nu}$ the second term on the right-hand side of (A14) is of order α, and hence, to zero order in α

$$V_{\lambda\mu\nu}(k, p-p', p', p) = -\frac{\partial}{\partial k_\lambda} V_{\mu\nu}(k,p',p). \quad \text{(A17)}$$

To simplify the algebra let us restrict ourselves to spin-zero hadrons while arguing that the contribution of the vector current to the term involving the derivative with respect to q_λ in (A14) is of order α. It should be clear from the discussion that the argument can be generalized readily to hadrons of arbitrary spin.

For spin-zero hadrons the expression for $q_\alpha \delta M_{\alpha\mu\nu}$ is

$$q_\alpha \delta M_{\alpha\mu\nu} = -\frac{q_\alpha \langle p'|t_\alpha|p'+q\rangle \delta M_{\mu\nu}{}^2(k)}{(q+p')^2+M^2} - \frac{\delta M_{\mu\nu}{}'^2(k) q_\alpha \langle p-q|t_\alpha|p\rangle}{(p-q)^2+M'^2}, \quad \text{(A18)}$$

where the matrix elements should be understood as continuations off the mass shells in the momenta $p'+q$ and $p-q$; quantities such as $\delta M_{\mu\nu}{}^2(k)$ are the coefficients of the photon propagator in the integrands whose integrals give the electromagnetic mass (squared) shifts of the initial and final hadrons. Contracting out the particle of momentum $p'+q$, we have

$$\langle p'|t_\alpha|p'+q\rangle = i\int dx\, e^{i(p'+q)\cdot x}(\square^2 - M^2) \times \langle p'|T(t_\alpha(0)\phi(x))|0\rangle, \quad \text{(A19)}$$

where ϕ is the renormalized field of the contracted particle. Employing translational invariance and integration by parts, it is easy to obtain

$$q_\lambda \langle p'|t_\lambda|p'+q\rangle = i\int dx\, e^{i(p'+q)\cdot x}(\square^2 - M^2) \times \langle p'|T(-i\partial_\lambda T_\lambda(0)\phi(x))|0\rangle - [(p'+q)^2+M^2] \times \int dx\, e^{-iq\cdot x}\delta(x_0)\langle p'|[t_0(x),\phi(0)]|0\rangle. \quad \text{(A20)}$$

In local field theory

$$\delta(x_0)[t_0(x),\phi(0)] \sim \delta^4(x),$$

and Eq. (A20) can be written as

$$q_\lambda \langle p'|t_\lambda|p'+q\rangle = \langle p'|-i\partial_\lambda t_\lambda|p'+q\rangle + [(p'+q)^2+M^2]K, \quad \text{(A21)}$$

where K is independent of q. Similarly, the second matrix element in Eq. (A18) can be written as

$$q_\lambda \langle p-q|t_\lambda|p\rangle = \langle p-q|-i\partial_\lambda t_\lambda|p\rangle + [(p-q)^2+M'^2]K'. \quad \text{(A22)}$$

When Eqs. (A21) and (A22) are substituted into (A18), and the result substituted into the second term on the right of Eq. (A14), the terms proportional to K and K' contribute nothing; since the derivative with respect to q_λ in (A14) acts on an expression independent of q. The remaining part of the second term on the right of (A14) is "proportional" to the divergence $\partial_\lambda t_\lambda$. Since the vector current is conserved by the strong interactions, the part of this term coming from the vector current is of order α. Q.E.D.

We need to know the forms of the tensors to zero order in α near $k^2=0$. In this region the tensors are given correctly by their Born approximations at small k. Ignoring terms of order $p-p'$, for $V_{\mu\nu}$ this is

$$V_{\mu\nu}{}^B = \langle p'|V_\mu|p\rangle 2p_\nu\left(\frac{Q}{k^2-2p\cdot k} + \frac{Q+1}{k^2+2p\cdot k}\right), \quad \text{(A23)}$$

where Q and $Q+1$ are the charges of the initial and final hadron. Since k^2 is negligible compared to $2p\cdot k$ near $k^2=0$, instead of (A23) we can also use the simpler form

$$V_{\mu\nu}{}^B = -\langle p'|V_\mu|p\rangle 2p_\nu(k^2-2p\cdot k)^{-1}. \quad \text{(A24)}$$

This is the expression we actually use in the calculations discussed in the text.

APPENDIX B

In this Appendix we outline how to compute some of the integrals encountered in Sec. V. The general method is based upon the introduction of Feynman parameters, shifting the integration variable, and rotating the k_0 contour to obtain a Eulcidean metric as discussed, for example, in the book by Jauch and Rohrlich (Ref. 18). These procedures are standard, we mention here only an occasional step which we found unobvious.

In Eq. (5.2) there appears the integral equivalent[16] to

$$I_1 = \int \frac{dk}{(k^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{2p\cdot k}{k^2-2p\cdot k}k_\alpha, \quad \text{(B1)}$$

where it should be remembered that here, and elsewhere, all factors in the denominator should be understood to have a $-i\epsilon$ added. The photon mass may be dropped, as I_1 is not infrared divergent. Write $2p\cdot k = (2p\cdot k - k^2)+k^2$ noting that the first term gives an odd integrand, and thus does not contribute. Therefore,

$$I_1 = \int \frac{dk}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2p\cdot k}k_\alpha. \quad \text{(B2)}$$

By standard techniques, this is

$$I_1 = -i\pi^2 2^{-1} p_\alpha[\beta_+{}^2 \ln(1-\beta_+{}^{-1}) + \beta_-{}^2 \ln(1-\beta_-{}^{-1}) + \beta_+ + \beta_-], \quad \text{(B3)}$$

where $\beta_\pm$ are the two zeros of $\beta^2+\Lambda^2M^{-2}(1-\beta)$. In the limit $\Lambda^2\gg M^2$,

$$I_1=i\pi^2 p_\alpha[\ln(\Lambda/M)-\tfrac{1}{4}]. \tag{B4}$$

The expression for B in Eq. (4.14) contains two intergals. The first is

$$I_2=\int \frac{dk}{(k^2+\lambda^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}k_\alpha. \tag{B5}$$

This integral is not ultraviolet divergent; so the factor $\Lambda^2(k^2+\Lambda^2)^{-1}$ may be omitted. Also, no error is made in the limit $\lambda^2\to 0$ if the last factor in the denominator is replaced by $k^2-2e\cdot k+\lambda^2$. One then obtains in the standard way

$$I_2=i\pi^2m^{-2}e_\alpha[\ln(m/\lambda)-1]. \tag{B6}$$

The second integral in (4.14) is of the form

$$I_3=2\int \frac{dk}{(k^2+\lambda^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{(e\cdot k)^2}{k^2-2e\cdot k}. \tag{B7}$$

Again, write $2e\cdot k=-(k^2-2e\cdot k)+k^2$ and note that only the second term contributes. The infrared cutoff can then be omitted, and we obtain

$$I_3=\int \frac{dk}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{e\cdot k}{k^2-2e\cdot k}, \tag{B8}$$

which is essentially the same as I_1. Hence,

$$I_3=i\pi^2m^2[\tfrac{1}{4}-\ln(\Lambda/m)]. \tag{B9}$$

The next to last term in Eq. (5.7) contains the integral

$$I_4=\int \frac{dk}{k^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{1}{k^2-2e\cdot k}. \tag{B10}$$

The exact result for this is

$$I_4=-i\pi^2[\beta_+\ln(1-\beta_+^{-1})+\beta_-\ln(1-\beta_-^{-1})], \tag{B11}$$

where $\beta_\pm$ are the same as used in discussing I_1. In the limit $\Lambda^2\gg m^2$,

$$I_4=i\pi^2[1+\ln(\Lambda^2/m^2)]. \tag{B12}$$

The integral occurring in Eq. (5.9) is

$$\int \frac{dk}{(k^2+\lambda^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2}\frac{\mathbf{k}}{k^2-2e\cdot k}\frac{2e\cdot k}{k^2-2p\cdot k}. \tag{B13}$$

Writing, for $\lambda^2\to 0$,

$$\frac{2e\cdot k}{(k^2+\lambda^2)(k^2-2e\cdot k)}=\frac{1}{k^2-2e\cdot k}-\frac{1}{k^2+\lambda^2},$$

we are led to two integrals, neither of which is ultraviolet divergent; so we can omit the factor $\Lambda^2(k^2+\Lambda^2)^{-1}$. The second integral is identical in form to I_2. The first is

$$I_5=\int \frac{dk}{k^2}\frac{\mathbf{k}}{k^2-2e\cdot k}\frac{1}{k^2-2p\cdot k}. \tag{B14}$$

This can be transformed by the method of Feynman parameters into

$$I_5=-i\pi^2\int_0^1 dx[x\mathbf{p}+(1-x)\mathbf{e}][xp+(1-x)e]^{-2}, \tag{B15}$$

and the exact result of the integration is

$$I_5=\frac{-i\pi^2}{(p-e)^2}\left\{\frac{1}{2M|e|}[\mathbf{p}(m^2-ME_e)+\mathbf{e}(M^2-ME_e)]\times\ln\left[\frac{M-|e|-E_e}{M+|e|-E_e}\frac{M(E_e+|e|)-m^2}{M(E_e-|e|)-m^2}\right]+\mathbf{p}\ln(M/m)+\mathbf{e}\ln(m/M)\right\}, \tag{B16}$$

where E_e and $\mathbf{e}$ are the electron energy and momentum in the hadron rest system, $p=0$. Ignoring terms of order mM^{-1}, this is

$$I_5=i\pi^2M^{-2}\{\mathbf{p}[\ln(M/m)-\beta^{-1}\tanh^{-1}\beta]+\mathbf{e}ME_e^{-1}\beta^{-1}\tanh^{-1}\beta\}, \tag{B17}$$

where $\beta=|e|/E_e$.

There remains only the integral

$$I_6=\int \frac{dk}{k^2+\lambda^2}\frac{1}{k^2-2e\cdot k}\frac{1}{k^2-2p\cdot k}. \tag{B18}$$

Again, λ^2 may be added to the last two factors in the denominator without changing the result for small λ^2. In this limit, one obtains

$$I_6=\frac{-i\pi^2}{2}\int_0^1\frac{dx}{Q(x)}\ln\frac{Q(x)}{\lambda^2}, \tag{B19}$$

where

$$Q(x)=[xe+(1-x)p]^2. \tag{B20}$$

A substitution of the form $x=a-b\coth\alpha$, as described in Appendix B of Ref. 8, then permits the integral to be performed in terms of the Spence function (5.12). In the approximation $m^2/M^2\ll 1$, $E_e/M\ll 1$,

$$I_6=\frac{i\pi^2}{2ME_e\beta}\left[(\tanh^{-1}\beta)^2+2\ln(m/\lambda)\times\tanh^{-1}\beta-L\left(\frac{2\beta}{1+\beta}\right)\right]. \tag{B21}$$

The exact result is given in Ref. 8.

APPENDIX C

We have seen that if it were not for the presence of the axial current, the β-decay amplitude would be universally renormalized because of the cancellation between the parts of the corrections in Figs. 1(a) and 1(b) which depend upon the details of the strong interactions. In this Appendix, we indicate the relationship between this cancellation and the cancellation known to exist in a Yang-Mills theory,[14] where the Ward identity applies to all three components of the isotopic charge density.

Considering the graph of Fig. 1(b), the cancellation involves the part of the right hand side of Eq. (4.8) coming from the first term in (4.9), with the electron momentum set equal to zero in the denominator of the electron propagator. That is, the relevant part of (4.8) is

$$\frac{i\alpha G}{4\pi^3\sqrt{2}}\bar{u}\gamma_\lambda(1+\gamma_5)v\int\frac{dk}{(k^2)^2}k_\lambda V_{\mu\mu}(k,p',p). \quad \text{(C1)}$$

To zero order in the hadron momentum transfer, this expression cancels exactly against the correction to the matrix element of the vector hadron current shown in Fig. 1(a). Namely, it cancels against

$$\frac{i\alpha G}{8\pi^3\sqrt{2}}\bar{u}\gamma_\lambda(1+\gamma_5)v\int\frac{dk}{k^2}V_{\lambda\mu\mu}(k,\,p-p',\,p',\,p). \quad \text{(C2)}$$

Let us compare this situation with that which exists in a theory in which the photon is the neutral member of an isotopic triplet of massless vector mesons.[33] If these mesons are coupled with strength e, it is well known that at zero momentum transfer the matrix element of $V_\lambda^{(+)}$ (the isospin components are indicated by superscripts) is given exactly, to all orders in α, by its value to zero order in α. In particular, the order α corrections to this matrix element cancel.

The order-α corrections to the matrix element of $V_\lambda^{(+)}$ in a Yang-Mills theory are shown in Fig. 5. The corrections of Fig. 5(a) are

$$\frac{i\alpha}{8\pi^3}\int\frac{dk}{k^2}\hat{V}_{\lambda\mu\mu}(k,\,p-p',\,p',\,p), \quad \text{(C3)}$$

where $\hat{V}_{\lambda\mu\nu}$ is the same as $V_{\lambda\mu\nu}$ defined in Appendix A, except that the product of electric currents $j_\mu(y)j_\nu(0)$, which occurs in (A8), is replaced by

$$\mathbf{V}_\mu(y)\cdot\mathbf{V}_\nu(0)=\tfrac{1}{2}[V_\mu^{(-)}(y)V_\nu^{(+)}(0)+V_\mu^{(+)}(y)V_\nu^{(-)}(0)]+V_\mu^3(y)V_\nu^3(0). \quad \text{(C4)}$$

It is not difficult to show that only the isotopic vector part of the electric current contributes to (C1) and (C2). Taking this into account, it is easy to check that to zero order in $p-p'$ the expression in (C3) is equal to

$$\frac{i\alpha}{4\pi^3}\int\frac{dk}{k^2}V_{\lambda\mu\mu}(k,\,p-p',\,p',\,p); \quad \text{(C5)}$$

that is, to within a factor of 2 coming from the isospin Clebsch-Gordan coefficients, it is unchanged by replacing $\hat{V}_{\lambda\mu\mu}$ by $V_{\lambda\mu\mu}$. This can be seen, for example, by going through the steps [Eqs. (A12)–(A14)] leading to (A17) for $\hat{V}_{\lambda\mu\nu}$ instead of $V_{\lambda\mu\nu}$.

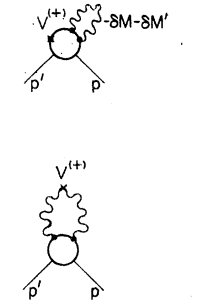

FIG. 5. The order α corrections to the isospin current in a theory where the photon is a member of an isotopic triplet.

The corrections of Fig. 5(b) are naturally expressed in terms of $\hat{V}_{\lambda\mu}$, which is the part of $V_{\lambda\mu}$ coming from the isovector electric current. However, as we have just mentioned, the isoscalar part does not contribute to the integrals, so that $\hat{V}_{\lambda\mu}$ can be replaced by $V_{\lambda\mu}$. To zero order in the momentum transfer, the corrections of Fig. 5(b) are then

$$\frac{i\alpha}{2\pi^3}\int\frac{dk}{k^2}k_\lambda V_{\mu\mu}(k,p',p). \quad \text{(C6)}$$

The Ward identity applied to the Yang-Mills theory requires that the expressions in (C5) and (C6) cancel. Algebraically, this is the same as our cancellation between the expressions in (C1) and (C2).

APPENDIX D

We discuss here to what extent our calculations are affected by the presence of operator Schwinger terms in the commutators of the currents given in (A2).[34] We need consider only the commutators involving the vector currents. This is because the axial currents contribute to the Fermi part of the decay amplitude only through the term containing $A_{\lambda\mu}$ in Eq. (4.11); that is, only through the antisymmetric part of $A_{\lambda\mu}$, whereas a Schwinger term addition to $A_{\lambda\mu}$ (see below) is symmetric in the tensor indices.

Considering the vector currents, suppose that instead of (A2a), we have

$$[j_0(\mathbf{x},0),V_\mu(0)]=V_\mu(0)\delta^3(\mathbf{x})+S^{(+)}(0)\partial_\mu\delta^3(\mathbf{x}), \quad \text{(D1a)}$$

$$[V_0(\mathbf{x},0),j_\mu(0)]=-V_\mu(0)\delta^3(\mathbf{x})+S^{(+)}(0)\partial_\mu\delta^3(\mathbf{x}). \quad \text{(D1b)}$$

A model theory for which these relations hold is one in which the vector currents have contributions coming from a triplet of spin-zero fields ϕ, ϕ^*, and ϕ_3. In such a model, the contribution to $S^{(+)}$ arising from these fields is $i\sqrt{2}\phi^*\phi_3$.

The general expressions which describe the correc-

[33] J. C. Ward and A. Salam, Nuovo Cimento **11**, 568 (1959).

[34] See, for example, the discussion in S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966).

tions of Figs. 1(a) and 1(b) are altered by the presence of Schwinger terms from those given in the text. Thus, the vector part of the tensor $T_{\mu\nu}$ defined in (A3) acquires,[35] in addition to the time-ordered product, an additional k-independent term related to the "sea-gull" graphs which appear in the Compton amplitude. In particular, instead of (A3), the part of $T_{\mu\nu}$ coming from the vector current becomes

$$V_{\mu\nu}(k,p',p)=i\int dx\, e^{-ik\cdot x}\langle p'|T(V_\mu(0)j_\nu(x))|p\rangle -\langle p'|D_{\mu,\nu}{}^{(+)}(0)|p\rangle, \quad \text{(D2)}$$

where $D_{\mu,\nu}{}^{(+)}$ is essentially the variational derivative of V_μ with respect to the electromagnetic field $\mathfrak{A}_\nu$ (keeping constant all other coordinate fields and all canonical conjugate fields),

$$\delta V_\mu(x)/\delta\mathfrak{A}_\nu(y)=D_{\mu,\nu}{}^{(+)}(x)\delta^4(x-y). \quad \text{(D3)}$$

Using a model theory with a triplet of spin-zero fields as a guide, we assume that the $D_{\mu,\nu}{}^{(+)}$ are related to the Schwinger term $S^{(+)}$ according to

$$D_{\mu,\nu}{}^{(+)}(x)=i(\delta_{\mu\nu}-\delta_{\mu4}\delta_{\nu4})S^{(+)}(x). \quad \text{(D4)}$$

From (D1), (D2), and (D4) it is easy to check that the vector part of the relations (A4)–(A7) hold for the modified tensor $V_{\mu\nu}$.

The third-rank tensor in (A8) also acquires additional terms when there are operator Schwinger terms in the theory. For the part coming from the vector current, the analog of (A8) becomes

$$\begin{aligned} V_{\lambda\mu\nu}(k,q,p',p) &= \int dxdy\, e^{-iq\cdot x+ik\cdot y}\langle p'|T(V_\lambda(x)j_\mu(y)j_\nu(0))|p\rangle \\ &+i\int dx\, e^{-iq\cdot x}\langle p'|T(V_\lambda(x)D_{\mu,\nu}(0))_+|p\rangle \\ &+i\int dy\, e^{ik\cdot y}\langle p'|T(D_{\lambda,\mu}{}^{(+)}(0)j_\mu(y))|p\rangle+i\int dx \\ &\times e^{-i(q-k)\cdot x}\langle p'|T(D_{\lambda,\mu}{}^{(+)}(x)j_\nu(0))|p\rangle-\delta M_{\lambda\mu\nu}. \end{aligned} \quad \text{(D5)}$$

[35] L. Brown, Phys. Rev. **150**, 1338 (1966).

Here $D_{\mu,\nu}$ is the same as $D_{\mu,\nu}{}^{(+)}$, but with the isospin current replaced by the electric current, and the $\delta M_{\lambda\mu\nu}$ is the same as the explicit pole term subtracted off in (A8) (with t_λ replaced by V_λ).

From (D2), (D4), and (D5), it is easy to derive that the vector part of (A11) is satisfied for the modfied tensor $V_{\lambda\mu\nu}$. Further, by performing the manipulations associated with Eqs. (A12)–(A14), the relation (A17) can be obtained with $V_{\mu\nu}$ and $V_{\lambda\mu\nu}$ given by (D2) and (D5).

These observations allow us to conclude that the results obtained in the text are unaffected by the presence of operator Schwinger terms with the exception of the finite correction arising from the derivative with respect to Λ^2 in Eq. (4.6). That is, $V_{\mu\mu}$ as given by (D2) clearly does not go according to Eq. (5.1) for large k_0. In fact, the leading term in this limit is a constant, and in the next order it goes as k_0^{-1} but with a coefficient that cannot be expected to be that given by (5.1). For example, in a theory involving only pions and photons, to zero order in the strong interactions the $V_{\mu\nu}$ appropriate to the decay $\pi^-\to\pi^0 e\bar{\nu}$ is

$$V_{\mu\nu}{}^\pi=-\sqrt{2}\left[\frac{(p+p'-k)_\mu(2p-k)_\nu}{k^2-2p\cdot k}-\delta_{\mu\nu}\right] \quad \text{(D6)}$$

and

$$V_{\mu\mu}{}^\pi \xrightarrow[k_0\to\infty]{} 3\sqrt{2}+(1/k_0)\langle p'|V_0|p\rangle. \quad \text{(D7)}$$

The difference between the second term of (D7) and the expression in Eq. (5.1) has the effect of replacing the 11/8 appearing in the first term of Eq. (5.7) by 1. This change would not alter appreciably the conclusions in Sec. VIII concerning the comparison of our results with the predictions of universality. However, given that operator Schwinger terms exist, there is of course no compelling reason why their numerical contribution can be estimated by looking at a simple model.

Finally, we mention that the constant term on the right side of (D7) will contribute additional logarithmic divergences in the two integrals involving $V_{\mu\nu}$ in Eq. (5.7). However, the coefficient of the logarithm is of order $\alpha|e|=O(\alpha^2)$, so that such terms can be consistently ignored within our approximation of keeping corrections only to order α.