Are There Equal-Amplitude Points on the Backward Cone?

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We search for at least three energies such that the invariant amplitudes A and B and their relative phase on the backward cone are the same for π^+p scattering. Certain relations between the energies and differential cross sections are obtained which have to be satisfied if such energies exist.

THE differential cross section for π^+p scattering is given in terms of the invariant amplitudes A and B as

$$\frac{d\sigma}{d\Omega}=\frac{4}{64\pi^2 s}(M^2-\tfrac{1}{4}t)|A|^2+\frac{2M}{64\pi^2 s}(s-M^2-1+\tfrac{1}{2}t)$$
$$\times(A^*B+B^*A)+\frac{1}{64\pi^2 s}[(s-M^2-1)^2+t(s-M^2)]|B|^2,$$

where M is the proton mass and $m_\pi=1$ units are used. This equation takes the following form on the backward cone defined by $u=0$ (using $s+t+u=2M^2+2$):

$$\frac{d\sigma}{d\Omega}=\frac{2M^2-2+s}{64\pi^2 s}|A|^2+\frac{M}{64\pi^2}2|A||B|\cos(\alpha-\beta)+\frac{1+sM^2-M^4}{64\pi^2 s}|B|^2. \quad (1)$$

Thus the cross section is given in terms of the magnitudes of A and B, and the relative phase angle between them. Here $A=|A|e^{i\alpha}$ and $B=|B|e^{i\beta}$.

Experimentally, the differential cross section alone is not sufficient to determine A, B, and $\cos(\alpha-\beta)$. One needs polarization experiments to measure these quantities.

In this paper we ask the following question: Are there points on the backward cone at which the A and B amplitudes have the same magnitudes and the same relative phase? That is, are there different energies $s_1, s_2, s_3, \cdots$, at which

$$|A|_{s_1}=|A|_{s_2}=|A|_{s_3}\cdots,$$
$$|B|_{s_1}=|B|_{s_2}=|B|_{s_3}\cdots,$$
$$\cos(\alpha-\beta)_{s_1}=\cos(\alpha-\beta)_{s_2}=\cos(\alpha-\beta)_{s_3}\cdots.$$

The motivation for this question comes from the following observations.

First, if we look at the complex diagram of the forward scattering amplitude we notice that there are points at which the amplitude crosses itself. That is, it has the same magnitude and phase at two different energies. Actually there is more than one such loop, the magnitude turning around and approaching the first loop again. In the case of the forward scattering amplitude a single linear combination of A and B amplitudes determines the differential cross section. But still the phase is relevant since the imaginary part of this combination determines the total cross section.

On the other hand, if one tries to express the direct channel amplitudes in terms of the crossed u-channel amplitudes, applying the Watson-Sommerfeld transformation to these, they become functions of u and $\cos\theta_u$. But these variables are constants on the backward cone. This makes one suspect that the amplitudes themselves may be constant there, the energy dependence coming from the kinematical multiplicative factors alone. It turns out that this is not the case, the transformation being singular between s,u, and $u,\cos\theta_u$. But beyond the direct channel resonances the behavior of the cross section seems to be in agreement with the expected one. Thus even though the cross section is falling, there might be energies at which the amplitudes are equal.

The number of the unknowns being three, if there were three such points they would be solutions of Eq. (1) for these energies.

Let us call

$$|A|^2=x,$$
$$AB^*+BA^*=2|A|B|\cos(\alpha-\beta)=y,$$

and

$$|B|^2=z.$$

If we try to solve for x, y, and z in terms of the three energies s_1, s_2, and s_3 we find the interesting result that the determinant of these three equations vanishes[1]:

$$\begin{vmatrix}(2M^2-2+s_1)/s_1 & M & (1+s_1M^2-M^4)/s_1\\(2M^2-2+s_2)/s_2 & M & (1+s_2M^2-M^4)/s_2\\(2M^2-2+s_3)/s_3 & M & (1+s_3M^2-M^4)/s_3\end{vmatrix}=0. \quad (2)$$

This just means that the third equation is not linearly independent of the first two and can be written as a linear combination of them. For brevity we write the equations as

$$a_1x+b_1y+c_1z=d_1,$$
$$a_2x+b_2y+c_2z=d_2,$$
$$a_3x+b_3y+c_3z=d_3.$$

Here a_i, b_i, and c_i are the elements of the matrix (2) and d_1, d_2, d_3 are the cross sections multiplied by $64\pi^2$.

[1] We would like to point out that this determinant does not vanish for the forward direction $t=0$ nor for any $t=$const. line. On the other hand it vanishes not only for the backward cone $u=0$ but for all lines $u=$const.

Then we must have

$$a_1\lambda+a_2\mu=a_3\,,$$
$$b_1\lambda+b_2\mu=b_3\,,$$
$$c_1\lambda+c_2\mu=c_3\,,$$

and

$$d_1\lambda+d_2\mu=d_3\,,$$

where λ and μ are proportionality constants which can be determined as follows.

Since $b_1=b_2=b_3=M$, the second equation gives

$$\lambda+\mu=1\,.$$

From the first or third equation together with the second, we find

$$\lambda=s_1(s_3-s_2)/s_3(s_1-s_2) \quad\text{and}\quad \mu=s_2(s_1-s_3)/s_3(s_1-s_2)\,.$$

Hence the cross sections should satisfy the relations

$$d_1\frac{s_1(s_3-s_2)}{s_3(s_1-s_2)}+d_2\frac{s_2(s_1-s_3)}{s_3(s_1-s_2)}=d_3$$

or

$$d_3s_3(s_1-s_2)+d_1s_1(s_2-s_3)+d_2s_2(s_3-s_1)=0\,,$$

which can be written as follows:

$$\det\begin{vmatrix} d_3s_3 & d_2s_2 & d_1s_1 \\ 1 & 1 & 1 \\ s_3 & s_2 & s_1 \end{vmatrix}=0\,. \quad (3)$$

This condition is satisfied if $s_1=s_2=s_3$ for which case obviously $d_1=d_2=d_3$. It is also satisfied if $d_1=d_2=d_3$ regardless what s_1, s_2, and s_3 are as long as $d_1=d_2=d_3$ at these energies. But this does not give us any information about s_1, s_2, and s_3. Finally there is also the possibility that

$$d_1s_1=d_2s_2=d_3s_3=\text{const.} \quad (4)$$

$s(d\sigma/d\Omega)=\text{const}$ is a hyperbola in the plane s versus $d\sigma/d\Omega$. If we look at the experimental differential cross-section[2] curve we see that at least in the known energy region the value of this constant must be smaller than approximately 200 000 $\mu b m_\pi^2/\text{sr}$.

The vanishing of the determinant means that in general there will not be solutions x, y, and z to our equations, unless d_1, d_2, and d_3 satisfy the conditions we just discussed. On the other hand, the vanishing of the determinant is also the condition for the existence of nonzero solutions to the homogeneous equations. This means that there are solutions x, y, and z at three different energies s_1, s_2, and s_3 such that the differential cross section is zero. Whether these solutions are physical is a point we must investigate. The experimental data[2] indicate that in the known energy region there are at least two points ($s_1\cong125m_\pi^2$), ($s_2\cong250m_\pi^2$) at which the cross section seems to be coming close to zero. Let us write the set of homogeneous equations in the matrix form $\mathbf{L}\bar{v}=0$, where $\det\mathbf{L}=0$:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0\,.$$

By solving this, we find

$$x/z=\tfrac{1}{2}(M^2+1) \quad\text{and}\quad y/z=-(3M^2+1)/2M$$

so that

$$\bar{v}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=z\begin{bmatrix} \tfrac{1}{2}(M^2+1) \\ -(3M^2+1)/2M \\ 1 \end{bmatrix}.$$

From this it is also seen that[3] $|A|/|B|=[\tfrac{1}{2}(M^2+1)]^{1/2}$ and

$$\cos(\alpha-\beta)=-\frac{3M^2+1}{4M}\left(\frac{2}{M^2+1}\right)^{1/2}=-1.06\,.$$

This shows that the solutions of the homogeneous equations are not physical. There are no three points on the backward cone where the amplitudes are the same and the cross sections are zero.

The inhomogeneous equations can have a solution only if the vector (d_1,d_2,d_3) is orthogonal to every solution of the homogeneous equation. This gives us the condition

$$d_1(M^2+1)/2-d_2(3M^2+1)/2M+d_3=0$$

or

$$d_1M(M^2+1)-d_2(3M^2+1)+d_32M=0\,. \quad (5)$$

Polarization experiments should make it possible to verify Eqs. (3) and (5) if three such points exist.

It is interesting to note that the Eq. (4) is in agreement with the well known $1/s$ behavior of the backward differential cross section at high energies.

As a final note, we would like to add that one consequence of the relation (2) is that the differential cross section on the backward cone can not be parametrized by three parameters in the form (1), even if these parameters are allowed to be multiplied by some function of s, provided it is the same function for all three parameters.

[2] For a list of references on experimental data see I. A. Sakmar, Phys. Rev. **148**, 1408 (1966). More recent data are given in A. Ashmore, C. J. S. Damerell, W. R. Frisken, R. Rubinstein, J. Orear, D. P. Owen, F. C. Peterson, A. L. Read, D. G. Ryan, and D. H. White, Phys. Rev. Letters **19**, 460 (1967).

[3] The general formulas for an arbitrary but constant u are as follows:

$$\frac{x}{z}=\frac{1}{2}\left(\frac{M^4-2M^2u-1+u}{M^2-1+u}\right),$$

$$\frac{y}{z}=-\frac{3M^4-1-2M^2-u(2M^2-2+u)}{2M(M^2-1+u)},$$

$$\cos(\alpha-\beta)=-\frac{3M^4-1-2M^2-2M^2u+2u-u^2}{4M(M^2-1+u)}\times\left[\frac{2(M^2-1+u)}{M^4-1-u(2M^2-1)}\right]^{1/2}.$$