Rotational Invariance of Magnetic Monopoles†

ASHER PERES*
Department of Physics and Astronomy, University of Maryland, College Park, Maryland
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The Schwinger quantization relation for magnetic charges, $g=n\hbar c/e$ (n integer), is derived by purely group-theoretic methods, from the combined requirements of rotational invariance and gauge invariance. No mention is made of the controversial "singular strings" of magnetic monopoles.

IT was shown long ago by Dirac[1] that if magnetic monopoles exist, their magnetic charge g is quantized and satisfies[2] $g=n/2e$, where n is an integer and e is the fundamental electric charge. This result was later sharpened by Schwinger[3] who found $eg=n$. Roughly, the arguments of Dirac and Schwinger were that any solution for the vector potential **A** around the monopole would contain one (or more) singular lines, thereby violating the rotational symmetry of the system. However, rotational symmetry could be restored if $g=n/2e$ (or n/e), because in that case different orientations of the singular line (or lines) could be related by innocuous gauge transformations.

Recently, the nature of these singular lines has become rather controversial,[4] and it may be worthwhile to show that the Schwinger relation $eg=n$ can be derived directly from the combined requirements of rotational invariance and gauge invariance, without ever mentioning singular lines.

To illustrate our method, let us consider first the simpler problem of a uniform magnetic field **B**, and let us construct the generators **P** of the magnetic translation group[5] for a particle of charge e. We cannot take simply $\mathbf{P}=\mathbf{p}\equiv -i\nabla$, because this is not gauge invariant. On the other hand, $\mathbf{P}'=\mathbf{p}-e\mathbf{A}$, which is gauge invariant, is also unacceptable because it leads to $[P_m',P_n']=-ie\epsilon_{mns}B_s$, while translation generators must satisfy

$$[P_m,P_n]=0 \qquad (1)$$

for any system endowed with translational symmetry. The correct solution is

$$\mathbf{P}=\mathbf{p}-e\mathbf{A}-\tfrac{1}{2}\mathbf{r}\times\mathbf{B}\,,$$

which satisfies (1) and is gauge invariant.

Likewise, let us consider the magnetic field $\mathbf{B}=g\mathbf{r}/r^3$ due to a magnetic monopole, and let us construct the generators **J** of the "magnetic rotation group" for a particle of charge e. We cannot take simply $\mathbf{J}=\mathbf{r}\times\mathbf{p}$, because this is not gauge invariant. On the other hand, $\mathbf{J}'=\mathbf{r}\times(\mathbf{p}-e\mathbf{A})$, which is gauge invariant, is also unacceptable, because it leads to[6]

$$[J_m',J_n']=i\epsilon_{mns}(J_s'-er_sr_kB_k)\,,$$
$$=i\epsilon_{mns}(J_s'-egn_s)\,,$$

(where $\mathbf{n}=\mathbf{r}/r$), while rotation generators must satisfy

$$[J_m,J_n]=i\epsilon_{mns}J_s \qquad (2)$$

for any system endowed with rotational symmetry. The correct solution is

$$\mathbf{J}=\mathbf{r}\times(\mathbf{p}-e\mathbf{A})+eg\mathbf{n}\,, \qquad (3)$$

which satisfies (2) and is gauge invariant.

From (3), we have

$$eg=\mathbf{n}\cdot\mathbf{J}\,,$$

the eigenvalues of which are $0, \pm1, \pm2, \cdots$ as is easily seen by taking a representation where **n** and $\mathbf{n}\cdot\mathbf{J}$ are diagonal. This is just Schwinger's result for eg.[7]

Note that $\mathbf{J}^2$ cannot be diagonal if **n** is. The possible eigenvalues of $\mathbf{J}^2$ are $J(J+1)$, with $J\geqslant|eg|$. The fact that $J<|eg|$ is forbidden could also have been directly deduced from

$$J(J+1)=[\mathbf{r}\times(\mathbf{p}-e\mathbf{A})]^2+(eg)^2.$$

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948).

[2] We use natural units $\hbar=c=1$.

[3] J. Schwinger, Phys. Rev. **144**, 1087 (1966).

[4] A. Peres, Phys. Rev. Letters **18**, 50 (1967).

[5] J. Zak, Phys. Rev. **134**, A1602 (1964).

[6] The reader may be worried by the simultaneous use of $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$ and $\mathbf{B}=-g\nabla(r^{-1})$. These equations are incompatible at $r=0$, or, in general, in any multiply connected domain surrounding the monopole (even if the monopole itself is excluded from that domain). The point is that these equations *are* compatible in any simply connected domain where $\mathrm{div}\mathbf{B}=0$. Such a domain is sufficient to obtain the algebra of the generators of *infinitesimal* rotations, Eq. (2) below, which is the only thing needed for our proof.

[7] Introducing spin does not affect this result. We then have $eg=\mathbf{n}\cdot\mathbf{J}-\mathbf{n}\cdot\mathbf{S}$, but since $\mathbf{n}\cdot\mathbf{J}$ and $\mathbf{n}\cdot\mathbf{S}$ are either both integers, or both half-odd-integers, eg is always an integer.