α_1, which may become equal to zero; and α_M, which may become equal to 1. We conclude that $M=N$ and we have the following two extrema corresponding to whether $\alpha_1=0$ or $\alpha_N=1$:

(1) Setting $\alpha_1=0$, we have

$$\psi(\lambda)=c\frac{(1-\lambda\beta_1)\cdots(1-\lambda\beta_{N-1})}{(1-\lambda\alpha_2)\cdots(1-\lambda\alpha_N)}. \tag{A6}$$

The extremum value for $\psi(\lambda_i)$ is determined by eliminating the $2N-1$ parameters c, β_1, β_2, $\cdots$, β_{N-1}, α_2, $\cdots$, α_N from the $2N$ equations

$$\psi(\lambda_j)=\psi_j,\quad j=1,2\cdots 2N.$$

This elimination can be done easily if we write the above expression (A6) for $\psi(\lambda)$ in the form

$$[1+h_1\lambda+h_2\lambda^2+\cdots+h_{N-1}\lambda_j^{N-1}]\psi(\lambda)$$
$$=c+k_1\lambda+k_2\lambda^2+\cdots+k_{N-1}\lambda^{N-1}.$$

Eliminating the new parameters h_r and k_r from the system of equations

$$[1+h_1\lambda_j+h_2\lambda_j^2+\cdots+h_{N-1}\lambda_j^{N-1}]\psi_j$$
$$=c+k_1\lambda_j+k_2\lambda_j^2+\cdots+k_{N-1}\lambda_j^{N-1},\quad j=1,2\cdots 2N$$

leads to

$$D_{2N}[\psi]=0\quad\text{for}\quad \psi_i=\text{extremum}.$$

(2) Setting $\alpha_N=1$, we have

$$(1-\lambda)\psi(\lambda)=c\frac{(1-\lambda\beta_1)\cdots(1-\lambda\beta_{N-1})}{(1-\lambda\alpha_1)\cdots(1-\lambda\alpha_{N-1})},$$

and a similar argument leads to

$$D_{2N}[(1-\lambda)\psi]=0\quad\text{for}\quad \psi_i=\text{extremum}.$$

All allowed values of $\psi(\lambda)$ at $\lambda=\lambda_i$ will lie between the two given by $D_{2N}[\psi]=0$ and $D_{2N}[(1-\lambda)\psi]=0$; therefore the set (ψ_1,λ_1), (ψ_2,λ_2), $\cdots$, (ψ_{2N},λ_{2N}) is admissible if and only if $D_{2N}[\psi]$ and $D_{2N}[(1-\lambda)\psi]$ have definite signs. These signs can be easily determined by induction on m.

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Fixed Poles and Compositeness

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Using a perturbative model, we study the asymptotic behavior of weak amplitudes, looking in particular for the existence of fixed poles. The weakly interacting particles are considered as either elementary or composite. Our conclusions can be summarized as follows: The existence of fixed poles is model-dependent. In particular: (1) If both interacting particles are elementary, we find the usual fixed poles at $J=\sigma_1+\sigma_2-n$, $n\geq 1$, where σ_1 and σ_2 are the spins of the particles. (2) If the weakly interacting particles are composite, the amplitude is superconvergent (even being nonunitary) and there are no fixed poles, at least for $J\geq 0$. (3) In the case of photoproduction of a spinless composite particle, i.e., an amplitude with only one elementary particle with spin one, there is no fixed pole at $J=0$. If we consider that the elementary particle has spin σ, we argue that the pole at $J=\sigma-1$ disappears, while the ones at $J=\sigma-n$ with $n>1$ may or may not exist, depending on the wave function of the produced hadron. We conclude by discussing the implications of our results, and in particular the limitations on the hypothesis of partially conserved axial-vector current.

I. INTRODUCTION

THE study of fixed singularities in the angular momentum plane (J plane) has been shown to be important for the understanding of the asymptotic behavior of scattering amplitudes for particles with spin.[1] Interest arose recently after the discovery of their connection with current algebra sum rules for finite momentum transfer.[2] As a consequence some results have been derived, especially in the domain of nonunitary amplitudes.[3,4]

In a fundamental series of papers Mandelstam has studied the compatibility of Regge behavior of amplitudes with presence of elementary spinning particles in the theory.[1] It is not surprising that these concepts are relevant for our purposes since elementary particles in general introduce subtractions in the dispersion relations. Under appropriate conditions, i.e., absence of bilinear unitarity, these behave as fixed singularities in the J plane.

The main purpose of this paper is to apply Mandelstam ideas to the study of weak amplitudes.

Our conclusions are extracted from perturbative models. However, it is very encouraging that these models are able to reproduce the main features of the J-plane singularity structure.[5,6] Even more, there is by

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[1] See, in particular, the general discussion in S. Mandelstam, Nuovo Cimento **30**, 1113 (1963).

[2] J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters **18**, 32 (1967); Phys. Rev. **157**, 1448 (1967).

[3] See Ref. 1 and H. D. I. Abarbanel, F. E. Low, I. J. Muzinich, S. Nussinov, and J. H. Schwarz, Phys. Rev. **160**, 1329 (1967).

[4] For a discussion of fixed poles in unitary amplitudes, see S. Mandelstam and L. Wang, Phys. Rev. **160**, 1490 (1967); C. E. Jones and V. L. Teplitz, *ibid.* **159**, 1271 (1967).

[5] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, New York, 1966). We follow closely the notation of this book.

[6] Equivalent models have already been used to study some features of the fixed poles in Refs. 2 and 3. See also Ref. 12.

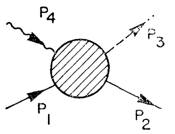

FIG. 1. General diagram contributing to the scattering og vector particle (wavy line) on a hadron (full line) leading to a scalar particle (dashed line) and a hadron.

now enough experience about these models as to know which are the diagrams needed in every case to generate the particular effect one is looking for.

Here, we are interested in fixed poles. First, we are concerned with the appearance of new terms in the asymptotic behavior as compared to the case when only Regge terms are present. Second, we are looking for the existence of singularities in the Regge residue functions and their eventual cancellation by means of killing factors.

These effects are indeed contained in the field-theoretical models presented here provided we include diagrams that contain an infinite number of ladders in the t channel. As nonplanar diagrams are not included in our calculations, all the complexities due to the third spectral function are not reproduced.[4]

Moreover, in this model we have exchange degeneracy for the trajectories. Hadrons will be supposed to interact through a ϕ^3-type Lagrangian while the weakly interacting particles will be considered to be either elementary or composite. In this last case the constituents will be taken always as spinless particles which need not be identified with the hadrons. All interactions, unless explicitly stated, are the most general ones consistent with Lorentz invariance and are trilinear in the fields of the particles.

Our conclusions can be summarized as follows: The existence of fixed poles is *model-dependent*. In particular:

(1) If both weakly interacting particles are elementary we find the usual fixed poles at $J=\sigma_1+\sigma_2-n$, $n\geq 1$,[1-3] where σ_1 and σ_2 are the spins of the particles. (2) If the weakly interacting particles are composite the amplitude is superconvergent (even being nonunitary) and there are no fixed poles at least for $J\geq 0$. (3) In the case of photoproduction of spinless composite particles, i.e. an amplitude with only one elementary particle with spin one, there is no fixed pole at $J=0$. If we consider that the elementary particle has spin σ, we argue that the pole at $J=\sigma-1$ disappears while the ones at $J=\sigma-n$ with $n>1$ may or may not exist depending on the wave function of the produced hadron.

We conclude by discussing the implications of our results and in particular the limitations on the hypothesis of partially conserved axial-vector current (PCAC).

II. DESCRIPTION OF THE METHOD AND RESULTS OF THE CALCULATIONS

In this section we describe the calculation and method used as well. We leave the detail for the Appendix.

We consider first the case in which the weakly interacting particles are spin 1 and spin 0. See Fig. 1.

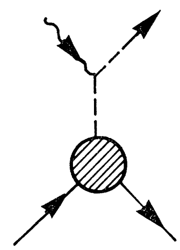

FIG. 2. Set of diagrams representing the exchange in the t channel of an elementary scalar particle.

The covariant scattering amplitude is:

$$T_\lambda=i\int e^{ik\cdot x}(\Box_x+m_1^2)(\Box_y+m_2^2)\times T\langle p_2|[V_\lambda(x)S(0)]|p_1\rangle d^4x\,, \quad (2.1)$$

where $V(x)$ and $S(x)$ are, respectively, the fields of the spin-1 particle and the zero one, p_1 and p_2 characterize the hadronic states, and m_1 and m_2 are the masses of the particles.[7]

Two different types of non-Regge singularities appear: (a) Kronecker-δ singularities stemming from graphs depicted in Fig. 2, and (b) fixed poles, which we will consider to be generated by the diagrams shown in Fig. 3.

Both kinds of singularities contribute to the asymptotic-behavior terms which are fixed powers in the energy in contradistinction to Regge poles. We make here a small digression to illustrate the relation of these singularities to the Dashen–Gell-Mann–Fubini sum rules. For this purpose we consider, for simplicity, that the spin-1 particle interacts with a conserved current. Current conservation implies

$$k^\lambda T_\lambda=0\,. \quad (2.2)$$

In the high-s limit the contribution from diagrams depicted in Figs. 2 and 3 cancel between themselves. Calling F_λ the amplitude corresponding to the diagram of Fig. 2 and defining $M_\lambda=T_\lambda-F_\lambda$, we get

$$k^\lambda M_\lambda=k^\lambda F_\lambda\,. \quad (2.3)$$

It has been shown[8] that this equation is identical to the one that appears in current algebra, $k^\lambda F_\lambda$ being inter-

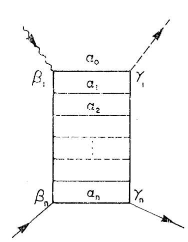

FIG. 3. Typical ladder diagram contributing to the asymtotic behavior.

[7] Our model can be definite by use of the following Lagrangian:
$$\mathcal{L}_{\rm int}=g(\bar K^+K^+-\bar K^0K^0)\sigma+f(\bar K^+\partial_\mu K^+)V^\mu+f'\partial_\mu(\bar K^+K^+)V^\mu+G\bar K^+K^0S+G\bar S\bar K^0K^++M(\bar S\overleftrightarrow{\partial}_\mu S)V^\mu.$$
$f'=0$ implies that the current $\partial\mathcal{L}/\partial V_\mu$ is conserved, K may be a scalar or pseudoscalar field, and σ must be scalar. The process we are considering is: $K^+ +$vector particle $(V^\mu)\to K^0+S$.

[8] S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966); and Ref. 12.

preted as the equal-time commutator. M_λ is identified with the current amplitude and is defined through the equation

$$M_\lambda = i\int e^{ik\cdot x} T\langle p_2|(\Box_x + m_1^2) \times V_\lambda(x)(\Box_y + m_2^2)S(0)|P_1\rangle d^4x. \quad (2.4)$$

Coming back to Eq. (2.1) we make the usual expansion of T_λ in invariant amplitudes:

$$T_\lambda = AP_\lambda + B\Delta_\lambda + Cp_{4\lambda}, \quad (2.5)$$

where

$$\begin{aligned} \Delta &= p_1 - p_2, \\ p &= p_1 + p_2, \\ t &= \Delta^2, \\ \nu &= p_4 \cdot p. \end{aligned} \quad (2.6)$$

To calculate the asymptotic behavior of A, B, and C we will compute the asymptotic behavior of the n ladder and then sum over all n (see Fig. 3). For this purpose we use the Mellin transform technique[9] proceeding through the following steps: We compute the Mellin transform of the contribution of each diagram. We extract the rightmost singularities which give the leading asymptotic behavior. Finally we sum formally these terms and analyze the singularities of the sum. In all steps we take advantage of the striking formal similarity between poles in complex angular momentum and those in the space of the Mellin transform.

As a result of this first part of the calculation we find a fixed pole at $J=0$ in T^λ which appears in a multiplicative way, such that it produces a singularity in the Regge residues of the moving poles.

Most of these results are contained in Refs. 2 and 3. What we want to stress is that in these calculations the weakly interacting particles are effectively treated as elementary ones. We want to study now how the compositeness affects the previous results.

As a first consequence, the Kronecker-δ terms disappear since, according to the Feynman rules, only lines corresponding to elementary fields must be included in the diagrams.

On the other hand, the diagrams of Fig. 3 become more complicated. To replace the external particle by a bound state we must consider the production amplitude shown in Fig. 4, and compute the asymptotic behavior of the term which has the pole at $(P_a+P_b)^2 = m_2^2$.

For that purpose we consider the class of diagrams shown in Fig. 4. Summing over n we try to reproduce the pole of the bound state, while summing over m we get the asymptotic behavior. In the Mellin transform method the calculation is strikingly symmetric in m and n because in both cases we are trying to reproduce a singularity of the amplitude.

The final result is that the asymptotic behavior of the residue at the pole corresponding to the A amplitude is

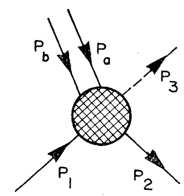

FIG. 4. Production amplitude in which the vector particle is seen as a bound state of two particles.

(see Appendix A35)

$$A(s,t) = [R(t,m_2^2)/\alpha_2'(m_2^2)]\Gamma(1-\alpha_1(t))(-s)^{\alpha_1(t)-1}, \quad (2.7)$$

where $\alpha_2(w)$ is the Regge trajectory corresponding to the external spinning particle, $\alpha_1(t)$ is the leading trajectory exchanged in the t channel, $R(t,m_2^2)/\alpha_2'(m_2^2)$ is the Regge residue and Γ is the usual gamma function whose poles at the negative integers reproduce the poles of the amplitude corresponding to physical one-particle states in the t channel. Notice that when $\alpha_1(t)$ goes through zero, V has no pole corresponding to a normal sense-nonsense transition.

The generalization to any integer spin σ is straightforward. In the decomposition in invariant amplitudes there will now be one of them, say A, which is the coefficient of $P_{\mu(1)}$, $P_{\mu(2)}\cdots P_{\mu(\sigma)}$. Repeating the arguments of the Appendix, its asymptotic behavior can be shown to be

$$A(s,t) \underset{s\to\infty}{=} R(t)\Gamma(\sigma-\alpha_1(t))(-s)^{\alpha_1(t)-\sigma}, \quad (2.8)$$

showing the absence of fixed poles and normal Regge behavior at the sense-nonsense points.

We next consider the case when the spinning particle is elementary while the spinless one is composite. For the physically important case of a spin-1 elementary particle (photoproduction of pions) we find the following asymptotic behavior of the A amplitude [see Appendix, (A47)]:

$$A(s,t) \underset{s\to\infty}{=} [\Gamma(1-\alpha_1(t))/\alpha_2'(m_2^2)](-s)^{\alpha_1(t)-1} \quad (2.9)$$

showing normal Regge behavior. But now the result depends strongly on the spin of the elementary particle: if the spin is σ then the first pole at $\sigma-1$ disappears when the spinless particle is composite, but that is not the case for the other poles at $\sigma-2$, $\sigma-3,\cdots$, etc. Nevertheless, the way these poles appear suggests that if the spinless particle was composed of three particles, then also the pole at $\sigma-2$ will disappear. So, if the particle constituents of the bound state are themselves considered to be composite (which will be in a certain way the translation of the bootstrap idea in our model), all the fixed poles will disappear.

III. CONCLUSIONS

We want to discuss here the possible consequences of our calculations. It has been thought[10] that, because

[9] J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963).

[10] See for instance, M. B. Halpern, Phys. Rev. **160**, 1441 (1967).

the photoproduction amplitude is not unitary, there could be a fixed pole at $J=0$. Our calculation shows that the criterion of nonunitarity is not sufficient for the existence of fixed poles. In fact, our model makes us believe that when the pion is a composite particle there is no fixed pole and the amplitude obeys Regge behavior.[11]

In the case of both weakly interacting particles being elementary we have found that, if we subtract the diagram of Fig. 2, then all invariant amplitudes obey unsubtracted dispersion relations. This result depends strongly on the model used as first stressed by Amati *et al.*[12] If we had accepted the existence of elementary particles among the hadrons, the conclusions would be completely different. This is the case of models discussed in Ref. 12 (elementary scalar particle in the t channel) and by Bardacki *et al.*[13] (elementary vector particle in the s channel) where non-Regge behavior in B and C appears. We can give another example in which this happens, namely, that of a target of spin-$\frac{1}{2}$ hadrons which have a nonminimal coupling to the vector current. It is interesting to notice that this non-Regge behavior in B and C demand subtractions in the dispersion relations while this is not the case for the A amplitude. These subtractions will appear as undetermined constants in the theory; so, it is plausible that only adding new parameters to the theory we should be able to generate non-Regge terms in the asymptotic limit of B and C.

Our conclusions concerning the absence of a fixed pole in photoproduction of pions imply limitations on the validity of the PCAC hypothesis because replacing the pion field by the divergence of the axial current introduces a fixed pole.[14]

Denoting by $F(t,s,p_3^2,p_4^2)$ the amplitude with the divergence of the current and by $F_\pi(t,s,p_3^2,p_4^2)$ the amplitudes for pion production, then

$$\partial_\mu A^\mu = c\phi_\pi \tag{3.1}$$

and

$$F(t,s,p_3^2,p_4^2)=c/(p_3^2-m_\pi^2)F_\pi(t,s,p_3^2,p_4^2)\,. \tag{3.2}$$

Developing F_π in powers of $p_3^2-m_\pi^2$, we find

$$F_\pi(t,s,p_3^2,p_4^2)=F_\pi^{(1)}(t,s,p_4^2)+F_\pi^{(2)}(t,s,p_4^2)(p_3^2-m_\pi^2)\,. \tag{3.3}$$

PCAC would tell us that the second term is negligible with respect to the first one near $p_3^2=0$. Our results imply that this is certainly wrong in the asymptotic region for the real part of the amplitude or for the imaginary part near a value of t for which the leading trajectory in the t channel passes by $J=0$. The reason is simply that, if the pion is composite F_π^1 will go to zero at this point because of the appearance of a killing factor. Nevertheless, we should keep in mind the possibility that the pion is elementary.

Note added in proof. R. Roskies has kindly pointed out to us that the multiplicative character of the fixed and Regge poles can be rigourously demonstrated. In (A8) the residue of the fixed pole at $\beta=-1$ can be explicitly evaluated. The integral over $d\alpha_0$ is then trivial and the result simply related to the original graphs with spinless particles only which of course possesses the Regge pole.

ACKNOWLEDGMENTS

We should like to acknowledge fruitful discussions with M. Ademollo, S. Adler, D. Amati, H. Dahmen, R. F. Dashen, B. Sakita, and K. C. Wali. Two of us (H. R. and M. A. V.) wish to thank the Niels Bohr Institute for its kind hospitality. Two of us (G. V. and M. A. V.) wish to express their gratitude to the Istituto di Fisica di Firenze for hospitality.

APPENDIX

In this Appendix we compute in detail the behavior of the relevant amplitudes discussed in the text.

(a) Scattering of an elementary particle that interacts weakly and carries spin 1 on a scalar particle producing a scalar weakly interacting particle and a strongly interacting one (Fig. 1).

As explained in Sec. I we are interested in adding the contributions of the diagrams associated with $n+1$ steps in the ladder (see Fig. 3).

The Feynman amplitude reads

$$M_\mu=(3n)!\int\prod_{i=1}^{n}d^4k_i\int_0^1\prod_{i=1}^{n}d\alpha_i d\beta_i d\gamma_i d\alpha_0 \times\delta(\textstyle\sum\alpha_i+\beta_i+\gamma_i+\alpha_0-1)(Mk_\mu^{(1)}+Np_\mu^{(4)})/D^{3n+1}, \tag{A1}$$

where the α's, β's, and γ's are the usual Feynman parameters. The expression $Mk_\mu^{(1)}+Np_\mu^{(4)}$ is the most general vertex for the spin-1 coupling.

The denominator reads:

$$\begin{aligned}D&=\sum_1^n[\gamma_ik_i^2+\beta_i(p_4-p_3+k_i)^2]+\sum_1^{n-1}\alpha_i(k_{i+1}-k_i)^2\\&\qquad+\alpha_0(k_1+p_4)^2+\alpha_n(p-k_n)^2-m^2\\&=\mathbf{k}^T\mathbf{A}\mathbf{k}-2\mathbf{k}^T\mathbf{B}\mathbf{p}+\mathbf{p}^T\mathbf{\Gamma}\mathbf{p}-m^2.\end{aligned} \tag{A2}$$

We are following the notation of Ref. 5. We perform the standard displacement $\mathbf{k}'=\mathbf{k}-\mathbf{A}^{-1}\mathbf{B}\mathbf{p}$ so that the nu-

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[11] In our calculation we study the production of a scalar weakly interacting particle. Nevertheless, (a) in Fig. 5 the external composite particle may be identified with the exchanged hadron (the class of diagrams considered is gauge invariant); (b) minor changes in the model allows us to extend our conclusions to the process $\gamma+$scalar charged particle $\rightarrow$ charged pseudoscalar particle $+$pseudoscalar neutral particle. Assuming that the nature of the target is unimportant in the mechanism of the generation of the fixed pole, as suggested by the model, since the relevant line is the one parametrized by α_0 (see Fig. 6 and Appendix), we conclude that our results apply to pions as well.

[12] D. Amati and R. Jengo, Phys. Letters **24B**, 108 (1967).

[13] K. Bardacki, M. B. Halpern, and G. Segré, Phys. Rev. **158**, 1544 (1967).

[14] The residue of the fixed pole does not vanish since the commutator $[j_0^{\text{el}},\partial_\mu A_\mu^+]\neq0$ as a consequence of $[\int j_0^{\text{el}}d^3(x),A_\mu^+]\neq0$.

merator reads, throwing away odd terms in k': The $\mathbf{k}$ are column and row vectors and the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{\Gamma}$ are of dimensions $n\times n$, $n\times 3$, and 3×3, respectively.

$$MC^{-1}(x)\prod_{1}^{n}\alpha_i p_\mu{}^{(1)}+F(x)p_\mu{}^{(3)}+G(x)p_\mu{}^{(4)}, \quad \text{(A3)}$$

where x is the set $(\alpha_i,\beta_i,\gamma_i)$, $C(x)=\det(A)$, and the two functions satisfy the condition

$$\begin{aligned}G(0,\beta_i,\gamma_i)&\neq 0,\\ F(0,\beta_i,\gamma_i)&\neq 0,\end{aligned} \quad \text{(A4)}$$

as can be shown explicitly by direct evaluation.

The scattering amplitude satisfies the invariant decomposition described in Sec. I. As it turns out, the only function that gets the contribution of the fixed pole in this model is A. This can be easily shown and from now on we forget about the other invariant functions.

The n-fold k integration can be performed to yield

$$A(s,t)=n!M\int dx\,\delta\Big(\sum_{1}^{n}\alpha_i+\beta_i+\gamma_i+\alpha_0-1\Big)\prod_{1}^{n}\alpha_i \times\frac{[C(x)]^{n+1-3}}{[\mathbf{p}^T(\mathbf{\Gamma}C-\mathbf{B}^T X\mathbf{B})\mathbf{p}-m^2C(x)]^{n+1}}, \quad \text{(A5)}$$

where $X=\mathbf{A}^{-1}C$.

We now write the new denominator as

$$D(x,s,t)=-g(\alpha)\sigma-J(x,s,t)C(x), \quad \text{(A6)}$$

with $\sigma=-s$ and $g(\alpha)=\prod_{0}^{n}\alpha_i$.

The Mellin transform in the variable σ of our expression may be easily computed using the standard methods for planar graphs and reads

$$L(\beta,t)=\Gamma(-\beta)(-1)^{n+1} \times\int_0^\infty dx\,M\prod_{1}^{n}\alpha_i[C(x)]^{-\beta-3}e^{-J(x,t)}\Big(\prod_{0}^{n}\alpha_i\Big)^{\beta}. \quad \text{(A7)}$$

By explicit calculation it can be proven that the function $e^{-J(x,t)}$ enjoys the following two properties: (a) It is vanishingly small for large values of x; and (b) it factorizes in β_i and γ_i when $\alpha_i=0$.

Since only end-point singularities will bring about the leading behavior of our integral, we are interested in the region $\alpha_i\cong 0$.

The explicit effect of the singularities can be made clear by means of partial integration. Formula (A7) reads

$$L(\beta,t)=\frac{-\Gamma(-\beta)}{\beta+1}(-1)^{n+1}\int_0^\infty dx\Big(\prod_{0}^{n}\alpha_i\Big)^{\beta+1}M \times\frac{\partial}{\partial\alpha_0}[C^{-3-\beta}e^{-J(x,t)}]. \quad \text{(A8)}$$

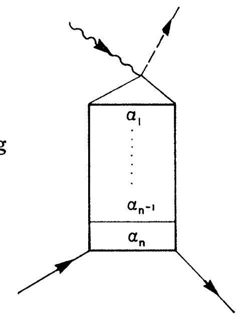

FIG. 5. Diagram leading to the fixed pole.

The singularity at $\beta=-1$ is the fixed pole. In terms of the language of contracting parameters it amounts to short circuit α_0 to bring the Fig. 3 to the form given in Fig. 5. This is nothing more than the result of Bronzan *et al.* The moving singularity can be made explicit too. By further partial integrations we obtain

$$\begin{aligned}L(\beta,t)&=\frac{-\Gamma(-\beta)}{\beta+1}\frac{M}{(\beta+2)^{n+1}}\int dx \times\Big(\prod\alpha_i\Big)^{\beta+2}\frac{\partial^{n+2}}{\partial^2\alpha_0\partial\alpha_1\cdots\partial\alpha_n}[C^{-3-\beta}e^{-J(x,t)}]\\ &\underset{\beta\approx-2}{\cong}\frac{\Gamma(-\beta)}{\beta+1}\frac{-M}{(\beta+2)^{n+1}}\int d\beta_i d\gamma_i \times\Big\{\frac{\partial}{\partial\alpha_0}\Big[\prod_{2}^{n}(\beta_i+\gamma_i)^{-1}(\alpha_0+\beta_1+\gamma_1)^{-1}e^{-J}\Big]_{\alpha_0\cdots\alpha_n=0}\Big\}. \end{aligned}\quad \text{(A9)}$$

We can now sum the set of diagrams

$$L(\beta,t)\approx\frac{\Gamma(-\beta)}{\beta+1}\frac{1}{\beta+2-F(t)}G(t), \quad \text{(A10)}$$

where we have used explicitly the fact that the integrals factorize. The functions $G(t)$ and $F(t)$ are some integrals that can be explicitly calculated.

Introducing the trajectory function $\alpha(t)=-1+F(t)$ and performing the inverse transform one gets

$$A(s,t)=\frac{1}{2\pi i}\int_C\frac{\Gamma(-\beta)}{\beta+1}\frac{G(t)}{\beta+1-\alpha(t)}s^\beta d\beta, \quad \text{(A11)}$$

where C is a parallel to the imaginary axis between $\mathrm{Re}\beta=0$ and $\mathrm{Re}\beta=-1$.

The contribution from the moving pole comes from the region $\beta+1-\alpha(t)\cong 0$. So its contribution to the amplitude is

$$A_{Mp}(s,t)=[\Gamma(1-\alpha(t))/\alpha(t)]s^{\alpha(t)-1}. \quad \text{(A12)}$$

However, when $\alpha(t)\cong 0$ we are trapped, and both poles contribute so that the total amplitude now reads

$$A(s,t)\cong\frac{\Gamma(1)G(t)s^{-1}}{-\alpha(t)}+G(t)\frac{\Gamma(1-\alpha(t))}{\alpha(t)}s^{\alpha(t)-1}. \quad \text{(A13)}$$

The first term is the fixed-pole contribution and the second the Regge pole. At $\alpha(t_0)=0$ the two contributions

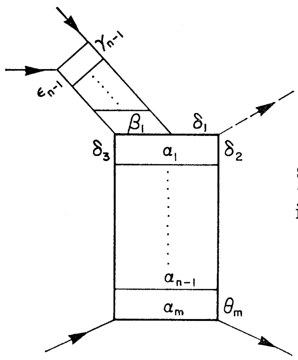

FIG. 6. Ladder diagrams assumed to represent the asymptotic behavior of process depicted in Fig. 4.

cancel each other. The multiplicative nature of the fixed pole is explicitly demonstrated in this model.

(b) Scattering of one spin-1 particle on a scalar hadron producing a spin-1 particle and a hadron.

The calculation is identical except for a different numerator function. The result for the corresponding amplitude $AP_\mu P_\nu$ is

$$\frac{\Gamma(-\beta)}{(\beta+1)(\beta+2)}\frac{G(t)}{\beta+2-\alpha(t)}. \qquad \text{(A14)}$$

There are two fixed poles now.

(c) Scattering of a composite spin-1 particle producing a scalar particle on a scalar hadron. The family of diagrams considered is shown in Fig. 6.

We introduce the following set of independent variables for our production amplitudes (see Fig. 4):

$$s=(p_2+p_3)^2, \quad t=(p_1-p_2)^2, \quad s_1=(p_a+p_1)^2, \qquad \text{(A15)}$$
$$t_1=(p_b-p_3)^2, \quad w=(p_a+p_b)^2, \quad \text{where} \quad p_4=p_a+p_b.$$

Since we are interested in the region where the external ladder is dominated by the pole of its composite state we write our amplitude as

$$F(t,t_1,s_1,s,\omega) \underset{w\to m_2^2}{\longrightarrow} (p_a-p_b)_\mu$$
$$\times[P_\mu A+\Delta_\mu B+p_\mu{}^{(4)}C]\,(1/w-m_2^2), \qquad \text{(A16)}$$

where A, B, C are the same we have in Sec. I. Performing the algebra we pick the function A in the following way:

$$F(t,t_1,s_1,s,\omega)=[(2s_1-s-t_1-\tfrac{3}{2}m^2+\tfrac{1}{2}t+\tfrac{1}{2}m_2^2)A$$
$$-(\tfrac{1}{2}t-t_1-m_2^2+\tfrac{3}{2}m^2)B]\frac{1}{w-m_2^2}, \quad A=\frac{1}{2}\frac{\partial F}{\partial s_1}. \qquad \text{(A17)}$$

B can be computed by $\frac{1}{2}\partial F/\partial s_1+\partial F/\partial t_1=+B$. C cannot be reached this way. The diagrams can be calculated as in the previous example:

$$F(t,t_1,s,s_1,w)=-\pi^2(m+n)!\left(\frac{-g^2}{16\pi^2}\right)^m\left(\frac{-g'^2}{16\pi^2}\right)^{n-1}G^2G'$$
$$\times\int_0^1\prod dx\frac{[C(x)]^{m+n-1}\delta(\sum x-1)}{[D(x,t,s,s_1,t_1,w)]^{m+n+1}}. \qquad \text{(A18)}$$

Here we denote by g the coupling constant (strong) between the hadrons, by g' the coupling constant corresponding to the binding force of the composite external particle, by G the weak coupling constant between the spin-1 particle and the hadrons, and by G' the weak coupling constant between the scalar particle and the hadron.

We make now double Mellin transform with respect to $-s$ and $-s_1$;

$$L(a,b)=\int_0^\infty ds ds_1 F(t,t_1,s,s_1,w)(-s)^{-b-1}(-s_1)^{-a-1} \qquad \text{(A19)}$$

$$F(t,t_1;\,s,s_1,w)$$
$$=\frac{1}{(2\pi i)^2}\int_{c_a c_b} L(a,b)(-s)^b(-s_1)^a da db, \qquad \text{(A20)}$$

with

$$L(a,b)=-\pi^2G'G^2\left(\frac{-g^2}{16\pi^2}\right)^m\left(\frac{-g'^2}{16\pi^2}\right)^{n-1}$$
$$\times(-1)^{m+n}\Gamma(-a)\Gamma(-b)\int_0^\infty dx[\alpha_1\cdots\alpha_m]^{a+b}$$
$$\times[\beta_1\cdots\beta_n]^a\Delta_1{}^bC^{-a-b-2}e^{-J(x,t,t_1,w)}, \qquad \text{(A21)}$$

where

$$D(t,t_1,s,s_1,w)=\alpha_1\cdots\alpha_m\beta_1\cdots\beta_n s_1$$
$$+\alpha_1\cdots\alpha_m\Delta_1(\beta,\gamma,\epsilon,\delta_1)s-J(x,t,t_1,w)C(x). \qquad \text{(A22)}$$

The only properties we need of Δ_1 are (a) that it does not vanish when one of the β's is 0, and (b) that when all β's vanish:

$$\Delta_1=\delta_1\prod_1^{n-1}(\beta_i+\epsilon_i). \qquad \text{(A23)}$$

It can be shown[15] that in the limit $s_1=s=-\infty$ and t_1 constant the production amplitude behaves like

$$(-s)^{\alpha(t)} \qquad \text{(A24)}$$

independently of the value of $\alpha(w)$, i.e., independently of the spin of the bound state. From (A17) we can deduce

$$A\sim s^{\alpha(t)-1}, \quad B\sim s^{\alpha(t)}, \qquad \text{(A25)}$$

so that we see already that there is no fixed-pole contribution to the asymptotic behavior in this case. Let us show how the killing factor appears explicitly in the Regge residue for the sense-nonsense transition.

By means of partial integration our expression reads:

$$L(a,b)=-\pi^2G'G^2\left(\frac{g^2}{16\pi^2}\right)^m\left(\frac{g'^2}{16\pi^2}\right)^{n-1}\frac{\Gamma(-a)\Gamma(-b)}{(a+b+1)^m(a+1)^n}$$
$$\times\int_0^\infty dx\,(\alpha_1\cdots\alpha_m)^{a+b+1}(\beta_1\cdots\beta_n)^{a+1}$$
$$\times\frac{\partial}{\partial\alpha_1\cdots\alpha_m\beta_1\cdots\beta_n}[\Delta_1{}^bC^{-2-a-b}e^{-J}]. \qquad \text{(A26)}$$

[15] The asymptotic behavior of production amplitudes have been studied in I. G. Halliday and J. C. Polkinghorne, Phys. Rev. **132**, 2741 (1963); J. C. Polkinghorne, Nuovo Cimento **36**, 857 (1965).

We extract the leading asymptotic behavior in Eq. (A26) by looking at the pole associated with $a+b$. On the other hand, to find the pole associated with the external particle we put $a+1\cong 0$.

Near the poles the integral factorizes as before, so that

$$L(a,b)\cong\Gamma(-a)\Gamma(-b)\frac{K_1(t)^{m-1}K_2(w)^{n-1}}{(a+b+1)^m(a+1)^n}R(t,w). \quad \text{(A27)}$$

Summing over n and m from 1 on, we obtain

$$L(a,b)\cong\frac{R(t,w)\Gamma(-a)\Gamma(-b)}{[a+b+1-K_1(t)][(a+1-K_2(w)]}, \quad \text{(A28)}$$

where in terms of the trajectory functions, $K_1(t)=\alpha_1(t)+1$ and $K_2(w)=\alpha_2(w)+1$. $R(t,w)$ is the residue function. Finally,

$$F(t,t_1,s_1,s,w)=-\frac{1}{(2\pi)^2}R(t,w) \times\int_{c_a c_b}\frac{\Gamma(-a)\Gamma(-b)(-s_1)^a(-s)^b da\,db}{[a+b-\alpha_1(t)][a-\alpha_2(w)]}. \quad \text{(A29)}$$

It is instructive to see how the pole in w appears. As w increases from $-\infty$ or the coupling is switched on, $\alpha_2(w)$ will start moving from -1, to the right. As the contour of integration is between -1 and 0 for $\mathrm{Re}(a)$, there will be a pinch when $\alpha_2(w_0)=0$, because it meets the pole of the function $\Gamma(-a)$. The same thing happens at any positive integer.

To calculate the contribution explicitly, we translate the contour to the left so as to isolate the moving pole

$$F(t,t_1,s,s_1,w)=\frac{1}{2\pi i}R(t,w) \times\int\frac{\Gamma(-b)\Gamma(-\alpha_2(w))}{[\alpha_2(w)-\alpha_1(t)+b]}(-s_1)^{\alpha_2(w)}(-s)^b db \quad \text{(A30)}$$

and

$$\lim_{w\to m_2^2}(w-m_2^2)F(t,t_1,s,s_1,w) =\frac{1}{2\pi i}\frac{R(t,m_2^2)}{\alpha_2'(m_2^2)}\int\frac{\Gamma(-b)}{[b+1-\alpha_1(t)]}s_1(-s)^b db. \quad \text{(A31)}$$

The method does not guarantee that we have picked the whole contribution from the pole in w but only the part with the highest power of s_1. As is known,[16] a single Regge pole produces more than one pole in the space of the Mellin transform. In our case there is a multiple pole near $a=-2$ in expression (A26) that will generate a moving pole

$$a(w)=-1+\alpha_2(w). \quad \text{(A32)}$$

[16] N. N. Khuri, Phys. Rev. **132**, 914 (1963).

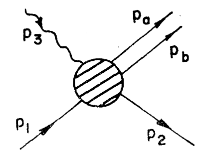

FIG. 7. Production amplitude in which the scalar particle is seen as a bound state of two particles.

So a will be zero near $w=m_2^2$, and

$$\lim_{w\to m_2^2}(w-m_2^2)F(t_1,t,s,s_1,w) =\frac{1}{2\pi i}\frac{R^{\rm II}(t,m^2)}{\alpha_2'(m_2^2)}\int\frac{\Gamma(-b)(-s)^b}{[b-\alpha_1(t)]}db. \quad \text{(A33)}$$

Nevertheless, as we are calculating that A amplitude that gets the contributions from the coefficient of the highest power in s_1, we simply use

$$A(s,t)=\frac{1}{2\pi i}\frac{R(t,m_2^2)}{\alpha_2'(m_2^2)}\int_{c_b}\frac{\Gamma(-b)(-s)^b}{[b+1-\alpha_1(t)]}db. \quad \text{(A34)}$$

This allows us to compute the leading asymptotic behavior in s, that is,

$$A(s,t)=\frac{R(t,m_2^2)}{\alpha_2'(m_2^2)}\Gamma(1-\alpha_1(t))(-s)^{\alpha_1(t)-1}. \quad \text{(A35)}$$

We see explicitly that $A(s,t)$ has no pole at $\alpha_1(t)=0$ corresponding to a sense-nonsense transition. In particular we have

$$\mathrm{Im}A(s,t)\underset{\alpha_1(t)\cong 0}{\cong}\frac{R(t,m_2^2)}{\alpha_2'(m_2^2)}2\pi\alpha_1(t) \times\Gamma(1-\alpha_1(t))(-s)^{\alpha_1(t)-1}, \quad \text{(A36)}$$

showing explicitly the killing factor.

(d) Photoproduction of a composite scalar particle by an elementary photon. The production amplitude is now linear in the photon polarization

$$\epsilon^\mu F_\mu(s,s_1,t,t_1,w). \quad \text{(A37)}$$

F_μ can be expressed in terms of invariant amplitudes, and we isolate the coefficient of P_μ that we call F. It corresponds to the helicity-flip amplitude. We refer to Fig. 7 for the kinematics.

As before, the changes affect the numerator of the Feynman amplitudes, and going through the same procedure, we find the following contribution to F:

$$-\pi^2\left(\frac{-g^2}{16\pi^2}\right)^m\left(\frac{-g'^2}{16\pi^2}\right)^{n-1}G'G^2(m+n)! \times\int_0^1\prod dx_i\left[\prod_1^m\alpha_i\frac{B}{C}\right]\frac{\delta(\sum x-1)[C(x)]^{m+n-1}}{[D(x,t,t_1,w,s,s_1)]^{m+n+1}}. \quad \text{(A38)}$$

As usual the only information needed about B is that it remains finite unless a whole set of parameters corresponding to a loop in the external ladder is made to vanish. Then

$$L(a,b)=G'(G\pi)^2\left(\frac{+g^2}{16\pi^2}\right)^m\left(\frac{+g'^2}{16\pi^2}\right)^{n-1}\Gamma(-a)\Gamma(-b)\times\int_0^\infty dx[\alpha_1\cdots\alpha_m]^{a+b+1}[\beta_1\cdots\beta_n]^a\times\Delta_1{}^bBC^{-a-b-3}\exp(-J). \quad (A39)$$

We make a change of variables

$$\beta_1=\alpha_0\bar\beta_1,\quad \delta_1=\alpha_0\bar\delta_1;\quad d\beta_1 d\delta_1=\alpha_0 d\alpha_0 d\bar\beta_1 d\bar\delta_1\delta(\bar\beta_1+\bar\delta_1-1). \quad (A40)$$

One can prove that

$$\Delta_1=\bar\beta_1\Delta_{12}+\delta_1\Delta_{11}. \quad (A41)$$

Hence,

$$L(a,b)=G'(G\pi)^2\left(\frac{g'^2}{16\pi^2}\right)^m\left(\frac{g^2}{16\pi^2}\right)^{n-1}\Gamma(-a)\Gamma(-b)\times\int_0^\infty d\bar x\,\delta(\bar\beta_1+\bar\delta_1-1)[\alpha_0\cdots\alpha_m]^{a+b+1}\times[\beta_1\cdots\beta_n]^a\bar\Delta_1{}^b\bar BC^{-3-a-b}e^{-J}. \quad (A42)$$

As before, we pick the poles at $a+b=-2$ and $a=-1$.

$$L(a,b)=G'(G\pi)^2\left(\frac{g^2}{16\pi^2}\right)^m\left(\frac{g'^2}{16\pi^2}\right)^{n-1}\times\frac{\Gamma(-a)\Gamma(-b)}{(a+1)^n(a+b+2)^{m+1}}\int_0^\infty d\bar x\times\delta(\bar\beta_1+\bar\delta_1-1)\bar\Delta_1\bar BC^{-1}e^{-J} \quad (A43)$$

$$=\Gamma(-a)\Gamma(-b)\frac{[K_2(w)]^n}{(a+1)^n}\frac{[K_1(t)]^m}{(a+b+2)^{m+1}}. \quad (A44)$$

Summing over m and n from $n=1$ and $m=0$,

$$L(a,b)=\Gamma(-a)\Gamma(-b)K_2(w)/[a+1-K_2(w)][a+b+2-K_1(t)], \quad (A45)$$

so that

$$F(t,t_1,s,s_1,w)=\frac{-K_1(w)}{(2\pi)^2}\int_{c_ac_b}\frac{\Gamma(-a)\Gamma(-b)(-s_1)^a(-s)^b da db}{[a+b+1-\alpha_1(t)][a-\alpha_2(w)]}. \quad (A46)$$

Near $w=m_2{}^2$, where $\alpha_2(w)=0$,

$$A(t,s,m_2{}^2)=\frac{1}{\alpha_2'(m_2{}^2)2\pi i}\int_{c_b}\frac{\Gamma(-b)(-s)^b}{b+1-\alpha_1(t)}db=\frac{\Gamma(1-\alpha_1(t))}{\alpha_2'(m_2{}^2)}(-s)^{\alpha_1(t)-1}, \quad (A47)$$

showing the absence of the pole at $\alpha_1(t)=0$, that is the existence of the killing factor.

It is interesting to see what happens when the elementary particle has spin greater than 1, say σ. In that case more terms of the form $\prod_1{}^n(\alpha_i)$ will appear in the numerator and the exponent will be $a+b+\sigma$. There will be a multiple pole of order $m-1$ at the point $a+b+\sigma=-1$. On the other hand, the variable α_0 will have an exponent $a+b+1$ independent of σ. This fact will produce a pole, of first order at $a+b=-2$, and when the summation is performed, a fixed pole at $b=-2$, since we put as before $\alpha_2(w)=0$. The asymptotic behavior will have now a non-Regge term of the form

$$F(t)(-s)^{-2} \quad (A48)$$

to be added to the moving-pole contribution

$$\frac{\Gamma(\sigma-\alpha_1(t))}{\alpha_2'(m_2{}^2)}\frac{1}{\sigma-2-\alpha_1(t)}(-s)^{\alpha_1(t)-\sigma}. \quad (A49)$$

In terms of J-plane singularities language we see that the fixed pole at $J=\sigma-1$ has disappeared but the ones at $J=\sigma-n$ are there for $n\geq 2$. Howevei, the calculation shows that their presence is linked to the parameters β_1 and δ_1 and we conjecture that introduction of diagrams with more and more particles forming the external spinless particle is going to push the poles to the left. It is amusing to think that for the case of a fully bootstrapped hadron the fixed poles of the type discussed in this paper all disappear.