Angular Dependence of Scattering at High Energies*

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Experimental evidence indicates that for nonforward (and nonbackward) scattering angles θ, differential scattering cross sections fall off exponentially with increasing barycentric energy $\sqrt{s}$; namely, $d\sigma/d\Omega \sim_{s\to\infty} e^{-\phi(s,\cos\theta)}$, where $\phi(s,\cos\theta)$ is a positive quantity increasing as some power of s. From assumptions of boundedness and analyticity in $\cos\theta$, we obtain certain constraints on the $\cos\theta$ dependence of the exponent function $\phi(s,\cos\theta)$.

THE restrictions on high-energy behavior of scattering amplitudes imposed by their presumed analyticity structure have been widely discussed, following the initial work of Froissart.[1] One supposes that for fixed barycentric energy $\sqrt{s}$, the amplitude is analytic in some domain in the $\cos\theta$ plane and that it has certain s-dependent boundedness properties in this domain. The customary assumption is that of uniform boundedness by a polynomial in s and the analyticity domain, depending on the application, is taken to be some portion of the full Mandelstam region—the whole $\cos\theta$ plane with the exceptions of the cuts (ρ,∞) and $(-\infty,-\rho)$, where $\rho\to 1+s_0/s$, $s\to\infty$.

For forward (backward) scattering, the experimental evidence suggests an asymptotic power-law dependence on the energy variable s. But for fixed, nonforward angles, what is strikingly indicated is a very rapid fall off with energy, of exponential form. More precisely, for nonforward angles differential scattering cross sections appear to behave like[2]

$$d\sigma/d\Omega \underset{s\to\infty}{\sim} e^{-\phi(s,z)}, \quad z\equiv\cos\theta\neq\pm 1 \tag{1}$$

where $\phi(s,z)$ is a positive quantity increasing as some power of s. Now it is intuitive that a function which is sufficiently smooth (analytic) in z, and polynomially bounded in s, cannot behave polynomially at the end points $z=\pm 1$ of the physical region and nevertheless fall arbitrarily fast over a range of neighboring points. Indeed, Cerulus and Martin[3] have shown that polynomial boundedness, taken together with certain z-plane analyticity assumptions, sets a bound on the rate of decrease with s, of the form

$$d\sigma/d\Omega > C\exp[-h(\theta)s^{1/2}\ln s].$$

It is therefore remarkable that recent evidence[4] on p-p scattering indicates an apparent violation of this bound. In the present paper we are concerned with a different aspect of the high-energy behavior for nonforward scattering; namely, the z dependence of the exponent $\phi(s,z)$ of Eq. (1), in contrast to the s dependence. We will allow some relaxation of the polynomial boundedness condition, but it will be necessary to introduce certain new assumptions about the zeroes in the z plane.

Let us consider a function $f(s,z)$, which we may take to be the full scattering amplitude in the case of spinless scattering or any one of the kinematic-singularity-free invariant amplitudes where spin is involved; or most generally, the spin-averaged differential cross section itself. The maximum domain of analyticity that we shall contemplate is the full z plane, with the cuts (ρ,∞) and $(-\infty,-\rho)$. It will be convenient to map this domain onto a unit circle. This is accomplished with the transformation

$$w=[(\rho+z)^{1/2}-(\rho-z)^{1/2}]/[(\rho+z)^{1/2}+(\rho-z)^{1/2}], \tag{2a}$$

where the square roots are taken positive in the gap, z real between $-\rho$ and $+\rho$. In the special case where $f(s,z)$ is identified as a differential cross section even in z (identical particle scattering), our results will be somewhat stronger if we exploit this symmetry by alternatively introducing

$$w=[\rho-(\rho^2-z^2)^{1/2}]/[\rho+(\rho^2-z^2)^{1/2}]. \tag{2b}$$

This maps the cut z^2 plane onto the unit circle.

We may now state our assumptions on $f(s,z)$ as follows:

(1) $f(s,z)$ is analytic in a domain D_R of the complex z plane defined by $|w|<R\leqslant 1$ [i.e., the image of D_R under the mapping of Eq. (2) is the interior of a circle of radius $R\leqslant 1$ centered at the origin of the w plane]. For $R<1$, it is clear that D_R is a *finite* region in the z plane and its boundaries, moreover, are at a nonzero distance from the Mandelstam cuts.

(2) $|f(s,z)|<e^{Q(s)}|1+(\rho^2-z^2)^{1/2}|^{M(s)}$.

(3) $f(s,z)$ has no zeros in D_R.

For $R=1$, assumption (1) is the often-invoked full cut plane analyticity hypothesis. We have introduced the parameter R to allow for the possibility of weakening it. The boundedness condition (2) we make more specific by assuming that the exponent $\phi(s,z)$ in Eq. (1) grows faster with s than both $Q(s)$ and $M(s)$ do. The usual assumption of polynomial boundedness cor-

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[1] M. Froissart, Phys. Rev. 123, 1053 (1961).
[2] G. Cocconi, V. T. Cocconi, A. D. Krisch, J. Orear, R. Rubinstein, D. B. Scarl, B. T. Ulrich, W. F. Baker, E. W. Jenkins, and A. L. Read, Phys. Rev. 138, B165 (1965); J. V. Allaby, G. Belletini, G. Cocconi, M. L. Good, A. N. Diddens, G. Mattiae, E. J. Sacharidis, A. Silverman, and A. M. Wetherell, Phys. Letters 23, 389 (1966); see also Ref. 7.
[3] F. Cerulus and A. Martin, Phys. Letters 8, 80 (1964).
[4] C. W. Akerlof, R. H. Hieber, A. D. Krisch, K. W. Edwards, L. G. Ratner, and K. Ruddick, Phys. Rev. Letters 17, 1105 (1966).

responds to $M=\text{const}$, $Q(s)=N\ln s$. Assumption (3) on the zeroes of $f(s,z)$ will be relaxed later on.

We begin our discussion by noting that the function

$$G(s,z)=-\ln f(s,z)+Q(s)+M(s)\ln(1+(\rho^2-z^2)^{1/2}) \quad (3)$$

is analytic in D_R and has a non-negative real part. By the Herglotz theorem,[5] $G(s,z)$ has a Poisson integral representation, which we conveniently express in terms of the w variable:

$$G(s,z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Re^{i\varphi}+w}{Re^{i\varphi}-w}d\sigma_s(\varphi)+i\mathrm{Im}G(s,0). \quad (4)$$

The integral is of the Stieltjes type and $d\sigma_s(\varphi)$ is a positive finite measure. In the remaining discussion we employ the representation of Eq. (4) for real values of w, between $-R$ and R, i.e., for real values of z in the domain D_R. More explicitly,

$$-\rho\frac{2R}{1+R^2}\leqslant z\leqslant\rho\frac{2R}{1+R^2} \quad (5a)$$

or

$$0\leqslant z\leqslant\rho\frac{2\sqrt{R}}{1+R}, \quad (5b)$$

depending on whether the mapping of Eq. (2a) or of Eq. (2b) is used. Taking the real part of Eq. (4) we obtain

$$\mathrm{Re}G(s,z)=-\ln|f(s,z)|+Q(s)+M(s)\ln|1+(\rho^2-z^2)^{1/2}|$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-w^2}{R^2-2Rw\cos\varphi+w^2}d\sigma_s(\varphi). \quad (6)$$

We take f to be the differential cross section of Eq. (1), so that $-\ln f=\phi(s,z)$. If $\phi(s,z)$ grows with s faster[6] than $Q(s)$ and $M(s)$, as we assume for $-1<z<1$, then

$$\phi(s,z)\simeq\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-w^2}{R^2-2wR\cos\varphi+w^2}d\sigma_s(\varphi);\qquad -1<z<1 \quad (6')$$

an equation which provides an integral representation for the experimentally measurable exponent $\phi(s,z)$. The positivity of the measure $d\sigma$ imposes stringent conditions on the z dependence of the exponent function. The restrictions imposed on $\phi(s,z)$ are most conveniently expressed in terms of auxiliary quantities which we now introduce (for brevity the s dependence is suppressed):

$$\lambda=4wR/(w+R)^2, \quad (7)$$

$$\psi(\lambda)=[(R+w)/(R-w)]\phi(s,z). \quad (8)$$

In terms of these quantities, Eq. (6′) reads

$$\psi(\lambda)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\sigma(\varphi)}{1-\lambda\cos^2(\frac{1}{2}\varphi)}. \quad (9)$$

From the positivity of the measure $d\sigma(\varphi)$ one now readily obtains the following derivative conditions on $\psi(\lambda)$:

$$(d^n/d\lambda^n)\psi(\lambda)\geqslant 0, \quad (10)$$

$$(d^n/d\lambda^n)1/\psi(\lambda)\leqslant 0, \quad (11)$$

$$(d^n/d\lambda^n)(1-\lambda)\psi(\lambda)\leqslant 0. \quad (12)$$

These derivative conditions are subject to direct experimental test. Suppose that $\psi(\lambda)$ is known at $n+1$ different points, $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$; then Eq. (10), for example, implies that

$$\sum_{i=1}^{n+1}\psi(\lambda_i)\prod_{j\neq i}(\lambda_i-\lambda_j)^{-1}\geqslant 0, \quad (10')$$

and of course similar relations are obtained from Eqs. (11) and (12).

The above expressions represent necessary conditions on the function $\psi(\lambda)$. The following question now arises naturally: Given a set of m positive numbers $\psi_1, \psi_2, \cdots, \psi_m$, what are the necessary *and sufficient* conditions for the existence of a function $\psi(\lambda)$ which has a representation of the form of Eq. (9) and which takes on the values $\psi_1, \psi_2, \cdots, \psi_m$ at the respective points $\lambda_1, \lambda_2, \cdots, \lambda_m$? We have obtained these necessary and sufficient conditions for $m=2, 3, 4$ by direct calculation (see Appendix for general formulas).

For $m=2$, taking $\lambda_2>\lambda_1$, we find

$$\psi_2\geqslant\psi_1, \quad (13)$$

$$(1-\lambda_2)\psi_2\leqslant(1-\lambda_1)\psi_1. \quad (14)$$

These conditions are already contained in Eqs. (10)–(12), which, for $n=1$, are sufficient as well as necessary.

For $m=3$, the conditions are that Eqs. (13) and (14) must be satisfied for one pair of points and in addition, with $\lambda_3>\lambda_2>\lambda_1$,

$$(\lambda_3-\lambda_2)/\psi_1+(\lambda_1-\lambda_3)/\psi_2+(\lambda_2-\lambda_1)/\psi_3\leqslant 0, \quad (15)$$

$$(\lambda_3-\lambda_2)(1-\lambda_1)\psi_1+(\lambda_1-\lambda_3)(1-\lambda_2)\psi_2+(\lambda_2-\lambda_1)(1-\lambda_3)\psi_3\leqslant 0. \quad (16)$$

Again, these results are already contained in Eqs. (10)–(12).

For $m=4$ our conditions, though of course consistent with Eqs. (10)–(12), are the stronger ones recorded below, with the convention $\lambda_4>\lambda_3>\lambda_2>\lambda_1$:

[5] See, for example, G. C. Evans, *The Logarithmic Potential* (American Mathematical Society Colloquium Publications, New York, 1927), Vol. VI, Chap. II; R. Nevanlinna, *Eindeutige Analytische Funktionen* (Springer-Verlag, Berlin, 1953), Chap. VII, p. 196.

[6] This would be true, for example, under the assumption of polynomial boundedness: $Q(s)\sim\ln s$, $M(s)=\text{const}$, if $\phi(s,z)$ grows like a power of s as indicated by experiment.

$$(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)(\psi_1\psi_2+\psi_3\psi_4)+(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)(\psi_2\psi_3+\psi_1\psi_4)+(\lambda_1-\lambda_3)(\lambda_4-\lambda_2)(\psi_1\psi_3+\psi_2\psi_4)\geqslant 0 \qquad (17)$$

and

$$(\lambda_2-\lambda_1)(\lambda_4-\lambda_3)[(1-\lambda_1)(1-\lambda_2)\psi_1\psi_2+(1-\lambda_3)(1-\lambda_4)\psi_3\psi_4] + (\lambda_3-\lambda_2)(\lambda_4-\lambda_1)[(1-\lambda_2)(1-\lambda_3)\psi_2\psi_3+(1-\lambda_1)(1-\lambda_4)\psi_1\psi_4] + (\lambda_1-\lambda_3)(\lambda_4-\lambda_2)[(1-\lambda_1)(1-\lambda_3)\psi_1\psi_3+(1-\lambda_2)(1-\lambda_4)\psi_2\psi_4]\leqslant 0. \qquad (18)$$

In addition the $m=3$ conditions must be satisfied for any one triplet of points.

To summarize, our basic assumptions lead to a variety of practical conditions on the exponent function $\phi(s,z)$ of Eq. (1), all of them stemming from Eq. (9). These conditions are most conveniently expressed in terms of the related function $\psi(\lambda)$, which is connected to $\phi(s,z)$ by Eqs. (2), (7), and (8). To give a concrete illustration in terms of the original variables, let us consider the case of identical particle scattering [Eq. (2b) then provides the relevant mapping]. For our illustrative purposes, suppose that we can choose $R=1$ in Eq. (4); this corresponds to full cut plane analyticity. Then, with $1>z_2>z_1>0$, Eqs. (13) and (14) are equivalent to

$$\phi(s,z_2)/(\rho^2-z_2^2)^{1/2}\geqslant\phi(s,z_1)/(\rho^2-z_1^2)^{1/2}, \qquad (13')$$

$$(\rho^2-z_2^2)^{1/2}\phi(s,z_2)\leqslant(\rho^2-z_1^2)^{1/2}\phi(s,z_1). \qquad (14')$$

In the above equations one can in effect set $\rho=1$ as long as $z_i<1$. These two-point conditions, and the analogous three-point conditions of Eqs. (15) and (16), taken, respectively, for all pairs and triplets of experimental points, should already in themselves provide a severe enough test of the basic assumptions under discussion here.

In our discussion so far, we have assumed that the function $f(s,z)$ has no zeroes in its domain D_R of analyticity in z. This is unreasonable. But if the number of zeroes is finite, and remains so even as $s\to\infty$, then we can factor out the zeroes:

$$f(s,z)=F(s,z)\prod_{i=1}^{N(s)}\frac{w-\alpha_i}{w\alpha_i^*-1},$$

where, in general, the zeroes α_i vary with s, as does their number N. As long as $N(s)$ remains bounded in the limit $s\to\infty$, all of our previous assumptions now apply to the zeroless function $F(s,z)$. But in the physical region of z, where $-\ln f=\phi$ is growing with s, $\ln F$ and $\ln f$ will approach each other in the limit $s\to\infty$, with the exception of possible isolated physical values of z which are zeroes of $f(s,z)$. Notice that our results would still hold if the number of zeroes $N(s)$ grows indefinitely with s, provided $N(s)/\phi(s,z)\to 0$ as s goes to infinity.

It is questionable whether the energies presently available in particle-collision experiments are high enough for nonforward differential cross sections to be in the asymptotic domain. Nevertheless, a test of our conditions on the basis of existing evidence would be most interesting. In the case of proton-proton scattering, the exponent function $\phi(s,z)$ appears to be well fitted by an expression of the form[7] $\phi=g(s)\sin\theta$, where $g(s)$ grows at least as fast as $\sqrt{s}$ and perhaps as fast as s. What is of particular interest here, however, is the $\sin\theta=(1-z^2)^{1/2}$ angular dependence suggested by experiment. This dependence is rather special, in the following sense. Consider the case, firstly, of identical particle scattering, and suppose the analyticity domain is so large that we can set $R=1$. (Little would change in our kinematics if R were less than unity by a small, finite amount, provided we restrict ourselves to physical values of z not too close to the end point $z=1$; and for the same reason, we can set $\rho=1$.) It follows from the three-point Eqs. (15) and (16) that if at any two points

$$\phi(s,z_1)/(1-z_1^2)^{1/2}=\phi(s,z_2)/(1-z_2^2)^{1/2}, \quad 0\leqslant z_1<z_2<1$$

then at all points in the physical region

$$\phi(s,z)=g(s)(1-z^2)^{1/2}=g(s)\sin\theta.$$

For scattering of nonidentical particles the same result holds, as we see from Eqs. (17) and (18), if for any three points ϕ is interpolated by the function $\sin\theta$.

The unique aspect of the z dependence embodied in $(1-z^2)^{1/2}$ can be brought out in another way (we again set $R=1$ and $\rho=1$). A function $f(s,z)$ which satisfies our assumptions, and which does not fall off exponentially with s along the z cuts, can have an exponential fall off for physical z only if it has the form $f\sim\exp[-g(s)\sin\theta]$. This follows immediately from the Poisson representation of Eq. (6′), since, under the above conditions, the dominant contributions for large s come from $\varphi=\pm\frac{1}{2}\pi$. This corresponds to $z=\infty$ under the mapping of Eq. (2a). For the case of identical particle scattering the dominant contribution comes from $\varphi=\pi$, again corresponding to $z=\infty$, under the mapping of Eq. (2b). In both cases we then obtain $\phi(s,\cos\theta)=g(s)\sin\theta$.

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We are grateful to Professor V. Bargmann for valuable advice and discussion.

[7] J. Orear, Phys. Rev. Letters **12**, 112 (1964); J. V. Allaby, G. Cocconi, A. N. Diddens, A. Klovning, G. Matthiae, E. J. Sacharidis, and A. M. Wetherell, Phys. Letters **25B**, 156 (1967).

APPENDIX

A set of m data, i.e., m pairs of numbers (ψ_1,λ_1), (ψ_2,λ_2), $\cdots$, (ψ_m,λ_m), where $\psi_i>0$ and $\lambda_i<1$, will be called *admissible* if there exists a function of the form

$$\psi(\lambda)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\sigma(\varphi)}{1-\lambda\cos^2(\frac{1}{2}\varphi)},$$

with $d\sigma(\varphi)$ a finite positive Stieltjes measure, such that

$$\psi(\lambda_1)=\psi_1,\quad \psi(\lambda_2)=\psi_2,\quad \cdots,\quad \psi(\lambda_m)=\psi_m.$$

The following are necessary and sufficient conditions. A set of m data (ψ_1,λ_1), $\cdots$, (ψ_m,λ_m) is admissible if and only if (1) one subset consisting of $m-1$ data is admissible *and* (2) the following 2 inequalities are satisfied:

For m even $(=2N)$

$$D_{2N}[\psi]\equiv(-1)^N\begin{vmatrix}\psi_1 & \lambda_1\psi_1 & \lambda_1^2\psi_1\cdots\lambda_1^{N-1}\psi_1 & 1 & \lambda_1 & \lambda_1^2\cdots\lambda_1^{N-1}\\ \psi_2 & \lambda_2\psi_2 & \lambda_2^2\psi_2\cdots\lambda_2^{N-1}\psi_2 & 1 & \lambda_2 & \lambda_2^2\cdots\lambda_2^{N-1}\\ \vdots & & \cdots & & & \vdots\\ \psi_{2N} & \lambda_{2N}\psi_{2N} & \cdots & & & \lambda_{2N}^{N-1}\end{vmatrix}\geqslant 0; \tag{A1}$$

and

$$D_{2N}[(1-\lambda)\psi]\leqslant 0; \tag{A2}$$

for m odd $(=2N+1)$

$$D_{2N+1}[\psi]\equiv(-1)^N\begin{vmatrix}\psi_1 & \lambda_1\psi_1 & \lambda_1^2\psi_1\cdots\lambda_1^{N}\psi_1 & 1 & \lambda_1 & \lambda_1^2\cdots\lambda_1^{N-1}\\ \psi_2 & \lambda_2\psi_2 & \lambda_2^2\psi_2\cdots\lambda_2^{N}\psi_2 & 1 & \lambda_2 & \lambda_2^2\cdots\lambda_2^{N-1}\\ \vdots & & & & & \\ \psi_{2N+1} & \lambda_{2N+1}\psi_{2N+1} & \cdots & & & \lambda_{2N+1}^{N-1}\end{vmatrix}\geqslant 0; \tag{A3}$$

and

$$D_{2N+1}[1/(1-\lambda)\psi]\geqslant 0. \tag{A4}$$

The above results are obtained from the representation of Eq. (9) by variational procedures. The method, in outline, is as follows. Consider, for example, the case where m is even and equal to $2N$. (The case where m is odd can be treated in a similar manner.) Our problem is equivalent to the problem of finding the maximum and minimum value of ψ at a particular point λ_i, given the values of ψ at the other $m-1$ points:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\sigma(\varphi)}{1-\lambda_i\cos^2(\frac{1}{2}\varphi)}=\text{extremum},$$

under the constraints

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\sigma(\varphi)}{1-\lambda_j\cos^2(\frac{1}{2}\varphi)}=\psi_j,$$

$$j=1,\,2,\,\cdots,\,i-1,\,i+1,\,\cdots,\,m.$$

Under variations of $\sigma(\varphi)$, the Lagrange multiplier method shows that an extremum is attained by a *discrete* measure $d\sigma=\sum_r c_r\delta(\varphi-\varphi_r)d\varphi$, or equivalently,

$$\psi(\lambda)=\sum_{r=1}^{M}c_r/(1-\lambda\alpha_r); \tag{A5}$$

$$M\leqslant m-1,$$

where

$$0\leqslant\alpha_1<\alpha_2<\cdots<\alpha_M\leqslant 1,$$
$$c_r>0.$$

The positivity of the c_r implies that $\psi(\lambda)$ in Eq. (A5) can also be written in the form

$$\psi(\lambda)=c\frac{(1-\lambda\beta_1)(1-\lambda\beta_2)\cdots(1-\lambda\beta_{M-1})}{(1-\lambda\alpha_1)(1-\lambda\alpha_2)\cdots(1-\lambda\alpha_M)},$$

where

$$0\leqslant\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots<\beta_{M-1}<\alpha_M\leqslant 1$$

and

$$c>0.$$

Taking logarithms, for convenience, we have

$$\ln\psi(\lambda)=\ln c+\sum_{r=1}^{M-1}\ln(1-\lambda\beta_r)-\sum_{r=1}^{M}\ln(1-\lambda\alpha_r).$$

Under variations of the α's and β's, we require that $\ln\psi(\lambda_i)$ be extremum under the $m-1$ constraints

$$\ln\psi(\lambda_j)=\ln\psi_j,\quad j\neq i.$$

Using the Lagrange method we find that

$$2M-1\leqslant m-1=2N-1.$$

We notice that the only parameters that are allowed to attain boundary values in their domain of variation are

α_1, which may become equal to zero; and α_M, which may become equal to 1. We conclude that $M=N$ and we have the following two extrema corresponding to whether $\alpha_1=0$ or $\alpha_N=1$:

(1) Setting $\alpha_1=0$, we have

$$\psi(\lambda)=c\frac{(1-\lambda\beta_1)\cdots(1-\lambda\beta_{N-1})}{(1-\lambda\alpha_2)\cdots(1-\lambda\alpha_N)}. \tag{A6}$$

The extremum value for $\psi(\lambda_i)$ is determined by eliminating the $2N-1$ parameters c, β_1, β_2, $\cdots$, β_{N-1}, α_2, $\cdots$, α_N from the $2N$ equations

$$\psi(\lambda_j)=\psi_j\,,\quad j=1,2\cdots 2N.$$

This elimination can be done easily if we write the above expression (A6) for $\psi(\lambda)$ in the form

$$[1+h_1\lambda+h_2\lambda^2+\cdots+h_{N-1}\lambda_j^{N-1}]\psi(\lambda) =c+k_1\lambda+k_2\lambda^2+\cdots+k_{N-1}\lambda^{N-1}.$$

Eliminating the new parameters h_r and k_r from the system of equations

$$[1+h_1\lambda_j+h_2\lambda_j^2+\cdots+h_{N-1}\lambda_j^{N-1}]\psi_j =c+k_1\lambda_j+k_2\lambda_j^2+\cdots+k_{N-1}\lambda_j^{N-1},\quad j=1,2\cdots 2N$$

leads to

$$D_{2N}[\psi]=0 \quad \text{for} \quad \psi_i=\text{extremum}.$$

(2) Setting $\alpha_N=1$, we have

$$(1-\lambda)\psi(\lambda)=c\frac{(1-\lambda\beta_1)\cdots(1-\lambda\beta_{N-1})}{(1-\lambda\alpha_1)\cdots(1-\lambda\alpha_{N-1})},$$

and a similar argument leads to

$$D_{2N}[(1-\lambda)\psi]=0 \quad \text{for} \quad \psi_i=\text{extremum}.$$

All allowed values of $\psi(\lambda)$ at $\lambda=\lambda_i$ will lie between the two given by $D_{2N}[\psi]=0$ and $D_{2N}[(1-\lambda)\psi]=0$; therefore the set (ψ_1,λ_1), (ψ_2,λ_2), $\cdots$, (ψ_{2N},λ_{2N}) is admissible if and only if $D_{2N}[\psi]$ and $D_{2N}[(1-\lambda)\psi]$ have definite signs. These signs can be easily determined by induction on m.

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Fixed Poles and Compositeness

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Using a perturbative model, we study the asymptotic behavior of weak amplitudes, looking in particular for the existence of fixed poles. The weakly interacting particles are considered as either elementary or composite. Our conclusions can be summarized as follows: The existence of fixed poles is model-dependent. In particular: (1) If both interacting particles are elementary, we find the usual fixed poles at $J=\sigma_1+\sigma_2-n$, $n\geq 1$, where σ_1 and σ_2 are the spins of the particles. (2) If the weakly interacting particles are composite, the amplitude is superconvergent (even being nonunitary) and there are no fixed poles, at least for $J\geq 0$. (3) In the case of photoproduction of a spinless composite particle, i.e., an amplitude with only one elementary particle with spin one, there is no fixed pole at $J=0$. If we consider that the elementary particle has spin σ, we argue that the pole at $J=\sigma-1$ disappears, while the ones at $J=\sigma-n$ with $n>1$ may or may not exist, depending on the wave function of the produced hadron. We conclude by discussing the implications of our results, and in particular the limitations on the hypothesis of partially conserved axial-vector current.

I. INTRODUCTION

THE study of fixed singularities in the angular momentum plane (J plane) has been shown to be important for the understanding of the asymptotic behavior of scattering amplitudes for particles with spin.[1] Interest arose recently after the discovery of their connection with current algebra sum rules for finite momentum transfer.[2] As a consequence some results have been derived, especially in the domain of nonunitary amplitudes.[3,4]

In a fundamental series of papers Mandelstam has studied the compatibility of Regge behavior of amplitudes with presence of elementary spinning particles in the theory.[1] It is not surprising that these concepts are relevant for our purposes since elementary particles in general introduce subtractions in the dispersion relations. Under appropriate conditions, i.e., absence of bilinear unitarity, these behave as fixed singularities in the J plane.

The main purpose of this paper is to apply Mandelstam ideas to the study of weak amplitudes.

Our conclusions are extracted from perturbative models. However, it is very encouraging that these models are able to reproduce the main features of the J-plane singularity structure.[5,6] Even more, there is by

[1] See, in particular, the general discussion in S. Mandelstam, Nuovo Cimento **30**, 1113 (1963).

[2] J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters **18**, 32 (1967); Phys. Rev. **157**, 1448 (1967).

[3] See Ref. 1 and H. D. I. Abarbanel, F. E. Low, I. J. Muzinich, S. Nussinov, and J. H. Schwarz, Phys. Rev. **160**, 1329 (1967).

[4] For a discussion of fixed poles in unitary amplitudes, see S. Mandelstam and L. Wang, Phys. Rev. **160**, 1490 (1967); C. E. Jones and V. L. Teplitz, *ibid.* **159**, 1271 (1967).

[5] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, New York, 1966). We follow closely the notation of this book.

[6] Equivalent models have already been used to study some features of the fixed poles in Refs. 2 and 3. See also Ref. 12.