High-Energy Bounds for Production Reactions*

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(Received 31 October 1967)

Bounds on the high-energy behavior of multiparticle production reactions are set on the basis of high-energy properties thought to hold for elastic scattering reactions. The connection which is exploited for the purpose is provided by the principle of unitarity. Stronger results, which would follow from certain analyticity conjectures concerning the production processes, are also described.

THE Froissart[1] bounds on high-energy behavior of elastic scattering amplitudes stem from the principle of unitarity and certain assumptions of analyticity. For a multiparticle inelastic process, $a+b\rightarrow c+d+e+\cdots$, the high-energy behavior is in turn delimited through the unitarity connection with the elastic reaction $a+b\rightarrow a+b$. We treat this connection in the following discussion of high-energy bounds for such inelastic processes.[2]

Consider a process with two incoming and n outgoing particles ($n>2$), all taken to be spinless. The amplitude depends on $3n-4$ kinematic variables. One of these, s, we take to be the square of the barycentric energy. This being fixed, a given configuration of the final state is specified by $3n$–7 independent variables describing the magnitude and relative orientation of the barycentric momenta of the outgoing particles. Let us denote the configuration variables collectively by the symbol v. Two remaining variables describe the orientation of the collision axis with respect to a system of axes defined by the final-state momentum vectors. We choose as polar axis the momentum vector of a particular one of the outgoing particles, the "distinguished" particle, and introduce the polar and azimuthal angles θ and φ. Thus θ is the angle formed in the center-of-mass system between the momentum vector of the distinguished particle and the collision axis.

In order to exploit the unitarity connection, partial wave by partial wave, with the corresponding elastic channel amplitude, we expand the inelastic amplitude $T(s,v,\theta,\varphi)$ in a spherical harmonic expansion

$$T(s,r,\theta,\varphi)=\sum_{l,m}(2l+1)d_l{}^m(\theta)\frac{e^{im\varphi}}{(2\pi)^{1/2}}T_l{}^m(s,v)\,,\qquad(1)$$

where

$$\int d_l{}^m d_{l'}{}^m d\cos\theta=\frac{2}{2l+1}\delta_{ll'}\,.\qquad(2)$$

We shall focus on the differential cross section $d\sigma/d\Omega$ for production of the distinguished particle at angle θ, integrated over all other final-state variables (including the energy of the distinguished particle). Let $\rho(s,v)$ be the final-state phase-space density (it is, obviously, independent of the angles θ and φ); and let $k=(\frac{1}{4}s-m^2)^{1/2}$ be the incident-particle barycentric momentum. Then

$$\frac{d\sigma}{d\Omega}\equiv\frac{1}{2\pi}\frac{d\sigma}{d\cos\theta}=\frac{1}{k}\sum_v\rho(s,v)\sum_{l,l',m}(2l+1)(2l'+1)\times d_l{}^m d_{l'}{}^m T_l{}^{m*}T_{l'}{}^m\,.\qquad(3)$$

We may at this point generalize and understand the above sum to run over all channels which contain a particle of the type selected here as the distinguished one; so that $d\sigma/d\Omega$ refers to the production of the distinguished particle at angle θ independent of the number and types of other particles accompanying it.

Now for the elastic channel reaction, unitarity relates the imaginary part of the lth partial-wave amplitude $A_l(s)$ to a sum of contributions from all coupled channels, including the elastic channel, the inelastic channels summed over in Eq. (3), and the rest. We are normalizing so that the elastic contribution to $\mathrm{Im}A_l$ is just $|A_l|^2$, hence $|A_l|\leqslant 1$. The contribution from the channels of Eq. (3) is easily worked out. Since $\mathrm{Im}A_l$ bounds this contribution from above, one has

$$\frac{1}{k}\mathrm{Im}A_l\geqslant\sum_{v,m}|T_l{}^m(s,v)|^2\rho(s,v)\,,\qquad(4)$$

where the sum over the variables symbolized by v includes the summation over all relevant channels. It follows from Schwarz's inequality applied to Eq. (3) that

$$\frac{d\sigma}{d\Omega}<\frac{1}{k}\{\sum_l(2l+1)[\sum_{v,m}|d_l{}^m T_l{}^m|^2\rho]^{1/2}\}^2.\qquad(5)$$

But

$$\sum_m|d_l{}^m T_l{}^m|^2\leqslant M_l{}^2(\theta)\sum_m|T_l{}^m|^2,\qquad(6)$$

where $M_l(\theta)$ is the maximum value attained by $|d_l{}^m(\theta)|$, for given θ and l, as m ranges over its allowed values. Combining results, we then have

$$\frac{d\sigma}{d\Omega}<\frac{1}{k^2}\{\sum(2l+1)(\mathrm{Im}A_l)^{1/2}M_l\}^2.\qquad(7)$$

* Supported by the U. S. Air Force Office of Research, Air Research and Development Command under Contract No. AF49(638)-1545.
[1] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[2] After the present work was completed we received a report by A. A. Logunov, M. A. Mestvirishvili, and Nguyen van Hieu (unpublished) dealing with the same topic. But the scope and results are somewhat different.

For forward (or backward) angles M_l is obviously given by

$$M_l=1, \quad \cos\theta=\pm 1. \tag{8}$$

For $\theta \neq 0$ or π, one finds the upper bound[3]

$$M_l(\theta) \sim l^{-1/3}(\cos\theta \sin^2\theta)^{-1/6}, \tag{9}$$

the maximum in $|d_l^m|$ occuring, for large l, at $m=l\sin\theta$. We shall however also wish to consider bounds on $d\sigma/d\Omega$ averaged over some fixed, finite range of angles θ. Performing the average directly on Eq. (3), then proceeding as above with Schwarz's inequality, one finds for $d\sigma/d\Omega$ an expression analogous to that of Eq. (7), with M_l replaced by the bound on the angular average of d_l^m. We denote this by $\langle M_l \rangle$ and find

$$\langle M_l \rangle \sim l^{-1/2}, \tag{10}$$

the proportionality factors, here suppressed, depending on the range of angles involved.

The bounds on $d\sigma/d\Omega$, and on $\langle d\sigma/d\Omega \rangle$, now rest on the properties of the elastic channel partial-wave amplitudes A_l. Following Froissart, suppose that the elastic amplitude $A(s,\cos\theta)$ is analytic, for fixed physical s, in an ellipse in the $\cos\theta$ plane, with foci at $\cos\theta=\pm 1$ and semi major axis $z_0=1+t_0/2k^2$; and suppose that it is uniformly bounded in this domain by a polynomial $B(s)$ in the variable s. These properties, and of course more, are implied by the Mandelstam representation. Then as Froissart shows,[1] the partial-wave amplitudes are bounded according to

$$|A_l(s)| \leqslant \frac{B(s)}{l} e^{-l\ln[z_0+(z_0^2-1)^{1/2}]}. \tag{11}$$

On the other hand, unitarity supplies the bound

$$|A_l(s)| \leqslant 1. \tag{12}$$

The bound of Eq. (11) becomes the more restrictive one for sufficiently large l, namely (in the limit of large s), for $l > \lambda_0 s^{1/2} \ln s$, where $\lambda_0 = \ln B(s)/\ln s$. Defining

$$L(s) \equiv \lambda s^{1/2} \ln s, \tag{13}$$

where the constant λ is chosen large compared to λ_0, one finds that the sum in Eq. (7) can effectively be cutoff at $l=L$, the remainder giving a contribution which falls with s like $s^{-(\lambda-\lambda_0)}$. Thus

$$\frac{d\sigma}{d\Omega} < \frac{1}{k^2} F(s,\theta)^2, \tag{14}$$

$$F(s,\theta)=\sum_{l=0}^{L(s)} (2l+1)[\operatorname{Im} A_l(s)]^{1/2} M_l(\theta). \tag{14$'$}$$

[3] The results of Eqs. (10) and (11) follow from a WKB analysis of the differential equation satisfied by the d_l^m functions.

The ensuing bounds now follow directly:

$$\frac{d\sigma}{d\Omega} < C_1 s (\ln s)^4, \quad \cos\theta=\pm 1, \tag{15}$$

$$\frac{d\sigma}{d\Omega} < C_2 s^{2/3} (\ln s)^{10/3} (\cos\theta \sin^2\theta)^{-1/3}, \quad \cos\theta \neq \pm 1, \tag{16}$$

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle < C_3 s^{1/2} (\ln s)^3, \tag{17}$$

where C_1 and C_2 are constants, and C_3 depends on the range of angles over which $d\sigma/d\Omega$ is averaged.

The bounds obtained here can all be improved slightly if the elastic scattering amplitude displays a Regge-like behavior at high energies. Such behavior implies that

$$A(s,t)=\frac{8\pi\sqrt{s}}{k} \sum (2l+1) A_l(s) P_l(\cos\theta) \underset{s\to\infty}{\longrightarrow} \beta(t) s^{\alpha(t)}, \tag{18}$$

where $t=-2k^2(1-\cos\theta)$ and (we assume) $\alpha(0)=1$, $d\alpha/dt>0$ for $t \leqslant 0$. Since

$$A_l(s)=\frac{k}{16\pi\sqrt{s}} \int A(s,t) P_l d\cos\theta,$$

the assumed Regge behavior yields the high-energy bound

$$|A_l(s)| < \text{const}/\ln s. \tag{19}$$

Using this in connection with Eq. (14′) in place of $|A_l| \leqslant 1$, we find for the nonforward differential cross sections the improved bounds

$$d\sigma/d\Omega < C_2' s^{2/3} (\ln s)^{7/3} (\cos\theta \sin^2\theta)^{-1/3}, \quad \cos\theta \neq \pm 1, \tag{16$'$}$$

$$\langle d\sigma/d\Omega \rangle < C_3' s^{1/2} (\ln s)^2. \tag{17$'$}$$

For forward production, $|\cos\theta|=1$, we can exploit Regge behavior to still better advantage as follows. Here, from Eqs. (8) and (14′) we have

$$F=\sum_{l=0}^{L} (2l+1)(\operatorname{Im} A_l)^{1/2} \leqslant \sum_{l=0}^{L} (2l+1)(\operatorname{Im} A_l)^{1/\mu}, \quad \mu \geqslant 2. \tag{20}$$

Introducing a set of positive parameters b_l, otherwise unspecified for the moment, we write

$$\sum_0^L (2l+1)(\operatorname{Im} A_l)^{1/\mu} = \sum_0^L [(2l+1) b_l (\operatorname{Im} A_l)^{1/\mu}] b_l^{-1}$$

$$\leqslant [\sum_0^L (2l+1)^\mu b_l^\mu \operatorname{Im} A_l]^{1/\mu} (\sum_0^L b_l^{-\nu})^{1/\nu}, \quad \frac{1}{\mu}+\frac{1}{\nu}=1. \tag{21}$$

The last step employs Hölder's inequality. Let us now choose the b_l according to

$$\{(2l+1)b_l\}^{\mu}=\tfrac{1}{2}(2l+1)l(l+1).$$

Evaluating the second factor on the right-hand side of Eq. (21), for $\mu>2$, we find

$$F\leqslant[L(s)]^{(2-4/\mu)}\left[\sum_0^L (2l+1)\frac{l(l+1)}{2}\operatorname{Im}A_l\right]^{1/\mu}. \quad (22)$$

But from Eq. (18) it follows that

$$(dA/d\cos\theta)_{\theta=0}=\frac{8\pi\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\frac{l(l+1)}{2}\times\operatorname{Im}A_l\underset{s\to\infty}{\longrightarrow}\sim s^2\ln s. \quad (23)$$

We are permitted to truncate the sum in Eq. (23) at $l=L$ and thus identify the sums appearing in Eq. (23) and (22). Finally we choose for the parameter μ the value $2/(1-\epsilon)$, where ϵ is an arbitrarily small positive quantity, and obtain

$$d\sigma/d\Omega<C_1's(\ln s)^{1+\epsilon},\quad \cos\theta=\pm1. \quad (15')$$

All of our bounds, even the improved versions based on the Regge picture for elastic scattering, are of course incredibly weak. The *total* cross section summed over all possible channels, grows with s at most like $(\ln s)^2$ according to Froissart; on the Regge picture it is bounded by a constant. So $d\sigma/d\Omega$, when integrated over all angles, cannot in any case grow more rapidly than $(\ln s)^2$; and if Regge behavior obtains it is in fact bounded by a constant. The bounds obtained here therefore serve only to set limits on the sharpness of diffraction peaks for production reactions. For example, in the case of forward production, and on the basis of the bounds set by elastic Regge behavior, a diffraction peak in $d\sigma/d\Omega$ cannot shrink so rapidly with increasing s that $d\sigma/d\Omega$ grows more rapidly than the integral over $d\sigma/d\Omega$ by the factor $s(\ln s)^{1+\epsilon}$. To be sure, such behavior is in any case not expected on any physical model known to us; but it is nevertheless of some comfort to rule out the possibility.

More restrictive bounds, in particular for nonforward production angles, would follow if one could demonstrate suitable "smoothness" or analyticity properties in the variable $\cos\theta$ for the inelastic processes under discussion. So far we have only made use of the unitarity connection to elastic scattering. Now for elastic processes one imagines that the amplitude has simple analytic properties in the $\cos\theta$ variable—let us say those properties implied by the Mandelstam representation, though less was needed for the Froissart results. Whatever analytic properties hold for the elastic amplitude also of course hold for the absolute square of the amplitude, i.e., for the elastic channel differential cross section. For a multiparticle-production process, with its many variables, corresponding analyticity properties are hard even to formulate in a reasonable way, let alone to prove. However, for the inelastic differential cross section, summed over all final-state variables except $\cos\theta$, it is perhaps not unreasonable to conjecture analyticity properties analogous to those presumed to hold for the elastic scattering case. Let us consider what would follow. We imagine, essentially, analyticity in the full $\cos\theta$ plane, with the exception of the Mandelstam cuts, and we demond uniform boundedness there by a polynomial in the s variable.[4]

Consider the inelastic differential cross section expanded directly in a Legendre series

$$\frac{d\sigma}{d\Omega}=\sum(2l+1)c_l(s)P_l(\cos\theta), \quad (24)$$

where

$$c_l(s)=\tfrac{1}{2}\int\frac{d\sigma}{d\Omega}P_l d\cos\theta,$$

and therefore

$$|c_l(s)|<\tfrac{1}{2}\int\frac{d\sigma}{d\Omega}d\cos\theta=\frac{1}{4\pi}\sigma_{\text{in}}(s). \quad (25)$$

Here $\sigma_{\text{in}}(s)$ is the cross section integrated over all final-state variables for the inelastic processes under discussion (a sum over all channels containing the distinguished particle is implied). The situation that now confronts us parallels that considered by Kinoshita, Loeffel, and Martin[4] for elastic scattering, once our analyticity assumptions are granted. The only differences are that in the elastic case the above authors deal with the amplitude, whereas here we deal with the inelastic differential cross section directly; and the Legendre coefficients c_l are bounded according to Eq. (25). Making the appropriate changes, we then find from the analysis of Kinoshita *et al.* the bound

$$\frac{d\sigma}{d\Omega}<C_4\sigma_{\text{in}}(s)\frac{(\ln s)^{3/2}}{\sin^2\theta},\quad \cos\theta\neq\gtrless1. \quad (26)$$

Since $\sigma_{\text{in}}(s)$ grows at most like $(\ln s)^2$, and is in fact bounded by a constant on the Regge picture, the bound of Eq. (26) would constitute a considerable improvement over those of Eqs. (16) and (16′). It would be interesting to learn if our analyticity conjectures can be sustained in a study of perturbation-theoretic diagrams.

The improved bounds conjectured above for nonforward production may still appear to be very weak if one considers the situation for elastic and other two-particle reactions. There the experimental evidence strongly indicates, for fixed nonforward angles, that the differential cross section falls very rapidly with s, in the exponential form $d\sigma/d\Omega\sim\exp[-\phi(s,\cos\theta)]$, where $\phi(s,\cos\theta)$ grows at least as fast as $\sqrt{s}$ and perhaps faster.

[4] Actually we need only assume the somewhat weaker analyticity conditions employed in the paper by T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. Letters **10**, 460 (1963).

That is, for nonforward angles the Froissart bounds, and even the improved bounds of Kinoshita *et al.*, prove to be excessively weak. For our production differential cross section $d\sigma/d\Omega$, however, the experimental situation is less clear. But if Regge behavior is relevant for production processes, it may be that $d\sigma/d\Omega$ at nonforward angles falls only polynomially with s, rather than exponentially. Consider for example a production reaction with three particles in the final state: $p_1+p_2 \to k+k_1+k_2$. Let k be the four-momentum of the distinguished particle; and define the subenergies $s_1=(k+k_1)^2$, $s_2=(k+k_2)^2$, as well as the momentum transfers $t_1=(p_1-k_1)^2$, $t_2=(p_2-k_2)^2$. It has been conjectured that the production amplitude displays Regge-like behavior, in the form $\text{Amp}\sim s_1^{\alpha(t_1)}s_2^{\alpha(t_2)}$, for the limit $s\to\infty$, with t_1, t_2 held finite. Now it is kinematically possible, even when the production angle θ of the distinguished particle is nonforward, for t_1, t_2 to be finite as $s\to\infty$, with $s_1\sim\sqrt{s}$, $s_2\sim\sqrt{s}$. Since $d\sigma/d\Omega$ involves an integration over all final-state variables other than θ, the differential cross section receives contributions from this Regge-like region of phase space. The corresponding s dependence of $d\sigma/d\Omega$ would then be polynomial rather than exponentially falling.

PHYSICAL REVIEW VOLUME 167, NUMBER 5 25 MARCH 1968

Use of Analyticity in the Calculation of Nonrelativistic Scattering Amplitudes*†

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(Received 25 October 1967)

A new method of calculating nonrelativistic scattering amplitudes is presented. The scattering amplitude is first calculated as a function of the complex energy below the scattering threshold, and the numerical results are then analytically continued to the physical region. The method is used to calculate two-body and two-channel scattering amplitudes. The numerical analytic continuation is accomplished by a rational-fraction representation similar to the Padé method. Several techniques of numerical analytic continuation by rational fractions are described, and some examples are discussed.

I. INTRODUCTION

The analytic properties of the solutions of the Schrödinger equation have been extensively studied, but they have rarely been used in an actual calculation. In an earlier communication,[1] we described the preliminary results of a method which uses the analytic properties of the nonrelativistic scattering amplitude $T(W)$ as a function of the complex-energy variable W to calculate physical scattering amplitudes. In this paper, we give a more complete discussion of the method, and we apply it to the calculation of one- and two-channel scattering problems.

The method consists of two steps. First, $T(W)$ is found for a number of unphysical ($W<0$) values of W for fixed and physical values of the external momenta. Then these numerical results are analytically continued, using a rational-fraction approximation, to the physical energy region to obtain scattering phase shifts and amplitudes. Because the amplitude is calculated in the unphysical energy region where momentum-space integral equations of the Lippmann-Schwinger[2] type are nonsingular or coordinate-space variation principles of the Kohn[3] type have no complicated scattered-wave terms, the first step in the calculation is considerably easier than the direct solution of the Schrödinger equation. The rational-fraction approximation, which constitutes the second step of the method, is a simple and accurate technique for accomplishing the analytic continuation. This approximation is an important part of our method, and because it has a number of advantages over the standard methods of numerical analytic continuation, we discuss the methods we have devised to represent a function by a rational fraction in some detail.

The method of calculating scattering amplitudes is straightforward to use and yields accurate results. Moreover, it can be applied with only minor modifications to two-body, two-channel, and three-body scattering. Although the two-body problem has been discussed and solved many times,[4–6] and the results obtained here are not new, the discussion of this method in the

* Supported in part by the National Aeronautics and Space Adminstration and AFOSR, Office of Aerospace Research Grant No. AF-AFOSR-130-66.

† Work based on a thesis submitted by the author to the faculty of the University of California, Berkeley, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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