Kinematics and Saturation of the Sum Rules and Inequalities for Inelastic Electron-Nucleon Scattering*

Frederick J. Gilman

Stanford Linear Accelerator Center, Stanford University, Stanford, California

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The kinematics, sum rules, and inequalities for inelastic electron-nucleon scattering are reviewed, with emphasis on the relations between the sum rules and inequalities and the assumptions needed in their derivation. What can be learned from the manner of saturation of the inequalities is discussed. Limits are put on possible modifications of the sum rules at large values of q^2 from what we already know at small values of q^2, and for $q^2 = 1$ BeV^2 it is shown that the inequality for electron scattering is roughly satisfied by summing over the inelastic scattering spectrum up to final hadron masses of 2 BeV.

I. INTRODUCTION

ALTHOUGH there have been many impressive successes for sum rules or low-energy theorems[1] derived from the commutation relations of the integrals of the time components of the vector and axial-vector weak-interaction current densities[2]

$$[F_a(t), F_b(t)] = i f_{abc} F_c(t),$$
$$[F_a(t), F_b{}^5(t)] = i f_{abc} F_c{}^5(t), \qquad (1)$$
$$[F_a{}^5(t), F_b{}^5(t)] = i f_{abc} F_c(t),$$

where

$$F_a(t) = -i \int \mathfrak{F}_{a4}(x) d^3x,$$
$$\qquad (2)$$
$$F_a{}^5(t) = -i \int \mathfrak{F}_{a4}{}^5(x) d^3x,$$

the commutation relations of the current densities themselves,

$$[\mathfrak{F}_{a4}(x), \mathfrak{F}_{b4}(y)]_{x_0=y_0} = -f_{abc}\mathfrak{F}_{c4}(x)\delta(\mathbf{x}-\mathbf{y}), \quad (3a)$$
$$[\mathfrak{F}_{a4}(x), \mathfrak{F}_{b4}(y)]_{x_0=y_0} = -f_{abc}\mathfrak{F}_{c4}{}^5(x)\delta(\mathbf{x}-\mathbf{y}), \quad (3b)$$
$$[\mathfrak{F}_{a4}{}^5(x), \mathfrak{F}_{b4}{}^4(y)]_{x_0=y_0} = -f_{abc}\mathfrak{F}_{c4}(x)\delta(\mathbf{x}-\mathbf{y}), \quad (3c)$$

have yet to be subjected to similar tests through the sum rules which they imply. It was first shown by Adler[3] that Eqs. (3) can be directly tested in high-energy neutrino reactions, where they lead to sum rules which imply that

$$d\sigma_T(\bar{\nu}+N)/dq^2 - d\sigma_T(\nu+N)/dq^2$$

goes to a constant which is independent of the four-momentum transfer q^2 as the incident neutrino energy goes to infinity. This q^2 independent constant is the same as the result one obtains for $d\sigma_T(\bar{\nu}+N)/dq^2 - d\sigma_T(\nu+N)/dq^2$, assuming a pointlike nucleon whose $V-A$ weak current is coupled to the leptons in the usual current-current interaction form.

By an isospin rotation, Bjorken[4] has shown how the part of Adler's results coming only from the vector current can be transformed into a useful inequality for inelastic electron scattering on nucleons. This inequality essentially states that as the incident electron energy goes to infinity, the sum, $d\sigma_T(e+p)/dq^2 + d\sigma_T(e+n)/dq^2$, of the total electron-proton and electron-neutron cross sections is greater than one-half the cross section for electrons scattering off pointlike (spinless) protons.

All of these results depend only on the commutation relations (3) of the time components of the currents. More recently, Bjorken[5] has derived an inequality for backward electron-proton scattering valid for large q^2 only which depends on the commutator of the space components of the vector currents. Using the chiral $U(6) \times U(6)$ algebra, for large q^2 the sum of neutron and proton backward scattering is predicted to be greater than one-half the result for a pointlike Dirac particle.

Both of these inequalities should soon be subject to direct test using data from inelastic electron scattering at SLAC. In fact, data from CEA and DESY already allows a very preliminary estimate[6] of contributions from the low and intermediate energy resonance region to the inequalities for values of q^2 up to 1 or 2 BeV^2. In this paper we first review the kinematics of electron scattering where only the final electron is detected. The relation between the invariant amplitudes and the total (massive) photon-nucleon cross sections is given, as is their relation to sums of squares of multipole amplitudes for single-pion electroproduction. Since each experimental and theoretical paper on inelastic electron scattering seems to try to invent a new notation, we relate to each other some of the more common expressions for the form factors and cross sections used by various authors. We then consider the sum rules and inequalities of Adler and Bjorken, stressing the assumptions needed to derive them and their justification. We show that the inequality for backward scattering derived by Bjroken follows from his original inequality (derived by an isospin rotation

[1] See the review given by R. F. Dashen, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, Calif., 1966* (University of California Press, Berkeley, Calif., 1967), p. 51.
[2] M. Gell-Mann, Physics 1, 63 (1964).
[3] S. L. Adler, Phys. Rev. 143, 1144 (1966).

[4] J. D. Bjorken, Phys. Rev. Letters 16, 408 (1966).
[5] J. D. Bjorken, Phys. Rev. 163, 1767 (1967).
[6] J. D. Bjorken, talk given at the 1967 International Symposium on Electron and Photon Interactions at High Energies, Stanford Linear Accelerator Center, 1967 (unpublished).

from Adler's neutrino sum rule) *if* for large q^2 the longitudinal or scalar photon-nucleon total cross sections are small (in a sense to be made more precise later) compared to the transverse cross sections. Finally, we consider the convergence and saturation of the sum rules and inequalities and how that convergence depends on q^2. We discuss what can be learned from the manner of saturation of the inequalities. Some limits on possible modifications of the sum rules at large q^2 are given from what we already know for q^2 near zero, as well as from the CEA and DESY data. For a value of $q^2 = 1$ BeV^2, we show that the inequality for electron scattering is quite possibly satisfied by summing over the inelastic scattering spectrum up to final hadron masses of 2 BeV.

II. KINEMATICS

Let us consider inelastic electron scattering (Fig. 1) where k and k' are the initial and final electron four-momenta, $q = k - k'$ is the four-momentum transfer, and p is the target nucleon's four-momentum. The final hadronic state n then has four-momentum $p_n = p + q$ and invariant mass squared[7] $W^2 = -(p+q)^2$. In the laboratory frame, where E and E' are the initial and final electron energies, we have

$$-p \cdot q/M_N = q_0 = E - E' = (W^2 - M_N{}^2 + q^2)/2M_N, \quad (4)$$

and q^2, the invariant four-momentum transfer squared (neglecting the electron mass) is

$$q^2 = 4EE' \sin^2(\tfrac{1}{2}\theta), \quad (5)$$

where θ is the scattering angle of the final electron relative to the incident beam direction.

If we only observe the energy and scattering angle of the final electron, then we may express the double differential cross section in terms of two invariant form factors which are functions of q_0 and q^2:

$$\frac{d^2\sigma}{d\Omega' dE'}$$

$$= \frac{4\alpha^2 E'^2}{q^4}[2\sin^2(\tfrac{1}{2}\theta)\alpha(q_0,q^2) + \cos^2(\tfrac{1}{2}\theta)\beta(q_0,q^2)] \quad (6a)$$

or equivalently,

$$-\frac{d^2\sigma}{dq^2 dE'} = \frac{4\pi\alpha^2}{q^4}\frac{E'}{E}[2\sin^2(\tfrac{1}{2}\theta)\alpha(q_0,q^2)$$
$$+ \cos^2(\tfrac{1}{2}\theta)\beta(q_0,q^2)] \quad (6b)$$

$$= \frac{4\pi\alpha^2}{q^4}\frac{1}{E^2}[\tfrac{1}{2}q^2\alpha(q_0,q^2) + (E^2 - Eq_0 - \tfrac{1}{4}q^2)$$
$$\times \beta(q_0,q^2)].$$

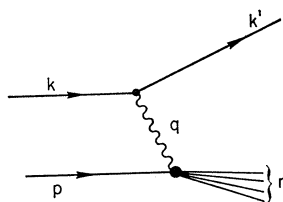The functions $\alpha(q_0,q^2)$ and $\beta(q_0,q^2)$ are the vector-current

FIG. 1. The kinematics of inelastic electron-nucleon scattering.

parts of functions first defined by Adler for neutrino scattering.[3] They have a rather simple interpretation as follows. Consider forward Compton scattering of massive photons (four-momentum q, mass$^2 = -q^2$, laboratory energy $q_0 > 0$) on nucleons, with the nucleon spin averaged over. If we call the Feynman amplitude for this process $(\epsilon_\mu')^* T_{\mu\nu}(q_0,q^2)\epsilon_\nu$, where ϵ_ν and ϵ_μ' are the initial and final photon polarization vectors which satisfy $\epsilon \cdot q = \epsilon' \cdot q = 0$, then[8]

$$\frac{1}{4\pi^2\alpha}\text{Im}T_{\mu\nu}(q_0,q^2) = \alpha(q_0,q^2)(\delta_{\mu\nu} - q_\mu q_\nu/q^2) + \beta(q_0,q^2)$$

$$\times \left(p_\mu - \frac{p \cdot q q_\mu}{q^2}\right)\left(p_\nu - \frac{p \cdot q q_\nu}{q^2}\right)\Big/ M_N{}^2$$

$$= (2\pi)^3 \overline{\sum_{s_N}} \sum_n \langle N(p) | J_\mu(0) | n\rangle$$

$$\times \langle n | J_\nu(0) | N(p)\rangle \delta^{(4)}(p_n - p - q), \quad (7)$$

where $\overline{\sum}_{s_N}$ and $\sum_n$ denote averaging over the nucleon spin and summing over final states n. The lowest-lying state which contributes to $\sum_n$ is the one-nucleon state which gives[9]

$$\alpha(q_0,q^2) = (q^2/4M_N{}^2)[F_1(q^2) + \mu F_2(q^2)]^2\delta(q_0 - q^2/2M_N)$$
$$= (q^2/4M^2)[G_M(q^2)]^2\delta(q_0 - q^2/2M_N) \quad (8a)$$

and

$$\beta(q_0,q^2) = \{[F_1(q^2)]^2 + (q^2\mu^2/4M_N{}^2)[F_2(q^2)]^2\}$$

$$\times \delta(q_0 - q^2/2M_N) \quad (8b)$$

$$= \frac{[G_E(q^2)]^2 + (q^2/4M_N{}^2)[G_M(q^2)]^2}{1 + q^2/4M_N{}^2}$$

$$\times \delta(q_0 - q^2/2M_N). \quad (8c)$$

It is easily verified that on putting these one-nucleon-state contributions to α and β in Eq. (6) and integrating over dE', one obtains the Rosenbluth formula for elastic electron-nucleon scattering.

Since $\alpha(q_0,q^2)$ and $\beta(q_0,q^2)$ are related to the imaginary part of forward Compton scattering of photons of

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[7] We use a metric where $p^2 = p_\mu p_\mu = |\mathbf{p}|^2 - p_0{}^2$, so that $p_N{}^2 = -M_N{}^2$ and $q^2 > 0$ corresponds to a spacelike four-vector q_μ. $\alpha = e^2/4\pi = 1/137\alpha$.

[8] The quantity $J_\mu(x)$ is the Heisenberg electromagnetic current operator divided by the electronic charge e. By the conserved-vector-current hypothesis, $J_\mu(x)$ is just the F-spin current, i.e., $J_\mu{}^{(a)}(x) = \mathfrak{F}_{\alpha\mu}(x)$.

[9] $F_1(q^2)$ and $F_2(q^2)$ are the usual Dirac and Pauli electromagnetic form factors of the nucleon, normalized so that $F_1(0) = F_2(0) = 1$, and μ is the anomalous magnetic moment in Bohr magnetons. $G_E = F_1 - (q^2\mu/4M_N{}^2)F_2$ and $G_M = F_1 + \mu F_2$ are the Sachs electric and magnetic form factors of the nucleon.

mass$^2=-q^2$, and the imaginary part of forward Compton scattering is related to total photon-nucleon cross sections, we may relate $\alpha(q_0,q^2)$ and $\beta(q_0,q^2)$ to the total cross sections, $\sigma_{\text{trans}}(q_0,q^2)$ and $\sigma_{\text{long}}(q_0,q^2)$, for the absorption of transverse and longitudinal (virtual) photons on nucleons. We find

$$\alpha(q_0,q^2)=\frac{|\mathbf{q}|}{4\pi^2\alpha}\sigma_{\text{trans}}(q_0,q^2) \qquad (9a)$$

and

$$\beta(q_0,q^2)=\frac{q^2}{4\pi^2\alpha|\mathbf{q}|}(\sigma_{\text{trans}}(q_0,q^2)-\sigma_{\text{long}}(q_0,q^2)), \qquad (9b)$$

where $|\mathbf{q}|=(q^2+q_0^2)^{1/2}$ is the magnitude of the laboratory photon three-momentum. As $q^2\to 0$, $\sigma_{\text{long}}\to 0$ and we find

$$\beta(q_0,q^2)\xrightarrow[q^2\to 0]{}\frac{q^2}{4\pi^2\alpha q_0}[\sigma_{\text{trans}}(q_0,0)], \qquad (10)$$

where $\sigma_{\text{trans}}(q_0,0)$ is the total photoabsorption cross section for *real* photons with laboratory energy q_0.

If the state n consists of a nucleon and a pion, i.e., we have electroproduction of a single pion, we can express $\sigma_{\text{trans}}(q_0,q^2)$ and $\sigma_{\text{long}}(q_0,q^2)$ in terms of squares of multipole amplitudes[10]:

$$\sigma_{\text{trans}}(q_0,q^2)=\frac{2\pi|\mathbf{p}_\pi|_{\text{c.m.}}}{|\mathbf{q}|_{\text{c.m.}}}\sum_{l=0}^{\infty}\{l(l+1)^2|M_{l+}|^2+(l+1)^2$$

$$\times(l+2)|E_{l+}|^2+(l+1)^2(l+2)|M_{(l+1)-}|^2$$

$$+l(l+1)^2|E_{(l+1)-}|^2\} \quad (11a)$$

and

$$\sigma_{\text{long}}(q_0,q^2)=-\frac{q^2}{q_0^2}\frac{4\pi|\mathbf{p}_\pi|_{\text{c.m.}}}{|\mathbf{q}|_{\text{c.m.}}}$$

$$\times\sum_l\{(l+1)^3|L_{l+}|^2+(l+1)^3|L_{(l+1)-}|^2\}, \quad (11b)$$

where

$$|\mathbf{p}_\pi|_{\text{c.m.}}=\{[W^2-M_N-M_\pi)^2]$$

$$\times[W^2-(M_N+M_\pi)^2]\}^{1/2}/2W$$

is the magnitude of the center-of-mass pion three-momentum and $|\mathbf{q}|_{\text{c.m.}}=(M_N/W)|\mathbf{q}|$ is the center-of-mass photon three-momentum. Note that in our notation $\sigma_{\text{long}}(q_0,q^2)$ is negative when $q^2>0$ (q^2 spacelike) so that the quantity $\beta(q_0,q^2)$ is actually the sum of two positive quantities in Eq. (9b) for $q^2>0$.

There are of course an infinite number of other ways

to write Eq. (6) in terms of two form factors. Drell and Walecka[11] write

$$\frac{d\sigma}{d\Omega'dE'}=\frac{4\alpha^2}{q^4}\frac{E'^2}{M_N}$$

$$\times[2W_1(q^2,q\cdot p)\sin^2(\tfrac{1}{2}\theta)+W_2(q^2,q\cdot p)\cos^2(\tfrac{1}{2}\theta)], \quad (12)$$

so that their functions W_1 and W_2 are related to α and β by

$$\alpha=W_1/M_N,$$

$$\beta=W_2/M_N. \qquad (13)$$

Another commonly used expression for the cross section is given by Hand,[12] who writes

$$\frac{d^2\sigma}{dl'dE'}=\frac{\alpha}{4\pi^2}\frac{K}{q^2}\frac{E'}{E}\left(\frac{2}{1-\epsilon}\right)(\sigma_T+\epsilon\sigma_S), \qquad (14)$$

where

$$K=q_0-\frac{q^2}{2M_N}=\frac{W^2-M_N^2}{2M_N}, \qquad (15)$$

and

$$\epsilon=\frac{1}{1+2(1+q_0^2/q^2)\tan^2(\tfrac{1}{2}\theta)},$$

or

$$\frac{2}{1-\epsilon}=2+\frac{\cot^2(\tfrac{1}{2}\theta)}{1+q_0^2/q^2}. \qquad (16)$$

Comparing with Eq. (6), we find

$$\alpha(q_0,q^2)=(K/4\pi^2\alpha)\sigma_T,$$

$$\beta(q_0,q^2)=\frac{K}{4\pi^2\alpha}\frac{q^2}{|\mathbf{q}|^2}(\sigma_T+\sigma_S). \qquad (17)$$

As $q^2\to 0$, σ_T becomes the real photon-nucleon total cross section, and becomes the same as $\sigma_{\text{trans}}(q_0,0)$ defined in Eq. (9). Generally, Hand's cross sections $\sigma_T=(|\mathbf{q}|/K)\sigma_{\text{trans}}$ and $\sigma_S=(-|\mathbf{q}|/K)\sigma_{\text{long}}$. Note that for spacelike q^2, σ_T, σ_S, σ_{trans}, and $-\sigma_{\text{long}}$ are all positive, as therefore are α and β.

III. SUM RULES AND INEQUALITIES FOR INELASTIC ELECTRON SCATTERING

The vector current part of the original sum rule of Adler for neutrino scattering can be written

$$\int_0^\infty dq_0[\beta^{(-)}(q_0,q^2)-\beta^{(+)}(q_0,q^2)]=1. \qquad (18)$$

The functions $\beta^{(\pm)}(q_0,q^2)$ are defined just as in Eq. (7) except that in place of the electromagnetic currents $J_\mu(0)$ and $J_\mu(0)$ we have put the isospin raising or

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[10] The relevant multipole and invariant amplitudes for pion electroproduction are defined by S. L. Adler and F. J. Gilman, Phys. Rev. **152**, 1460 (1966). Also see the discussion of electroproduction amplitudes of J. D. Bjorken and J. D. Walecka, Ann. Phys. (N. Y.) **38**, 35 (1966).

[11] S. D. Drell and J. D. Walecka, Ann. Phys. (N. Y.) **28**, 18 (1964).

[12] L. N. Hand, Phys. Rev. **129**, 1834 (1963).

lowering F-spin currents $\mathfrak{F}_{(1\pm i2)\mu}(0)$ [recall that $\mathfrak{F}_{3\mu}(0)$ is just the isovector part of the electromagnetic current]. If we explicitly separate out the nucleon Born term in Eq. (18), we have

$$[F_1^V(q^2)]^2+q^2\left(\frac{\mu^V}{2M_N}\right)^2[F_2^V(q^2)]^2$$
$$+\int_{M_\pi+(q^2+M_\pi^2)/2M_N}^{\infty}dq_0[\beta^{(-)}(q_0,q^2)-\beta^{(8)}(q_0,q^2)]=1,$$

(19)

where the superscript V denotes the fact that we are dealing with the isovector part of the current; the isovector anomalous magnetic moment $\mu^V=\mu_p'-\mu_n'$ $=3.70$. As $q^2\to 0$, we see from Eq. (10) or (17) that only the first term, $[F_1^V(q^2)]^2$, on the left-hand side of Eq. (19) survives, and as $q^2\to 0$ it goes to 1, in agreement with the left-hand side.

In the derivation[3] of Eq. (18) only two assumptions enter: (1) the commutation relation Eq. (3a) of the F-spin densities, and (2) an unsubtracted dispersion relation for the forward Compton scattering amplitudes (which are the coefficients of $p_\mu p_\nu$ and $q_\mu q_\nu$ in the expansion of $T_{\mu\nu}$) corresponding to $\beta(q_0,q^2)$. It is of course the second assumption which is most open to question. However, we note the following:

(a) The fact that as $q^2\to 0$ the left- and right-hand sides of Eq. (19) as it now stands automatically become equal rules out a q^2-independent subtraction. This just means we have done nothing grossly wrong, e.g., introduced a kinematic singularity in q^2 in one of our amplitudes.

(b) The assumption of an unsubtracted dispersion relation for the amplitude corresponding to β for two axial-vector currents, together with Eq. (3c), leads at $q^2=0$ directly to the Adler-Weisberger sum rule,[3,13] so it is very unlikely that there is a q^2-independent subtraction there either.

(c) Consider the derivative with respect to q^2 at $q^2=0$ of Eq. (19). From Eq. (10) we know that,

$$\beta^{(-)}(q_0,q^2)-\beta^{(+)}(q_0,q^2)\xrightarrow[q^2\to 0]{}\frac{1}{2\pi^2\alpha}\frac{q^2}{q_0}$$

$$\times[\sigma_T(\gamma^-+N)-\sigma_T(\gamma^++N)]=\frac{1}{2\pi^2\alpha}\frac{q^2}{q_0}$$

$$\times[2\sigma_T(\gamma^V+p\to I=\tfrac{1}{2})-\sigma_T(\gamma^V+p\to I=\tfrac{3}{2})],\quad (20)$$

where $\sigma_T(\gamma^-+N)$ and $\sigma_T(\gamma^++N)$ are the total, transverse cross sections on nucleons of the "fictitious massless photons" γ^- and γ^+ which correspond to the isospin indices $(1\mp i2)/\sqrt{2}$, while $\sigma_T(\gamma^V+p\to I=\tfrac{1}{2})$ and $\sigma_T(\gamma^V+p\to I=\tfrac{3}{2})$ are the total, transverse cross sections on protons which correspond to $I=\tfrac{1}{2}$ and $I=\tfrac{3}{2}$

[13] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); W. I. Weisberger, *ibid.* **14**, 1047 (1965).

final states and which come from the isovector part of the "real photon." Taking the derivative of Eq. (19) with respect to q^2 at $q^2=0$ we then find

$$2\frac{dF_1^V(q^2)}{dq^2}\bigg|_{q^2=0}+\left(\frac{\mu^V}{2M}\right)^2+\frac{1}{2\pi^2\alpha}\int_{M_\pi+M_\pi^2/2M}^{\infty}\frac{dq_0}{q_0}$$

$$\times[2\sigma_T(\gamma^V+p\to I=\tfrac{1}{2})-\sigma_T(\gamma^V+p\to I=\tfrac{3}{2})]=0.\quad (21)$$

This is of course just the Cabibbo-Radicati sum rule.[14] If there was a q^2-dependent subtraction in Eq. (19), the right-hand side of Eq. (21) would presumably no longer be zero. The fact that Eq. (21) appears to be satisfied[15,16] as it stands then sets limits on such a subtraction constant. We will return to this point in Sec. V.

Let us assume for the moment that Eq. (18) [or equivalently Eq. (19)] is true as it stands. Then, since $2(\beta_p+\beta_n)\geq\beta^{(-)}+\beta^{(+)}\geq\beta^{(-)}-\beta^{(+)}$, we can derive the inequality[4]

$$\int_0^\infty dq_0[\beta_p(q_0,q^2)+\beta_n(q_0,q^2)]\geq\tfrac{1}{2},\quad (22)$$

where β_p and β_n correspond to electron-proton and electron-neutron scattering, respectively. Equation (22) is only an inequality, both because we can say nothing about the contribution of the isoscalar part of the electromagnetic current from the commutation relation Eq. (3a) or the sum rule Eq. (18), *and* because $\sigma(\gamma^V+N\to I=\tfrac{3}{2})\geq 0$. In fact, at $q^2=0$ where only the proton Born term contributes, the left-hand side of Eq. (22) is equal to one, i.e., twice the right-hand side, since one-half the nucleon's charge is isoscalar, about which the sum rule Eq. (18) has no knowledge.

Equation (22) is just the inequality for inelastic electron scattering first derived by Bjorken. If we integrate over dE' $(=-dq_0)$ in Eq. (6b) and let $E\to\infty$, we can rewrite the inequality as

$$\lim_{E\to\infty}\int_0^\infty dq_0\left[\left(\frac{d^2\sigma_p}{dq_0dq^2}\right)+\left(\frac{d^2\sigma_n}{dq_0dq^2}\right)\right]$$

$$=\lim_{E\to\infty}\left(\frac{d\sigma_p}{dq^2}+\frac{d\sigma_n}{dq^2}\right)\geq\frac{2\pi\alpha^2}{q^4},\quad (23)$$

which is just one-half the result for $d\sigma/dq^2$ for a point (spinless) particle. The assumptions needed to derive Eq. (22) or (23) are of course just those needed to derive the sum rule Eq. (18).

[14] S. L. Adler (unpublished), and Ref. 3; J. D. Bjorken (unpublished), and Phys. Rev. **148**, 1467 (1966); N. Cabibbo and L. Radicati, Phys. Letters **19**, 697 (1966); R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1966).

[15] F. J. Gilman and H. J. Schnitzer, Phys. Rev. **150**, 1362 (1966).

[16] S. L. Adler and F. J. Gilman, Phys. Rev. **156**, 1598 (1967).

Bjorken's second inequality, however, depends on different assumptions. It reads[5]

$$\lim_{q^2 \to \infty} q^2 \int_0^\infty \frac{dq_0}{q_0^2} [\alpha_p(q_0,q^2) + \alpha_n(q_0,q^2)] \geq \tfrac{1}{2}, \qquad (24)$$

where α_p and α_n correspond to electron-proton and electron-neutron (backward) scattering, respectively. The right-hand side of Eq. (24) depends not on Eq. (3), but on the commutator of two *space components* of the currents. The number $\tfrac{1}{2}$ on the right-hand side of Eq. (24) corresponds to the $U(6) \times U(6)$ chiral algebra. It would be zero if the nucleon's isospin was carried (for large q^2) by spin-0 objects. As it stands, Eq. (24) predicts that at large q^2 the sum of proton and neutron backward scattering should be greater than one-half that of a point Dirac particle.

IV. RELATION BETWEEN THE TWO ELECTRON SCATTERING INEQUALITIES

It is clear from the kinematics presented in Sec. II that if σ_{long} or $\sigma_S = 0$, then, since in that case α and β would only depend on σ_{trans} or σ_T, α and β are related by

$$\beta(q_0,q^2) = \frac{q^2}{q^2 + q_0^2} \alpha(q_0,q^2).$$

The two inequalities presented in Sec. III are then related. We in fact now show the following[17]: If, for all q_0 and $q^2 \to \infty$,

$$\sigma_S(q_0,q^2) \leq \frac{q^2}{q_0^2} \sigma_T(q_0,q^2), \qquad (25)$$

then Bjorken's inequality for backward scattering using chiral $U(6) \times U(6)$ algebra follows from his original inequality derived from Adler's neutrino sum rule. This follows trivially since if we write Bjorken's original inequality as

$$\tfrac{1}{2} \leq \int_0^\infty dq_0 \beta(q_0,q^2),$$

and use Eqs. (17) and (25), we have

$$\tfrac{1}{2} \leq \int_0^\infty dq_0 \beta(q_0,q^2) = \int_0^\infty dq_0 \frac{K}{4\pi^2\alpha} \frac{q^2}{|\mathbf{q}|^2} (\sigma_T + \sigma_S)$$

$$\leq \lim_{q^2 \to \infty} \int_0^\infty dq_0 \frac{K}{4\pi^2\alpha} \frac{q^2}{q^2 + q_0^2} \left(\sigma_T + \frac{q^2}{q_0^2} \sigma_T \right)$$

$$= \lim_{q^2 \to \infty} \int_0^\infty dq_0 \frac{K}{4\pi^2\alpha} \frac{q^2}{q_0^2} \sigma_T$$

$$\leq \lim_{q^2 \to \infty} q^2 \int_0^\infty \frac{dq_0}{q_0^2} \alpha(q_0,q^2), \qquad (26)$$

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[17] We write Eqs. (25) and (26) without explicitly indicating that they are only supposed to be true for the sum of electron-proton

the last line of Eq. (26) being just the inequality for backward scattering derived using the chiral $U(6) \times U(6)$ algebra [Eq. (24)].

Thus, the inequality for backward scattering which depends on the commutator of two space components of the currents also follows from the original electron scattering inequality (which depends on the commutator of two time components of the currents) plus Eq. (25). It of course remains to establish the validity of Eq. (25). For the nucleon, dropping a common factor [including a $\delta(q_0 - q^2/2M_N)$], we have $\sigma_S = [G_E(q^2)]^2$ and $\sigma_T = (q^2/4M_N^2)[G_M(q^2)]^2$ so that Eq. (25) is true if

$$[G_E(q^2)]^2 = \sigma_S \leq \frac{q^2}{(q^2/2M_N)^2} \sigma_T = [G_M(q^2)]^2. \qquad (27)$$

This appears to be true experimentally for *both* the neutron and proton *for all* q^2. In the region of the $N^*(1238)$ resonance, recent data[18] indicate that Eq. (25) is true up to $q^2 \approx 1$ BeV^2 with σ_S consistent with zero above $q^2 \approx 0.4$ BeV^2. Thus Eq. (25) appears to be true for the nucleon and up through the region of the $N^*(1238)$. At higher energies there is presently a conclusive lack of data on σ_S.

V. CONVERGENCE AND SATURATION OF THE SUM RULES AND INEQUALITIES

Before discussing in detail how the various integrals in the sum rules and inequalities are (or are not) saturated by a few (or many) resonances, we might well ask whether the integrals converge at all. For fixed q^2, we see from Eq. (9) that for large q_0,[19]

$$\alpha(q_0,q^2) \propto q_0 \sigma_{\text{trans}}(q_0,q^2),$$

$$\beta(q_0,q^2) \propto (1/q_0)[\sigma_{\text{trans}}(q_0,q^2) - \sigma_{\text{long}}(q_0,q^2)], \qquad (28)$$

where $\sigma(q_0,q^2)$ is a total (massive) photon-nucleon cross section. Thus for large q_0, Adler's sum rule behaves as

$$\int \frac{dq_0}{q_0} [\sigma^{(-)}(q_0,q^2) - \sigma^{(+)}(q_0,q^2)],$$

where $\sigma^{(-)}$ and $\sigma^{(+)}$ correspond to antineutrino and

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and electron-neutron scattering. Also, the same argument goes through if we have $|\sigma_{\text{long}}(q_0,q^2)| \leq (q^2/q_0^2)\sigma_{\text{trans}}(q_0,q^2)$ for all q_0 and $q^2 \to \infty$. We have used Hand's cross sections σ_S and σ_T only because the experimental data are usually translated in terms of them.

[18] H. L. Lynch, J. V. Allaby, and D. M. Ritson, Phys. Rev. **164**, 1635 (1967).

[19] Independent of sum rules and inequalities, it would be very interesting to have experimental data on the total (massive) photon-nucleon cross section $\sigma(q_0,q^2)$ for large q^2 and various values of q_0. We might then answer directly questions considered here like: As $q_0 \to \infty$, do the cross sections approach a constant? Are the total cross sections q^2-dependent? If they are q^2-dependent, do they behave in a way understandable from vector-meson-dominance models? Do the longitudinal and transverse cross sections behave differently?

neutrino cross sections or to the cross sections of fictitious γ^- and γ^+ photons. By the Pomeranchuk theorem, these cross sections approach each other as $q_0 \to \infty$, and therefore, the integral converges. In fact, one expects in a Regge theory of high-energy scattering. that

$$\sigma^{(-)} - \sigma^{(+)} \propto q_0{}^{\alpha_\rho(0)-1},$$

where $\alpha_\rho(0)$ is the $t=0$ intercept of the ρ Regge trajectory and is numerically about 0.5.

For Bjorken's two inequalities, however, we have an integral like

$$\int^\infty \frac{dq_0}{q_0}(\sigma_p + \sigma_n),$$

where σ_p and σ_n are total (massive) photon-photon and photon-neutron cross sections. If σ_p and σ_n approach a constant as $q_0 \to \infty$, as we naively expect them to do, then *the integrals in Bjorken's inequalities diverge* logarithmically. Thus the inequalities for inelastic electron scattering are trivially satisfied if we integrate to high enough values of q_0.

What then can in fact be tested by the inequalities? *What can be tested is specific models* for the saturation of the integral in the inequalities, such as saturation by a few resonances or by states in a "quasielastic peak."[6] It is of course a perfectly definite and testable model to ask if the inequalities hold when we only integrate to say $q_0=5$ BeV or to where there are no longer any bumps in the inelastic scattering spectrum.[20] If such a simple model in fact works, we will have learned a great deal about the structure of the nucleon. *If such a specific model does not work*, however, *we can not immediately say that the inequality is wrong or that Adler's neutrino sum rule is wrong*. It may be that Adler's sum sum rule (and therefore the inequality) is saturated at large q^2 only when we include a substantial part of the high-energy tail. It may also be that Adler's sum rule is plain wrong at large q^2, but we will be unable to tell this by looking at the electron-scattering inequality because it diverges and will be satisfied simply by integrating to a large enough value of q_0. However, since two times the integrand in the inequality, Eq. (22), is an upper bound on the integrand in the Adler sum rule, Eq. (18), for each value of q_0 (and q^2), and if the saturation of the inequality by the resonance region is so poor that we have to integrate to, say, 10^3 BeV to satisfy the inequality, then the Adler sum rules are essentially useless, even if they do finally converge.

For small values of q^2, the convergence of the β sum rule has been studied by Adler and Gilman[15] who write[21]

$$\int_0^E dq_0\left(1 - \frac{q_0}{E} - \frac{q_0{}^2}{2E^2}\right)[\beta^{(-)}(q_0, q^2) - \beta^{(+)}(q_0, q^2)]$$

$$= 1 + F_2(E)(q^2/M_N{}^2) + O((q^2/M_N{}^2))^2, \quad (29)$$

so that the β sum rule, Eq. (18), becomes the statement, that if we take the limit of both sides of Eq. (29) as $E \to \infty$, then $F_2(E) \to 0$. Using existing photoproduction data and assuming that $F_2(E) \to_{E\to\infty} 0$, it was shown[16] that $F_2(E)$ is negative and that its magnitude is less than 0.5 for $E \gtrsim 5$ BeV. Thus for small q^2 (say, $q^2=0.1$ BeV^2) we find the sum rule is satisfied to within a few percent with an incident energy of ~ 5 BeV. Note that, although our expansion in powers of $(q^2/M_N{}^2)$ is strictly only good for small q^2, we would not be greatly surprised from the value $F_2(5 \text{ BeV}) \simeq -0.5$ if for $q^2 \simeq M_N{}^2$ we only obtained a 50% saturation of the sum rule for $E=5$ BeV, $q^2=M_N{}^2$.

The data at or near $q^2=0$ also limit possible modifications of the β sum rule, Eq. (18), Suppose, for example, we guess that Eq. (18) should be replaced by[22]

$$\int_0^\infty dq_0[\beta^{(-)}(q_0, q^2) - \beta^{(+)}(q_0, q^2)] = \left(\frac{M^2}{M^2+q^2}\right)^2. \quad (30)$$

This agrees with Eq. (18) at $q^2=0$, but differs elsewhere. It in fact corresponds to a q^2-dependent subtraction constant of the form $q^2(2M^2+q^2)/(M^2+q^2)^2$ in the (massive) photon-nucleon forward Compton scattering amplitudes corresponding to $\beta(q_0, q^2)$. If we take the derivative with respect to q^2 at $q^2=0$ of Eq. (30), we obtain a modification of the Cabibbo-Radicati sum rule [Eq. (21)]:

$$2\frac{dF_1{}^V(q^2)}{dq^2}\bigg|_{q^2=0} + \left(\frac{\mu^V}{2M}\right)^2$$

$$+ \frac{1}{2\pi^2\alpha}\int_{M_\pi+M_\pi{}^2/2M}^\infty [2\sigma_T(\gamma^V+p \to I=\tfrac{1}{2})$$

$$- \sigma_T(\gamma^V+p \to I=\tfrac{3}{2})] = -(2/M^2). \quad (31)$$

Available photoproduction data, which show that the Cabibbo-Radicati sum rule is well satisfied[15,16] (see Table I), then limit[23] $M \gtrsim 1.7$ BeV, i.e., greater than a ρ-meson mass by better than a factor of two. Such arguments, of course, apply only to subtraction con-

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[20] It is clear that if we integrate to some fixed mass W of the final hadron state, the inequality will no longer be satisfied if we increase q^2 sufficiently, because the contribution of any given resonance or finite sum of resonances is expected to be strongly damped at large q^2. One possible way the sum rule or inequalities might be satisfied is for the region of bumps in the inelastic cross section to move out in W as q^2 increases. This of course would be very interesting to look for experimentally (see also Ref. 6 in this connection).

[21] The extra factor of $(1-q_0/E+q_0{}^2/2E^2)$ is related to the fact that the authors of Ref. 16 were interested in the convergence of $d\sigma(\bar\nu+N)/dq^2-d\sigma(\nu+N)/dq^2$ and the extra factor comes from the connection of the double differential cross section to α and β, Eq. (6).

[22] The author thanks D. Ritson for a conversation on this point

[23] The limit on M is obtained by neglecting the contribution of the high-energy tail in Table I and attributing it to the $-2/M^2$ on the right-hand side of Eq. (31).

TABLE I. Contributions to the Cabibbo-Radicati sum rule,

$$2\frac{dF_1{}^V(q^2)}{dq^2}\bigg|_{q^2=0}+\left(\frac{\mu^V}{2M_N}\right)^2+\frac{1}{2\pi^2\alpha}\int_{M_\pi+M_\pi{}^2/2M_N}^{\infty}\frac{dq_0}{q_0}$$
$$\times[2\sigma_T(\gamma^V+p\to I=\tfrac{1}{2})-\sigma_T(\gamma^V+p\to I=\tfrac{3}{2})]=0.^{\text{a}}$$

$2[dF_1{}^V(q^2)/dq^2]\|_{q^2=0}$	$-1720\ \mu\mathrm{b}$
$(\mu^V/2M_N)^2$	$+1520\ \mu\mathrm{b}$
S-wave(E_{0+})	$+600\ \mu\mathrm{b}$
$N^*(1238)$	$-1200\ \mu\mathrm{b}$
$N^*(1520)$	$+400\ \mu\mathrm{b}$
$N^*(1688)$	$+100\ \mu\mathrm{b}$
High-energy tail starting at $q_0=1.1$ BeV to make sum rule satisfied	$+300\ \mu\mathrm{b}$

a See Refs. 15 and 16 for details.

stants which depend on q^2 linearly, i.e., we can say nothing using the above arguments about subtraction constants that behave like $(q^2/M^2)^2$ as $q^2 \to 0$.

Let us now look briefly at what the existing data on inelastic electron scattering[18,24,25] say about the inequality for $q^2=1$ BeV^2. In Table II we have calculated rough values for $\beta_p(q_0,q^2=1$ BeV$^2)$ at the peaks of the known resonances extracted mostly from the recent DESY data.[25] We see that $\beta_p(q_0,1$ BeV$^2)$ is slowly decreasing but has a mean value of $\simeq 0.2$/BeV in the resonance region. As the valleys between the resonances are rather shallow $\beta_p \simeq 0.2$/BeV gives a fair estimate of the integrand of the sum rule. Integrating from threshold $(W=M_N+M_\pi,\ q_0=0.68$ BeV) to around the region of the $N^*(1920)$ $(W=1.92$ BeV, $q_0=2.02)$, we find

$$\int_{0.68}^{2.02} dq_0\beta_p(q_0,q^2=1\ \mathrm{BeV}^2)\simeq 0.25.$$

[24] A. A. Cone et al., Phys. Rev. **156**, 1490 (1967).
[25] F. W. Brasse et al., DESY Report, 1967 (unpublished).

TABLE II. Values of $\beta(q_0,q^2=1$ BeV$^2)$ at the peaks of the known pion-nucleon resonances.

q_0 (BeV)	$\beta(q_0,1$ BeV$^2)$
0.88($W=1238$ MeV)	0.26/BeV
1.28($W=1512$ MeV)	0.25/BeV
1.58($W=1688$ MeV)	0.22/BeV
2.02($W=1920$ MeV)	0.13/BeV

Assuming a similar value for

$$\int_{0.68}^{2.02} dq_0\beta_n(q_0,q^2=1\ \mathrm{BeV}^2),$$

(which seems to be true at least for $q^2=0$) and adding on the Born-term contribution, which is $\simeq 0.1$, we find

$$\int_0^{2.02} dq_0[\beta_p(q_0,1\ \mathrm{BeV}^2)+\beta_n(q_0,1\ \mathrm{BeV}^2)]\simeq 0.6,$$

and the inequality is roughly satisfied by integrating over the resonance region.

It is of course clear that if we increase q^2 much higher (to say 2 BeV^2) the contribution of the low-mass region considered above will decrease and the inequality will no longer be satisfied by considering only this region. For information on what it takes to satisfy the inequality for large values of q^2 we must await the outcome of experiments underway at SLAC.

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