New Phase-Shift Parametrization Based on a Phase-Shift Dispersion Relation*

PHILIP W. COULTER
University of California, Irvine, California
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The dispersion relation for the phase shift used by Ball and Frazer is generalized to include the effects of zeros and poles in the S matrix. For complex S-matrix zeros near the physical cut, one obtains a new parametrization for a resonant state. For $l=0$, the usual Breit-Wigner phase shift is found. The method is easily extended to resonances in higher partial waves (in terms of two parameters for the case of pure elastic unitarity). The results are used to describe three partial waves in πN scattering: P_{33}, D_{13}, and P_{11}. Excellent agreement with the published P_{33} phase shifts is obtained by using our resonant form plus the integral over the inelastic cut. The inelastic contribution to the phase shift is small at low energies but becomes important for $E_L>300$ MeV (E_L is the incident-pion kinetic energy). Inelastic effects in the D_{13} partial wave can account for almost all of the phase shift up to $E_L=500$ MeV. A narrow resonance plus inelasticity gives a good fit to the phase shifts up to $E_L=900$ MeV. Our fit to the P_{11} phase shift depends on the nucleon pole, the Roper resonance, a zero in the S matrix below the elastic threshold, and inelastic effects. A good fit to the low-energy phase shifts can be obtained only if one assumes large inelasticity above 1 BeV; however, the model does account for the qualitative behavior of the P_{11} phase shift.

I. INTRODUCTION

A DISPERSION relation for the phase shift δ was used by Ball and Frazer[1] in an attempt to explain the higher πN resonances in terms of inelastic processes. Their procedure was to evaluate δ from (for S waves)

$$\delta(s)=-\frac{k}{2\pi}\int_{s_I}^{\infty}\frac{\ln[\eta(s')]}{k'(s'-\ )}ds', \tag{1}$$

where $\eta=\exp(-2\,\mathrm{Im}\delta)$, k is the momentum in the center-of-mass system, and s_I is the inelastic threshold. The left-hand cut in δ is neglected because they were only interested in a mechanism for explaining high-energy inelastic resonances.

If a Breit-Wigner form is used to fit a resonance, one finds a pair of zeros in the S matrix for complex values of s ($s\equiv$square of total energy in the center-of-mass system) near the resonance. The zero in the S matrix causes a cut in δ which was not considered by Ball and Frazer. We have generalized their dispersion relation to include S-matrix zeros for complex values of s and also zeros and poles for real s below the elastic threshold.

The result of including complex S-matrix zeros for S waves is the reproduction of the usual elastic Breit-Wigner resonant form for $l=0$. The advantage of this procedure is that it can easily be extended to higher l values. The resonant phase shift is found in terms of two parameters (the position of the zeros), and inelastic effects must be included by using the integral in Eq. (1).

The dispersion relation for δ is derived in Sec. II. In Sec. III, we discuss the application to three partial waves in πN scattering: P_{33}, D_{13}, and P_{11}. The P_{33} partial wave is mostly elastic, and we are able to get a good fit to the accepted phase shifts[2-4] up to $E_L=250$ MeV (E_L is the laboratory energy of the incident pion). By including inelasticity in the P_{33} partial wave, the phase shifts can be fitted to higher energies. In the D_{13} partial wave, we find that the major part of the phase shift up to 500 MeV can be accounted for by the integral over the inelastic cut. This contribution never rises above 30° and changes sign abruptly in the resonant region. The phase shift up to $E_L=900$ MeV can be accounted for very well in terms of a resonance plus the inelastic contribution. Inelastic effects are very important in the P_{11} partial wave also. We attempt to fit the phase shift in terms of the pole at the nucleon position, a zero in the S matrix below threshold, a resonance at $E_L\approx 590$ MeV, and the integral over the inelasticity. The major features of the phase shift can be explained, but the fit is very rough and is highly dependent on the inelasticity above $E_L=1$ BeV.

II. DISPERSION RELATION FOR THE PHASE SHIFT

The S matrix is given by

$$S=e^{2i\delta}=\eta e^{2i\,\mathrm{Re}\delta}. \tag{2}$$

Following Ball and Frazer, we write a dispersion relation for $\delta(s)/k$ for S waves. We separate this into several parts:

$$\delta(s)=\delta_L(s)+\delta_{\mathrm{inel}}(s)+\delta_R(s)+\delta_P(a), \tag{3}$$

where δ_L is the contribution from the unphysical cuts, δ_{inel} is the integral over the inelastic factor, δ_R includes the effects of complex zeros in the S matrix, and δ_P is the contribution from a pole in the S matrix for real s

* Supported in part by the National Science Foundation.
[1] J. S. Ball and W. R. Frazer, Phys. Rev. Letters 7, 204 (1961).
[2] L. D. Roper, Phys. Rev. Letters 12, 340 (1964); L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965); L. D. Roper and R. M. Wright, *ibid.* 138, B912 (1965).
[3] P. Bareyre, C. Brickman, A. Stirling, and G. Villet, Phys. Letters 18, 342 (1965).
[4] P. Auvil, C. Lovelace, A. Donnachie, and A. Lea, Phys. Letters 12, 76 (1964); A. Donnachie, A. Lea, and C. Lovelace, *ibid.* 19, 146 (1965).

below the elastic threshold;

$$\delta_L(s)=\frac{k(s)}{2\pi i}\int_L \Delta\left[\frac{\delta(s')}{k(s')}\right]\frac{ds'}{s'-s},\tag{4}$$

$$\delta_{\text{inel}}(s)=-\frac{k(s)}{2\pi}\int_{s_I}^{\infty}\frac{\ln[\eta(s')]}{k(s')(s'-s)}ds'.\tag{5}$$

$\Delta(f(s))$ means the discontinuity in $f(s)$, and s_I is the inelastic threshold.

Now, assume that the S matrix has zeros for complex values of s (see Fig. 1). If these zeros are at $s=a\pm ib$, we may factor them out:

$$S(s)=[s-(a+ib)][s-(a-ib)]f(s),$$

where $f(s)$ has no cuts near $a\pm ib$. Now when we write the dispersion relation for δ/k ($\sim\ln S$), we find a logarithmic cut in δ beginning at $a\pm ib$. The discontinuity across this logarithmic cut is easily computed, and for real s we find

$$\delta_R(s)=-k(s)\,\text{Im}\left[\int_{a+ib}^{a+i\infty}\frac{ds'}{k(s')(s'-s)}\right].\tag{6}$$

$\delta_R(s)$ is thus a function only of a and b.

The same procedure may be followed to obtain the contribution of a pole on the real axis below threshold. We obtain

$$\delta_P(s)=\tfrac{1}{2}ik(s)\int_{-\infty}^{s_P}\frac{ds'}{k(s')(s'-s)},\tag{7}$$

where s_P is the pole position. [The integral is independent of the path of integration, and for scattering of unequal-mass particles one can avoid cuts in $k(s')$ by doing the integration from s_P to $s_P+i\infty$.] The effect of a real zero in the S matrix is the negative of Eq. (7).

In the case of equal-mass particles, $k=\frac{1}{2}(s-s_E)^{1/2}$, where s_E is the elastic threshold, and the integration may be done explicitly. We find

$$\delta_R(s)=\tan^{-1}\left[\frac{\gamma(s-s_E)^{1/2}}{m_R^2-s}\right],\tag{8}$$

where

$$m_R^2=[(a-s_E)^2+b^2]^{1/2}+s_E,\tag{9}$$

$$\gamma=[2(m_R^2-a)]^{1/2},\tag{10}$$

and

$$\delta_P(s)=-\tan^{-1}\left[\left(\frac{s-s_E}{s_E-s_P}\right)^{1/2}\right].\tag{11}$$

In the case of elastic scattering of unequal-mass particles, the integration cannot be carried out explicitly. However, in this case we may simply write the dispersion relation for $\delta/[\frac{1}{2}(s-s_E)^{1/2}]$ to begin with, putting some more of our ignorance in the background term $\delta_L(s)$.

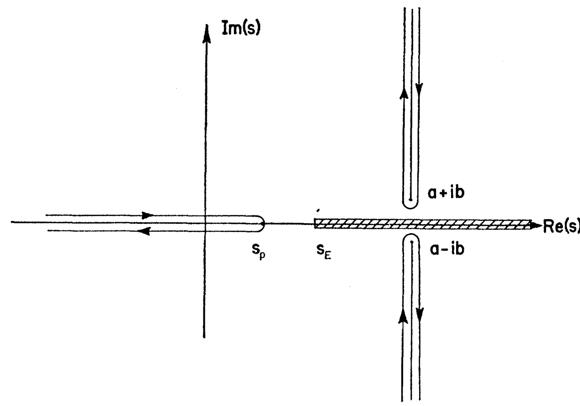

FIG. 1. Position of zeros and poles assumed in the S matrix and the corresponding cuts in δ.

We may follow the same procedure for P waves. In this case we write the dispersion relation for $s\delta(s)/[(s-s_E)^{3/2}]$. Our result depends on the kinematical factor we use. Another parameter could be introduced by dividing δ by $(s-s_E)^{3/2}/(s+s_0)$. However, we find that the results are insensitive to the value of s_0, and we choose $s_0=0$ in our calculation. For P waves we find

$$\delta_R=\tan^{-1}\left[\frac{\gamma(s-s_E)^{1/2}}{m_R^2-s}\right]-\frac{s_E}{s}\frac{\gamma(s-s_E)^{1/2}}{m_R^2-s_E},\tag{12}$$

$$\delta_P=-\tan^{-1}\left[\left(\frac{s-s_E}{s_E-s_P}\right)^{1/2}\right]+\frac{s_E}{s}\left(\frac{s-s_E}{s_E-s_P}\right)^{1/2},\tag{13}$$

where γ and m_R^2 are again given by Eqs. (9) and (10).

An expression for a D-wave resonance may also eb found by dividing δ by $(s-s_E)^{5/2}/s^2$. We get

$$\delta_R(s)=\tan^{-1}\left[\frac{\gamma(s-s_E)^{1/2}}{m_R^2-s}\right]-\frac{s_E^2}{s^2}\frac{\gamma(s-s_E)^{1/2}}{m_R^2-s_E}$$
$$-\frac{s_E\gamma(s-s_E)^{3/2}}{s^2(m_R^2-s_E)^2}\left(2m_R^2-s_E-\frac{1}{3}\frac{\gamma^2 s_E}{m_R^2-s_E}\right).\tag{14}$$

The equations for the phase shifts all have the property that $\delta\sim k^{2l+1}$ at threshold. The disagreeable feature is the appearance of the pole at $s=0$ for partial waves with $l>0$.[5] The equations should still be useful if we are far enough away from the pole. Note that $\delta_R\to\pi$ as $s\to\infty$, just as the Layson[6] resonant form does. Also, $\delta_P\to-\frac{1}{2}\pi$ as $s\to\infty$. This latter behavior seems to be inconsistent with Levinson's theorem,[7] which requires that δ approaches an integral multiple

[5] Of course, there is no pole in δ when all of the contributions to δ are added together.

[6] W. M. Layson, Nuovo Cimento 27, 724 (1963). The Layson form differs from the Breit-Wigner form only in the introduction of a slowly varying energy denominator in the definition of the width. See the Appendix for a comparison of our form and the Layson form.

[7] J. Hartle and C. Jones, Ann. Phys. (N. Y.) 38, 348 (1966); Phys. Rev. 140, B90 (1965); Phys. Rev. Letters 14, 801 (1965).

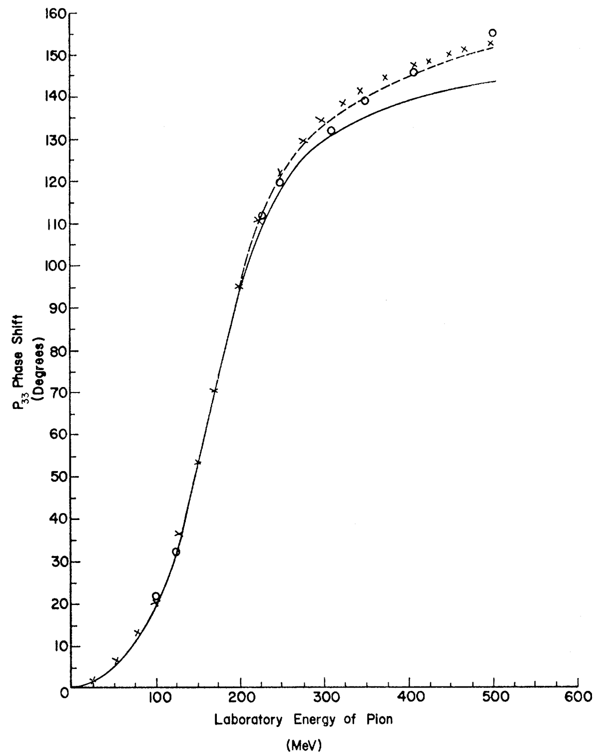

FIG. 2. P_{33} phase shift as a function of incident pion energy. The solid line is the phase shift found from Eq. (12), with $m_R = 8.72$, $\gamma = 1.45$. The dashed line is the sum of the resonant phase shift and the contribution from the inelastic cut. The crosses and circles are Roper's and Bareyre's results, respectively. The circles are used only to indicate possible errors made by the author in reading Bareyre's data from the graphs.

of π as $s \to \infty$. The reason for this apparent inconsistency may be seen by considering a simple example. If we solve the N/D equations for $l=0$ for the case where the left-hand cut in the amplitude is a simple pole, then it is easy to adjust the residue of the pole so that a bound state appears. We find that $\delta(\infty) = -\pi$, in agreement with Levinson's theorem. Our analysis also gives $\delta(\infty) = -\pi$, since the left-hand-cut pole, as well as the pole in the S matrix due to the bound state, gives a contribution of $-\frac{1}{2}\pi$ to $\delta(\infty)$.

In the Sec. III, the usefulness of these equations will be tested by applying them to some partial waves in πN scattering. Since the term $\delta_L(s)$ cannot be explicitly taken into account, we will neglect it, hoping that the main features of the phase shift are due to the zeros and poles of the S matrix and the inelasticity.

III. APPLICATIONS TO πN SCATTERING

A. P_{33} Partial Wave

Perhaps the best partial wave to use to test the resonant form for $l=1$ is the P_{33} partial wave in πN scattering, where it is known that the resonance is dominant and inelastic effects are small at low energies. The results of our fit are shown in Fig. 2, along with the phase shifts found by Roper and Bareyre. The solid line is the fit obtained from δ_R alone. The dotted line is the sum of δ_R and the integral over the inelasticity, which does not contribute significantly until $E_L \gtrsim 300$ MeV. The inelasticity was chosen to be a rough fit to that found by Bareyre up to $E_L = 1$ BeV. Various arbitrary forms were used above 1 BeV. The inelastic contribution to δ below 500 MeV was not strongly dependent on the form chosen above 1 BeV. The dashed line is only intended to be a rough estimate of the inelastic effect.

We see that Bareyre's phase shift can be fitted very well up to 500 MeV with our resonant form plus some inelasticity. It would also be possible to find a rough fit to his phase shifts at higher energies by assuming an appropriate form for η for $E_L > 1$ BeV. Roper's phase shifts were found by using the Layson resonant form plus a background term (which turns out to be negli-

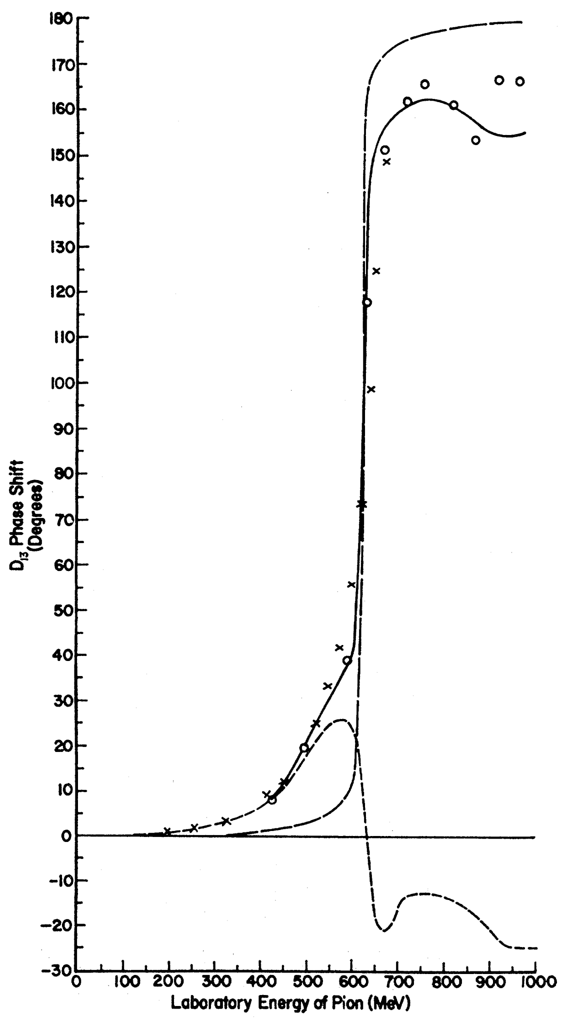

FIG. 3. D_{13} phase shift as a function of incident pion energy. The short-dashed curve is the contribution from the inelastic cut. The long-dashed curve is found from Eq. (14), with $m_R = 10.95$, $\gamma = 0.1$. The crosses and circles are defined in Fig. 2.

gible) to fit the P_{33} phase shifts. No inelastic effects were included in the Layson form, and none were found in the phase-shift analysis. It is clear, however, that there should be a contribution to δ from inelastic effects above $E_L=300$ MeV. Perhaps the use of the Layson fit serves to mask the inelasticity. The Layson fit for an elastic resonance involves three parameters, whereas our fit involves only two parameters and has a simple derivation from dispersion theory. The scattering length found by Roper is 0.24. We find a scattering length of 0.19 (from the resonance plus inelasticity), while the value obtained by Woolcock[8] from the experimental data is 0.215 ± 0.004.

B. P_{13} Partial Wave

Inelastic effects are known to be very important in the D_{13} partial wave. This is one of the resonances which Ball and Frazer suggested might be accounted for mainly in terms of the inelasticity. Again we fit the inelasticity by Bareyre's results up to $E_L=1$ BeV. According to Bareyre, the inelasticity becomes small at 1 BeV, and the inelastic contribution to δ, δ_{inel}, shown in Fig. 3, is found by letting η be very nearly one above $E_L=1$ BeV. The rapid variation in δ_{inel} near the resonance is not a function of the high-energy dependence of η, but the upper and lower limits on δ will be affected if there is appreciable inelasticity not too far above 1 BeV. With this form for η, we find that the phase shift below $E_L=500$ MeV is almost entirely due to the inelasticity. However, δ_{inel} increases only up to about 600 MeV, at which energy it decreases rapidly (independently of the high-energy behavior of η). One must add in a resonance near this energy to account for the high-energy phase shifts. It is clear that the resonant width which we obtain is appreciably narrower than that found by Roper by fitting the resonance with the Layson form. The agreement with Bareyre's phases is very good up to $E_L=900$ MeV.

C. P_{11} Partial Wave

Our fit to the P_{11} phase shifts is more ambiguous because we require a large amount of inelasticity for $E_L>1$ BeV in order to obtain a reasonably good fit (we use Bareyre's inelasticity up to 1 BeV). There is evidence that the inelasticity is large above 1 BeV,[9] but it is clear that our values for δ_{inel} are very rough. The inelasticity used above 1 BeV was chosen to give a fairly good fit to the low-energy phase shifts.

There are two well-known features of the P_{11} partial wave: (1) the nucleon pole below the elastic threshold and (2) the Roper resonance at $E_L\approx590$ MeV. The contribution from the nucleon pole, given by Eq. (13), is seen to depend only on the nucleon mass. There is, however, another feature of the P_{11} partial wave which is strongly dependent on the pion-nucleon coupling constant. We know that the S matrix is equal to one at the elastic threshold and approaches $-\infty$ near the nucleon pole (the residue of the scattering amplitude is positive because $l=1$). Thus the S matrix must have a zero between s_E and m^2. The zero must actually appear before the beginning of the short cut due to nucleon exchange if it is to appear at all. It is likely that the S matrix is negative before the short cut is reached, since it starts at $W=6.84$ [10,11] and the nucleon pole is at $W=6.7$, while the elastic threshold is at $W=7.7$ ($W=\sqrt{s}$). The position of the zero is clearly strongly dependent on the pion-nucleon coupling constant and has a large effect on the low-energy phase shifts. This is the reason why dynamical calculations of the P_{11} partial wave which give too large a value for the πN coupling also

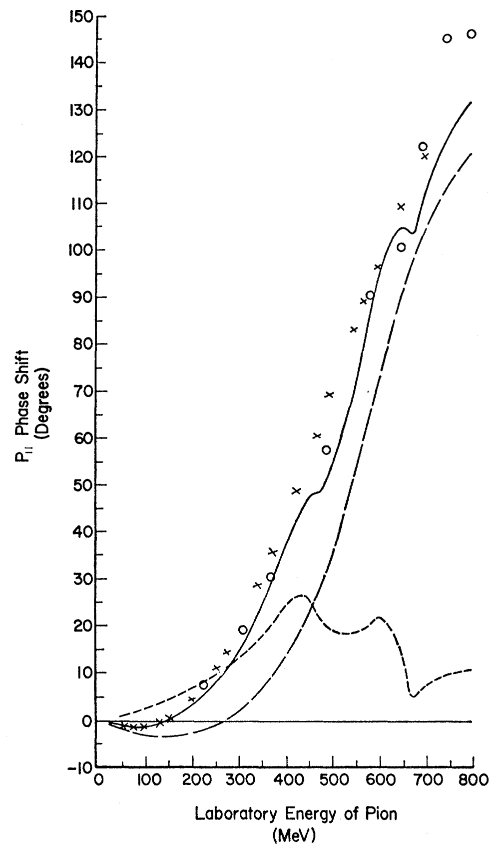

FIG. 4. P_{11} phase shift as a function of incident pion energy. The short-dashed curve is the contribution from the inelastic cut. The long-dashed curve is found from Eq. (12), with $m_R=10.78$ and $\gamma=1.75$, plus the contribution from the nucleon pole at $W=6.7$ and an S-matrix zero at $W=7.16$. The crosses and circles are defined in Fig. 2.

[8] W. S. Woolcock, in *Proceedings of the Aix-en-Provence Conference on Elementary Particles, 1961* (Centre d'Etudes Nucléaires de Saclay, Seine et Dise, 1961), Vol. I, p. 459.

[9] A phase-shift analysis up to $E_L=1300$ MeV suggests that η may be substantially less than 0.1 above 1 BeV [C. Johnson (private communication)]. We never let η get smaller than 0.1 in the form which we assume above 1 BeV.

[10] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960); S. C. Frautschi and J. D. Walacka, *ibid.* **120**, 1486 (1960).

[11] We use units $\hbar=c=m_\pi=1$.

fail to reproduce the low-energy phase-shift behavior.[12] In these cases the zero in the S matrix is too close to the elastic threshold, which causes a large negative phase shift at low energy.

Our fit to the phase shift, shown in Fig. 4, involves three parameters in addition to the inelasticity—the position of the zero in the S matrix near the Roper resonance and the position of the zero on the real axis below threshold.[13] The fit cannot be considered very impressive in view of all this freedom, but the model does account for the qualitative features of the phase shift.[14]

IV. DISCUSSION

The method proposed here can be a useful tool for interpreting phase-shift analyses. In the partial waves treated here, we obtain good fits to the phase shifts even though the left-hand–cut contribution δ_L is neglected. In any case, it is clear that δ_L is a smoothly varying function of s (for physical s), and any sudden variations in δ must be due to zeros in the S matrix or rapid changes in η. This method can be used as a consistency check for a phase-shift analysis: A phase-shift analysis showing rapid changes in η without corresponding fluctuations in δ cannot be self-consistent. (A reliable calculation of the fluctuation due to changes in η can be made without knowing η at all energies.)

An interesting point brought out in this analysis concerns the definition of the elastic width of a resonance. We obtain a considerably narrower width for the D_{13} resonance than Roper does. Inelastic effects due to the decay of a resonance into other channels can cause the elastic width to appear to be much larger than this analysis would imply. Thus it may be misleading to attempt to compute the elastic width of a highly inelastic resonance by simply fitting the resonance with a Breit-Wigner resonant form.

Finally, one might object to this analysis on the grounds that using $\frac{1}{2}(s-s_E)^{1/2}$ instead of k is the major reason for the difference between the Layson resonant form and the form used here (see the Appendix for a comparison). However, a moment's reflection will show that using the Layson form with k replaced by $\frac{1}{2}(s-s_E)^{1/2}$ will tend to give a smaller effective width above the resonance, thus increasing the discrepancy between the Layson form and the one used here.

[12] P. W. Coulter and G. L. Shaw, Phys. Rev. **141**, 1419 (1966).

[13] The phase shift δ_P in Eq. (13) is sensitive to the pole position, but the sum of the phase shift from the nucleon pole and the zero in the S matrix is not very sensitive to the pole position.

[14] An interesting feature of Fig. 4 is the fluctuations in δ associated with fluctuations in η. If η is changing rapidly in a given energy region, one could not expect to get a good fit to δ by using a parametrization that does not allow for sudden changes in δ.

ACKNOWLEDGMENT

It is a pleasure to thank Professor Gordon Shaw for helpful conversations.

APPENDIX

The relation between our resonant phase shift and the elastic Breit-Wigner or Layson phase shift will be demonstrated here. For $l=1$ the resonant form is

$$\delta_R=\tan^{-1}\left[\frac{\gamma(s-s_E)^{1/2}}{m_R^2-s}\right]-\frac{s_E}{s}\frac{\gamma(s-s_E)^{1/2}}{m_R^2-s_E}. \quad \text{(A1)}$$

If the second term is small, we may compute $\tan\delta_R$ by neglecting terms of order γ^3. We find

$$\tan\delta_R=\frac{m_R^2}{s(m_R^2-s_E)}\frac{\gamma(s-s_E)^{3/2}}{m_R^2-s}. \quad \text{(A2)}$$

The corresponding Layson phase shift for the elastic scattering of equal-mass particles is

$$\tan\delta_R=\frac{(s-s_E)^{3/2}}{1+a(s-s_E)}\frac{\gamma}{m_R^2-s}, \quad \text{(A3)}$$

where a is a "range" parameter. We can introduce another parameter in Eq. (A2) by letting the position of the pole be adjustable. If we do this, then the pole position can be chosen so that Eqs. (A2) and (A3) are identical. Thus the range parameter is closely related to the pole position of δ_R. We have chosen the pole position far enough away from the elastic threshold so that δ_R is not very sensitive to the pole position. We regard this insensitivity as a requirement for using our equation.

Thus the resonant phase shift which has been derived here is very nearly the same as the Layson elastic phase shift if the second term in Eq. (A1) is small. There are some differences, however. Near the position of the resonance in the P_{33} partial wave in πN scattering, the second term in Eq. (A1) contributes about $-16°$ to δ_R, so that the phase shift does not go through $90°$ until $s>m_R^2$, whereas the δ_R from the Layson fit is $90°$ at $s=m_R^2$.