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Some Consequences of the Wu-Yang Asymptotic Behavior of Form Factors

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We investigate the properties of a class $\mathfrak{A}$ of form factors $F(t)$ which include some with the asymptotic behavior suggested by Wu and Yang: $F(t)\sim A\exp[-a(-t)^{1/2}]$, $a>0$, for large negative t. A simple representation theorem for such form factors is proved. This theorem is then used as the main tool to obtain the results which follow. "Asymptotic lower bounds" are derived for any $F(t)$ of class $\mathfrak{A}$, both at large positive and at large negative t. It is shown that any form factor belonging to class $\mathfrak{A}$ satisfies a generalized unsubtracted dispersion relation. A whole set of sum rules for ImF and ReF on the cut are obtained, generalizing those which follow from a superconvergent dispersion relation. An asymptotic lower bound on the number of changes of sign of ImF and of ReF along the cut at large positive t is derived. As an illustration, we first construct according to a simple procedure $F(t)$'s which have the given asymptotic behavior and whose only singularities are simple poles. We then derive an asymptotic lower bound on the number of these poles at large positive t for any $F(t)$ of this type.

I. INTRODUCTION

RECENT electron-proton scattering measurements[1,2] are compatible with the suggestion of Wu and Yang[3] that the electromagnetic form factors of elementary particles might decrease like $\exp(-a|t|^{1/2})$, $a>0$, for large spacelike (that is, negative) values of the squared momentum transfer t.

The first comment we wish to make in this connection is the following. In the framework of our present theoretical description of elementary particles, it is assumed or proved, as the case may be, that form factors have properties of analyticity and moderate growth. In view of these properties, asymptotic behaviors of the Wu-Yang kind seem to have some special significance in the following sense. It has been proved under rather general conditions[4,5] that form factors cannot decrease faster than $\exp(-a|t|^{1/2})$ for every positive a as t tends to $-\infty$ (unless they vanish identically). We could express this fact by saying that the behavior postulated in Ref. 3 is an "asymptotic lower bound" for form factors at large negative t.

Our next comments apply only to a narrower class $\mathfrak{F}$ of form factors. Class $\mathfrak{F}$ is defined as the class of those form factors $F(t)$ which fulfill the following conditions: They are holomorphic in the t-plane cut along the real axis from some finite b to $+\infty$, they satisfy the reality condition $F(t^*)=[F(t)]^*$ (an asterisk means complex conjugation), and they are polynomially bounded as $|t|$ tends to ∞. [Further assumptions will be made in Sec. II to ensure that the F's have well-behaved boundary values on the cut. In this Introduction, we shall take for granted that these boundary values are continuous, and that they are smoothly reached.] It is expected for well-known mathematical reasons that the properties of a given form factor F of class $\mathfrak{F}$ will be very closely related to the properties of its discontinuity on the cut, which we shall call ImF, following a widespread usage. The hope is that the properties of ImF will have a reasonable physical interpretation. If we now require that the given F have the Wu-Yang asymptotic behavior, we have to describe the effect that these additional requirements have on the properties of ImF.

Let us state the Wu-Yang assumption in the weakened form we have found convenient to work with. We shall assume that F satisfies

$$\limsup_{t\to-\infty} |t|^{-1/2}\ln|F(t)|\leq -a \tag{1.1}$$

for some positive a. [From Sec. II on, we shall denote by $\mathfrak{A}$ the class of those form factors of class $\mathfrak{F}$ which satisfy a condition like (1.1).]

The known consequences of condition (1.1) for ImF can be summarized as follows:

(a) It has been proved by one of us[6] that if (1.1) is satisfied, then Im$F(t)$ cannot decrease as fast as $\exp(-\beta t^{1/2})$ as t tends to $+\infty$ (β: any given positive number, however small), unless F is identically zero. Therefore, (1.1) implies an asymptotic lower bound for ImF for large positive t.[7]

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[1] S. D. Drell, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, Calif., 1966* (University of California Press, Berkeley, Calif., 1967), p. 85.
[2] W. Albrecht, H. J. Behrend, F. W. Brasse, W. Flauger, H. Hultschig, and K. G. Steffen, Phys. Rev. Letters **17**, 1192 (1966).
[3] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).
[4] A. Martin, Nuovo Cimento **37**, 671 (1965).
[5] A. M. Jaffe, Phys. Rev. Letters **17**, 661 (1966); Phys. Rev. **158**, 1454 (1967).
[6] A. P. Balachandran, Nuovo Cimento **42A**, 804 (1966).
[7] Slightly better results have been obtained in Ref. 5.

(b) It has been proved by Martin in Ref. 4 that ImF determines F uniquely when (1.1) is satisfied.

(c) It is generally known that conditions of decrease of the kind (1.1) sometimes imply superconvergence sum rules for ImF. Sum rules of this and related types have been discussed and put to various uses in many places.[8,9] More precisely, the results one has are of the following nature. Assume that the form factor F is given by the equation

$$F(t)=\pi^{-1}\int_b^\infty (\tau-t)^{-1}\,\mathrm{Im}F(\tau)d\tau\,, \qquad (1.2)$$

where the real-valued measurable function ImF satisfies

$$\int_b^\infty |\mathrm{Im}F(\tau)|\,\tau^\alpha d\tau<\infty \qquad (1.3)$$

for all $\alpha\geq 0$. Then in order that

$$\lim_{t\to-\infty} |F(t)|\,|t|^\alpha=0$$

for all $\alpha\geq 0$, it is necessary and sufficient that the sum rules

$$\int_b^\infty \tau^N\,\mathrm{Im}F(\tau)d\tau=0 \qquad (1.4)$$

be fulfilled for all non-negative integers N.

(d) It is well known that conditions of decrease like (1.1) imply that ImF has to change sign an infinite number of times.[10]

The form factors F of the class $\mathfrak{F}$ just described are usually represented in terms of ImF by a dispersion integral like (1.2), with some subtractions if needed. When condition (1.1) is imposed, it follows from (b) that the subtraction polynomial will be uniquely given by ImF. This welcome feature, however, should not let us forget consequence (d), which indicates that ImF has of necessity a complicated asymptotic behavior for large positive t if (1.1) has to be satisfied.

The basic idea of the present article is to use a representation for a given form factor F satisfying the Wu-Yang assumption in terms of a (generalized) function S subjected only to a very simple support condition deduced from (1.1). This representation is explained in Sec. II. The simplicity of the condition on S is in striking contrast to the complicated nature of ImF as reflected in statement (d).

More precisely, our starting point is the following. If F is a form factor of the class $\mathfrak{F}$ which satisfies condition (1.1), then we show that we can write it as

$$F(t)=\int_{-\infty}^{+\infty} S(\xi)\,\exp[-(b-t)^{1/2}\xi]d\xi\,, \qquad (1.5)$$

where $0<\arg(t-b)<2\pi$ and S is a real-valued tempered distribution on the real line whose support is contained in $\{\xi\,|\,\xi\geq a\}$. Conversely, if S is a tempered distribution with these properties, formula (1.5) defines a function of class $\mathfrak{F}$ satisfying (1.1).

Notice that the asymptotic lower bound for F at large negative t follows immediately from this statement (see Sec. IIIB).

In Secs. III and IV, we use this representation to rederive, refine, and generalize the results listed above about the properties of ImF which follow from a condition like (1.1). We give here a brief and somewhat imprecise summary of our results.

The main result of Sec. III is the following generalized unsubtracted dispersion relation:

$$F(t)=\lim_{\delta\to+0}\pi^{-1}\int_b^\infty (\tau-t)^{-1}\,\mathrm{Im}F(\tau)e^{-\delta\tau}d\tau\,. \qquad (1.6)$$

Equation (1.6) may be considered as a refinement of statement (b) above, which follows from it as a corollary. Other results are asymptotic lower bounds for F.

In Sec. IV, we derive a whole set of generalized sum rules. Examples are

$$\lim_{\delta\to+0}\int_b^\infty \tau^N\,\mathrm{Im}F(\tau)e^{-\delta\tau}d\tau=0 \qquad (1.7)$$

for all non-negative integers N, and the continuous infinity of sum rules

$$\lim_{\delta\to+0}\int_b^\infty \mathrm{Re}[F(\tau+i0)\,\exp(-i\eta(\tau-b)^{1/2})]\times\exp[-\delta\tau]d\tau=0 \qquad (1.8)$$

for all $\eta<a$. Relations (1.7) are generalized versions of

[8] We give a short bibliography of the subject. For form factors, see: R. G. Sachs, Phys. Rev. **126**, 2256 (1962); A. P. Balachandran, P. G. O. Freund, and C. R. Schumacher, Phys. Rev. Letters **12**, 209 (1964); and Ref. 6. For partial waves, see: G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957); A. P. Balachandran and F. von Hippel, Ann. Phys. (N. Y.) **30**, 446 (1964); A. P. Balachandran, J. Math. Phys. **5**, 614 (1964); Ann. Phys. (N. Y.) **30**, 476 (1964); M. Kugler, Phys. Rev. Letters **17**, 1166 (1966); and Phys. Rev. **160**, 1574 (1967). For forward scattering amplitudes, see: M. L. Goldberger, H. Miyazawa, and R. Oehme, Phys. Rev. **99**, 986 (1955); A. P. Balachandran, Phys. Rev. **134**, B197 (1964). For forward and fixed-t scattering amplitudes, see: V. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966); L. D. Solov'ev, Yadern. Fiz. **3**, 188 (1966) [English transl.: Soviet J. Nucl. Phys. **3**, 131 (1966)]; T. L. Trueman, Phys. Rev. Letters **17**, 1198 (1966); P. Babu, F. J. Gilman, and M. Suzuki, Phys. Letters **24B**, 65 (1967); B. Sakita and K. C. Wali, Phys. Rev. Letters **18**, 29 (1967); M. B. Halpern, Phys. Rev. **160**, 1441 (1967); R. Musto and F. Nicodemi, Nuovo Cimento **49A**, 333 (1967); R. H. Graham and M. Huq, Phys. Rev. **160**, 1421 (1967); N. J. Papastamatiou and S. Pakvasa, Phys. Rev. **161**, 1554 (1967); and a very large number of other recent papers. For fixed-angle scattering amplitudes, see A. P. Balachandran, Phys. Rev. **137**, B177 (1965).

[9] Reference 6 makes use of the sum rules to prove the result mentioned under (a).

[10] See Ref. 3. In Ref. 4, Martin notices that this follows from a theorem of Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

(1.4) which hold even when (1.3) does not.[11] The distribution of "changes of sign" of ImF is then investigated with the aid of the sum rules and the asymptotic lower bound for ImF at large positive t. It is shown that given any real number T, however large, ImF must change sign infinitely often on the half-line $t \geq T$. We also show that if $n(\tau)$ denotes the number of changes of sign of ImF in $T \leq t \leq \tau$, then

$$\limsup_{\tau\to+\infty} \tau^{-1/2} n(\tau) \geq \pi^{-1} a\,, \tag{1.9}$$

where a is as in condition (1.1).

We observe that all the results on ImF stated in Secs. III and IV can be easily taken over to ReF on the cut by considering $F_1(t) = (b-t)^{1/2} F(t)$ instead of $F(t)$. Clearly, if F belongs to $\mathfrak{F}$ and satisfies condition (2.1), F_1 does so too, and

$$\mathrm{Im}F_1(t) = -(t-b)^{1/2}\theta(t-b)\ \mathrm{Re}F(t)\,.$$

Section V illustrates some of our results. The kind of examples we have chosen represents a concession to current fashion. We give a prescription to construct form factors F which have the Wu-Yang asymptotic behavior and whose only singularities are simple poles.[12] We also show for such F's that if $n_0(\tau)$ denotes the number of δ functions in ImF with support in $t \leq \tau$, then

$$\limsup_{\tau\to+\infty} \tau^{-1/2} n_0(\tau) \geq \pi^{-1} a\,.$$

If these poles are interpreted as resonances, and if the value of a found in Ref. 2 is used, one would conclude that it follows from the Wu-Yang asymptotic behavior that there are an infinite number of resonances with masses spaced (asymptotically) no more than about 2 BeV apart. We prefer to leave such an inference to the reader, however.

The Appendix contains a function-theoretic lemma used in one proof.

For mathematical convenience, many of our considerations involve distribution-theoretic arguments.[13] We believe, however, that a reader not familiar with this theory should be able to follow the proofs by the standard device of treating distributions as ordinary functions. In this spirit, we ourselves often use in what follows a symbolic functionlike notation to express distribution-theoretic relations.

We have aimed at mathematical rigor. However, many arguments are not spelled out completely and are given in an intuitive form.[14] It is our conviction that the interested reader will be able to complete our arguments without difficulty.

II. REPRESENTATION THEOREM

A. General Assumptions and Definitions

The form factors we consider in the present paper will all belong to the following class of analytic functions. Let a be a *positive* number, and let b be a real number. We denote by $\mathfrak{A}$ the class of those functions F which have the following properties:

(A1) F is holomorphic in the t-plane cut along the real axis from b to $+\infty$.

(A2) There are positive numbers α, β, and K, depending on F, such that, for all t in the cut plane,

$$|F(t)| < K(1+|t|)^{\alpha}[\Delta(t)]^{-\beta}\,,$$

where $\Delta(t)$ denotes the distance from t to the cut.

(A3) We have (τ real)

$$\limsup_{\tau\to-\infty} |\tau|^{-1/2} \ln|F(\tau)| \leq -a\,.$$

(A4) For $\tau < b$, $F(\tau)$ is real.

Notice that we do not require F to be actually discontinuous on the entire cut. Neither have we excluded poles on the half-line $\tau \geq b$. Remark also that if we multiply F in $\mathfrak{A}$ by a polynomial with real coefficients, we obtain again a function in $\mathfrak{A}$. Similarly, if we divide F in $\mathfrak{A}$ by a polynomial whose zeroes are all on the cut, then we obtain again a function in $\mathfrak{A}$.

It is convenient to introduce a second class of functions for reasons which will soon become apparent. Let us denote by $\mathfrak{B}$ the class of those functions G which satisfy the following conditions:

(B1) G is holomorphic in the upper-half u plane (Im$u>0$).

(B2) There are positive numbers γ, δ, and L, depending on G, such that, for all u with Im$u>0$,

$$|G(u)| < L(1+|u|)^{\gamma}(\mathrm{Im}u)^{-\delta}\,.$$

(B3) We have (σ real)

$$\limsup_{\sigma\to+\infty} \sigma^{-1} \ln|G(i\sigma)| \leq -a\,.$$

(B4) For $\sigma > 0$, $G(i\sigma)$ is real.

It is possible to associate each function F in $\mathfrak{A}$ with a function G in $\mathfrak{B}$ in the following way. The relation

$$t-b=u^2 \tag{2.1}$$

[11] The sum rules of R. F. Dashen and M. Gell-Mann [in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Co., San Francisco, 1966)] and S. Fubini [Nuovo Cimento **43A**, 475 (1966)] can be inserted into these equations to obtain constraints on the absorptive parts of scattering amplitudes.

[12] Models leading to form factors of the Wu-Yang type have been discussed by S. D. Drell, A. C. Finn, and M. H. Goldhaber, Phys. Rev. **157**, 1402 (1967); and J. D. Stack, Phys. Rev. **164**, 1904 (1967).

[13] Our main source on this subject is L. Schwartz, *Théorie des Distributions* (Hermann & Cie., Paris, 1957), Vols. I and II.

[14] For example, we speak freely of real-valued distributions without bothering to define what we mean by this concept. Also, in a symbolic expression like (2.3), we do not comment on the fact that $e^{iu\xi}$, considered as a function of ξ for a fixed u with Im$u>0$, is not actually in $\mathcal{S}$, and that (2.3) makes sense only because of the support properties of S.

establishes a one-to-one correspondence between the cut t plane and the upper-half u plane. For any function F defined on the cut t plane, we define the corresponding function G on the upper-half u plane by

$$G(u)=F(t)\,, \tag{2.2}$$

t and u being related by (2.1). Obviously, the correspondence so defined between G and F is one-to-one. We shall say that F and G are *associated* with each other. One shows easily that *G is in $\mathcal{B}$ if and only if its associated function F is in $\mathcal{A}$.*

B. A Useful Theorem

The reason behind our assumption (B2) [or (A2)] is to be found in our wish to make use of the following beautiful result.[15]

Theorem II.B.1

In order that a function G have properties (B1) and (B2) it is necessary and sufficient that there exist a tempered distribution S on the real line whose support is contained in $\{\xi|\xi\geq 0\}$, such that we have the following (symbolic) relation for $\operatorname{Im}u>0$:

$$G(u)=\int_{-\infty}^{+\infty} e^{iu\xi}S(\xi)d\xi\,. \tag{2.3}$$

This theorem is proved in Ref. 16. It can be reformulated in the following way. It is shown in Ref. 13 (Chap. VII, Th. VI) that any tempered distribution S is the nth derivative (in the sense of distributions; n depends on S) of some continuous function which is polynomially bounded. Therefore, *in order that a function G have properties (B1) and (B2), it is necessary and sufficient that there exist a non-negative integer n and a continuous function f on the real line such that*

$$f(\xi)=0$$

for $\xi\leq 0$,

$$|f(\xi)|<C(1+|\xi|)^{\alpha}\,,$$

for some positive C and α, and

$$G(u)=(-iu)^n\int_0^{\infty} f(\xi)e^{iu\xi}d\xi \tag{2.4}$$

for $\operatorname{Im}u>0$. The distribution S in (2.3) is the nth derivative of f in (2.4) in the sense of distributions.

Theorem II.B.1 has the following corollary.[17]

Theorem II.B.2

If a function G has properties (B1) and (B2), then there exists a tempered distribution G_+ on the real line such that, for all φ in $\mathcal{S}$,

$$G_+(\varphi)=\lim_{\sigma\to+0}\int_{-\infty}^{+\infty} G(\rho+i\sigma)\varphi(\rho)d\rho\,. \tag{2.5}$$

We shall sometimes write symbolically

$$G_+(\rho)=G(\rho+i0)\,.$$

The tempered distribution G_+ is the Fourier transform, in the sense of the theory of distributions, of the tempered distribution S. Symbolically,

$$G_+(\rho)=\int_{-\infty}^{+\infty} e^{i\rho\xi}S(\xi)d\xi\,,$$
$$S(\xi)=(2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-i\rho\xi}G_+(\rho)d\rho\,. \tag{2.6}$$

C. Representation Theorem

From Theorem II.B.1, we can derive the following generalization of the Paley-Wiener theorem.

Representation Theorem II.C.1

The function G belongs to $\mathcal{B}$ if and only if there exists a real-valued tempered distribution S on the real line whose support is contained in $\{\xi|\xi\geq a\}$, such that we have the following (symbolic) relation for $\operatorname{Im}u>0$:

$$G(u)=\int_{-\infty}^{+\infty} e^{iu\xi}S(\xi)d\xi\,. \tag{2.7}$$

Proof (sketch): (α) Sufficiency. If the condition is satisfied, it follows from Theorem II.B.1 that G satisfies (B1) and (B2). (B4) follows from the reality of S. Let us show that (B3) also is satisfied. In (2.4) we can assume that f is real and that

$$f(\xi)=0\,,$$

for $\xi\leq a$. Therefore,

$$\sigma^{-n}G(i\sigma)=\int_a^A f(\xi)e^{-\sigma\xi}d\xi+\int_A^{\infty} f(\xi)e^{-\sigma\xi}d\xi$$
$$=I_1+I_2,\quad(\sigma>0,\ A>a)\,.$$

Let M be the maximum of $|f(\xi)|$ in the interval $[a,A]$; then

$$|I_1|\leq M\int_a^A e^{-\sigma\xi}d\xi<M\sigma^{-1}\exp(-a\sigma)\,.$$

[15] For the notion of tempered distribution, see Ref. 13, Chap. VII.

[16] Seminar notes by students of M. Zerner, University of Marseille, France (unpublished).

[17] $\mathcal{S}$ is defined in Ref. 13, Chap. VII, Sec. 3. It is the set of those C^∞ functions $\varphi(\rho)$ on the real line [i.e., of those functions $\varphi(\rho)$ on the real line which are differentiable infinitely many times] which together with all their derivatives vanish at infinity more rapidly than any power of ρ^{-1}. $\mathcal{S}$ is equipped with the topology described in Ref. 13.

For any positive number ϵ, and for $\sigma>\epsilon$,

$$|I_2|=\left|\int_A^\infty f(\xi)\exp(-(\sigma-\epsilon)\xi)\exp(-\epsilon\xi)d\xi\right| \leq N\exp(-(\sigma-\epsilon)A),$$

where

$$N=\int_A^\infty |f(\xi)|\exp(-\epsilon\xi)d\xi.$$

(B3) follows from these estimates.

(β) *Necessity.* Since G satisfies (B1)–(B3), the function G_1, defined by

$$G_1(u)=(u+i)^{-m}G(u),$$

where m is a large enough integer, satisfies (B1), (B3), and, instead of (B4), an estimate of the form

$$|G_1(u)|<L(\mathrm{Im}u)^{-\delta}.$$

Using now Theorem 6.2.4 of Ref. 18 for any half-plane $\mathrm{Im}u\geq\sigma(>0)$, we find that

$$|e^{-iau}G_1(u)|<L(\mathrm{Im}u)^{-\delta}.$$

Therefore, $e^{-iau}G_1(u)$ satisfied (B1) and (B2). By Theorem II.B.1, we thus have (symbolically)

$$e^{-iau}G(u)=\int_{-\infty}^{+\infty}S_0(\xi)e^{iu\xi}d\xi,$$

where the tempered distribution S_0 has its support in $\{\xi|\xi\geq 0\}$. Putting (symbolically)

$$S(\xi)=S_0(\xi-a),$$

we get (2.7). The reality of S follows from (B4). Q.E.D.

In an obvious way, we get from this theorem the following.

Corollary II.C.2

F is in $\mathcal{A}$ if and only if there exists a real tempered distribution S on the real line whose support is contained in $\{\xi|\xi\geq a\}$, such that for all t in the cut plane we have the (symbolic) formula

$$F(t)=\int_{-\infty}^{+\infty}d\xi\, S(\xi)\exp[-\xi(b-t)^{1/2}]d\xi, \quad (2.8)$$

where $|\arg(b-t)|<\pi$.

D. Remarks on the Boundary Values of Functions Belonging to $\mathcal{A}$ and $\mathcal{B}$

In the sections which follow, we shall show how to make use of the representation theorem to obtain various properties of the values on the cut of form factors F which belong to class $\mathcal{A}$. We therefore conclude the present section by making a few remarks about the boundary values of functions in $\mathcal{A}$ and $\mathcal{B}$.

If F belongs to $\mathcal{A}$, it satisfies obviously (B1) and (B2). Therefore, by Theorem II.B.2, there is a tempered distribution F_+ such that, symbolically,

$$F(\tau+i0)=F_+(\tau).$$

A similar argument shows that there is a tempered distribution F_- such that, symbolically,

$$F(\tau-i0)=F_-(\tau).$$

F_- can be seen to be the complex conjugate of F_+, because of (A4). We define $\mathrm{Im}F$ as

$$\mathrm{Im}F=(2i)^{-1}(F_+-F_-). \quad (2.9)$$

It is a real-valued tempered distribution whose support is contained in $\{\tau|\tau\geq b\}$, because of (B4).

Similarly, if G is in $\mathcal{B}$, we have a tempered distribution G_+ such that, symbolically,

$$G_+(\rho)=G(\rho+i0). \quad (2.10)$$

We shall frequently consider the distribution G_i, given symbolically by

$$G_i(\rho)=(2i)^{-1}[G_+(\rho)-G_+(-\rho)]. \quad (2.11)$$

If F in $\mathcal{A}$ and G in $\mathcal{B}$ form a pair of associated functions (in the sense of Sec. IIA), then there is a one-to-one correspondence between $\mathrm{Im}F$ and G_i. To see this, assume for a moment that G can be extended to a continuous function on the closed upper half-plane $\mathrm{Im}u\geq 0$. Then $\mathrm{Im}F$ and G_i are ordinary functions, and

$$\begin{aligned}\mathrm{Im}F(b+\rho^2)&=+G_i(\rho) \quad \text{for} \quad \rho>0\\ &=-G_i(\rho) \quad \text{for} \quad \rho<0.\end{aligned} \quad (2.12)$$

(Remember that two locally integrable functions which are equal almost everywhere define the same distribution.[19]) In the general case, the rules which enable one to compute the distribution $\mathrm{Im}F$ given the distribution G_i, and conversely, are more complicated. We give them below, together with their heuristic justification.

Assume first that G_i is given. To find $\mathrm{Im}F(\varphi)$, or, symbolically, to find

$$\int_{-\infty}^{+\infty}\mathrm{Im}F(\tau)\varphi(\tau)d\tau$$

for any φ in $\mathcal{S}$, let us look at the result of a formal change of variables. We find, using (2.12),

$$\int_{-\infty}^{+\infty}\mathrm{Im}F(\tau)\varphi(\tau)d\tau=\int_{-\infty}^{+\infty}G_i(\rho)\rho\varphi(b+\rho^2)d\rho.$$

But if φ is in $\mathcal{S}$, $\Gamma(\varphi)$, defined by

$$\Gamma(\varphi)(\rho)=\rho\varphi(b+\rho^2)$$

[18] R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954).

[19] See Ref. 13, Chap. I.

is also in $\mathcal{S}$, so that the integral on the right-hand side is well defined. Thus we guess that the following is true.

Proposition II.D.1

If F in $\mathfrak{A}$ and G in $\mathfrak{B}$ are associated with each other, then

$$\operatorname{Im}F(\varphi)=G_i(\Gamma(\varphi))$$

for all φ in $\mathcal{S}$.

Assume next that ImF is given. Let φ be in $\mathcal{S}$. Then we formally get from (2.12)

$$G_i(\phi)=\int_{-\infty}^{+\infty} G_i(\rho)\varphi(\rho)d\rho=\int_b^{\infty} \operatorname{Im}F(\tau)\psi(\tau)d\tau\,, \tag{2.13}$$

where

$$\psi(\tau)=\tfrac{1}{2}(\tau-b)^{-1/2}[\varphi((\tau-b)^{1/2})-\varphi(-(\tau-b)^{1/2})]. \tag{2.14}$$

But the meaning of the integral on the right is ambiguous. One correct way to get around this difficulty is the following. We recall once more that, being a tempered distribution, ImF is the nth derivative in the sense of distributions of a continuous, polynomially bounded function f. Therefore, it can be extended by continuity from $\mathcal{S}$ to $\mathcal{S}_n$.[20] Let ψ_0, ψ_1, $\cdots$ be a sequence of functions in $\mathfrak{D}$ [21] such that

$$\psi_l^{(k)}(b)=\delta_{lk}.$$

We define for each non-negative m a continuous map Γ_m from $\mathcal{S}$ into $\mathcal{S}_m$ by

$$\Gamma_m(\varphi)(\tau)=\psi(\tau) \quad \text{for} \quad \tau>b$$
$$=c_0\psi_0(\tau)+\cdots+c_m\psi_m(\tau)\,, \quad \text{for} \quad \tau\leq b$$

where ψ is as in Eq. (2.14), and

$$c_k=\lim_{\tau\to b+0}\psi^{(k)}(\tau)=k![(2k+1)!]^{-1}\varphi^{(2k+1)}(0).$$

We are now ready to state the content of Eq. (2.13) in a precise way.

Proposition II.D.2

If F in $\mathfrak{A}$ and G in $\mathfrak{B}$ are associated with each other, and if

$$\operatorname{Im}F=d^nf/d\tau^n\,,$$

where f is a continuous, polynomially bounded function, then

$$G_i(\varphi)=\operatorname{Im}F(\Gamma_n(\varphi))$$

for all φ in $\mathcal{S}$.

We shall not give a proof of these two propositions. We hope that the heuristic arguments given above will convince the reader of their truth. A complete proof can be based on the following fact: *Let φ be the Fourier transform of a function in $\mathfrak{D}$* (φ is therefore an entire function of u, say, whose restriction to lines Imu= const is in $\mathcal{S}$). *Let G and G_+ be as in Theorem II.B.2. Then, for all $\sigma>0$,*

$$G_+(\varphi)=\int_{-\infty}^{+\infty} G(\rho+i\sigma)\varphi(\rho+i\sigma)d\rho\,.$$

III. APPLICATIONS OF THE REPRESENTATION THEOREM, ASYMPTOTIC LOWER BOUNDS, AND THE GENERALIZED DISPERSION RELATION

A. General Remarks

According to the representation theorem stated in Sec. IIC, if F is in $\mathfrak{A}$, then the boundary values of G_+ and G_i [Eqs. (2.10) and (2.11)] of its associated function G in $\mathfrak{B}$ are Fourier transforms of tempered distributions S and S_i which are related symbolically by

$$S_i(\xi)=(2i)^{-1}[S(\xi)-S(-\xi)]. \tag{3.1}$$

The results we obtain in this and the next section rest upon the important fact that since a is *positive*, S is known to have its support *in a closed half-line which does not contain the origin*. Therefore: (a) S_i also is zero in a neighborhood of the origin.[22] (b) S can be recovered unambiguously once S_i is given.[23]

B. Asymptotic Lower Bounds

As we already remarked in the Introduction, the following asymptotic lower bound for F at large negative t follows immediately from the representation theorem: *If F satisfies (A1), (A2), and (A4), and if for τ real*

$$\lim_{\tau\to-\infty}\tau^{-1/2}\ln|F(\tau)|=-\infty\,, \tag{3.2}$$

then F is identically zero. For it then follows from (3.2) that F satisfies (A3) for any positive a, however large. Therefore, by the corollary of Section II C, the support of the tempered distribution S appearing in (2.8) is empty. Thus, S is zero.[24] With it, F is (identically) zero, by (2.8).

Let F be in $\mathfrak{A}$. Let μ and ν be real numbers, and let $\nu<b$. Then

$$R_\mu(\tau)=\exp[\mu(\tau-\nu)^{1/2}]\operatorname{Im}F(\tau)$$

is a well-defined distribution (in $\mathfrak{D}'$ [25]) because of the support properties of ImF.[26]

[20] $\mathcal{S}_n$ is the set of those functions $\varphi(\rho)$ which have n continuous derivatives, and which together with their first n derivatives vanish at infinity more rapidly than any power of ρ^{-1}. $\mathcal{S}_n$ is equipped with the topology given by the seminorms $p_{k,l}(\varphi)=\sup\{(1+\rho^2)^l|\varphi^{(k)}(\rho)|\,|-\infty<\rho<+\infty\}$, k,l: non-negative integers, $k\leq n$. $\mathcal{S}$ is dense in $\mathcal{S}_n$.

[21] $\mathfrak{D}$ is the set of those functions $\varphi(\rho)$. which are infinitely differentiable, and which vanish outside a compact interval. See Ref. 13, Chap. I.

[22] For this concept, see Ref. 13, Chap. I, Sec. 3.

[23] Namely, because of (a), $\theta(\xi)S_i(\xi)$ has an unambiguous meaning and can be seen to be equal to $(2i)^{-1}S(\xi)$; cf. (3.1).

[24] This follows from the fact that $\mathfrak{D}$ is dense in $\mathcal{S}$ (Ref. 13, Chap. VII, Théorème III) and from the definition of the support of a distribution (Ref. 13, Chap. I., Sec. 3).

[25] Cf. Ref. 13, Chap. I., Sec. 2.

[26] Cf. Ref. 13, Chap. V, Sec. 1.

Proposition III.B.1

If there is a positive number γ such that R_γ is a tempered distribution, then F is identically zero.

Proof: Let G be the function in $\mathcal{B}$ which is associated with F. Put

$$T_\mu(\rho)=\exp(\mu\rho)G_i(\rho).$$

Because of Proposition II.D.2, it follows that T_γ and $T_{-\gamma}$ are tempered distributions. Therefore, the Fourier transform S_i of G_i is a function which is holomorphic in the strip $|\mathrm{Im}\xi|<\gamma$.[27] But it vanishes (almost everywhere) on the real axis between $-a$ and $+a$. Therefore, $S_i=0$. This implies $S=0$, i.e., $F=0$ by (2.8). Q.E D.

This result may be interpreted as giving an asymptotic lower bound for ImF at large positive t. As remarked in the Introduction, a similar result can be obtained for the real part of F on the cut by considering the function $F_1(t)=(b-t)^{1/2}F(t)$ instead of F.

More refined results might have been obtained by using the notion of quasianalyticity, as discussed in Ref. 5.

C. The Generalized Dispersion Relation

We are going to prove now that form factors of class $\mathcal{A}$ satisfy a *generalized dispersion relation without "subtractions."* This result is a refinement of a uniqueness theorem due to Martin[4] which can be formulated in the following way: *Let F_1 and F_2 be form factors belonging to the class $\mathcal{A}$. If*

$$\mathrm{Im}F_1=\mathrm{Im}F_2,$$

then

$$F_1=F_2.$$

We first derive the following lemma. Let G be a function in $\mathcal{B}$. For $\delta>0$, define the function G_δ by the symbolic expression

$$G_\delta(u)=\pi^{-1}\int G_i(\rho)(\rho-u)^{-1}\exp(-\delta\rho^2)d\rho \qquad (3.3)$$

for Im$u>0$.[28]

We then have

Lemma III.C.1

$$\lim_{\delta\to+0}G_\delta(u)=G(u),$$

uniformly in u when u lies in any compact set contained in $\{u|\mathrm{Im}u>0\}$.

Proof: For $\delta>0$, Im$u>0$, define a function $\chi_{\delta,u}$ by

$$\chi_{\delta,u}(\xi)=(2\pi)^{-1}\int_{-\infty}^{+\infty}\exp(i\rho\xi)(\rho-u)^{-1}\exp(-\delta\rho^2)d\rho.$$

This function belongs to $\mathcal{S}$. We have

$$G_\delta(u)=2S_i(\chi_{\delta,u}) \qquad (3.4)$$

[We have used Eqs. (2.6), (2.11), (3.1), and (3.3). Remember the definition of the Fourier transform of a tempered distribution.[29]] Because of the support properties of S_i,

$$S_i(\chi_{\delta,u})=S_i(\alpha\chi_{\delta,u}),$$

where α is any C^∞ function which is equal to 1 for $|\xi|\geq a$ and to 0 for $|\xi|\leq\frac{1}{2}a$, say. But $\alpha\chi_{\delta,u}$ tends to $\chi_{0,u}(\xi)=i\alpha(\xi)\theta(\xi)e^{iu\xi}$ in $\mathcal{S}$ when δ tends to 0 through positive values, uniformly in u for u in any compact set contained in the half-plane $\{u|\mathrm{Im}u>0\}$. The lemma now follows of the representation theorem with the help of (3.1) and (3.4). Q.E.D.

Let F be a form factor in the class $\mathcal{A}$. For $\delta>0$, define the function F_δ by the symbolic expression

$$F_\delta(t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Im}F(\tau)}{\tau-t}\exp(-\delta\tau)d\tau. \qquad (3.5)$$

Proposition III.C.2

We have as a corollary to the preceding lemma:

$$\lim_{\delta\to+0}F_\delta(t)=F(t)$$

uniformly in t when t lies in any compact set contained in the complement of the cut $\{\tau|\tau\geq b\}$.

Proof: Let G be the function in $\mathcal{B}$ which is associated F. Using Proposition II.D.1 and the symbolic relation

$$G_i(\rho)+G_i(-\rho)=0,$$

we find

$$F_\delta(b+u^2)=\exp(-\delta b)G_\delta(u).$$

Q.E.D.

From this result, it is easy to obtain the following (symbolic) formulas for the derivatives of F:

$$F^{(n)}(t)=\lim_{\delta\to+0}\frac{n!}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Im}F(\tau)}{(\tau-t)^{n+1}}\exp(-\delta\tau)d\tau, \qquad (3.6)$$

valid for t in the cut plane.

If ImF turns out to be a measurable function such that

$$\int_b^\infty|\mathrm{Im}F(\tau)|(1+|\tau|)^{-m-1}d\tau<\infty$$

for some positive integer m, then it is easy to derive from Proposition III.C.2 the usual subtracted dispersion relation:

$$F(t)=c_0+c_1(t-d)+\cdots+c_{m-1}(t-d)^{m-1}+\frac{(t-d)^m}{\pi}\int_b^{+\infty}\frac{\mathrm{Im}F(\tau)d\tau}{(\tau-d)^m(\tau-t)},$$

where d is any number in the cut plane, and where the

[27] L. Schwartz, Medd. Lunds Mat. Semin, Suppl., 196 (1952).

[28] For any $\delta>0$ and any u with Im$u\neq0$, the function $\rho\to(\rho-u)^{-1}\exp(-\delta\rho^2)$ belongs clearly to $\mathcal{S}$

[29] Cf. Ref. 13, Chap. VII, Sec. 6.

subtraction constants c_k are given in terms of ImF by

$$c_k = \lim_{\delta\to+0} \frac{1}{\pi} \int_b^\infty \frac{\mathrm{Im}F(\tau)}{(\tau-d)^{k+1}} \exp(-\delta\tau)d\tau$$

[cf. Eq. (3.6)].

IV. FURTHER APPLICATIONS OF THE REPRESENTATION THEOREM. GENERALIZED SUM RULES. AN ASYMPTOTIC LOWER BOUND FOR THE RATE OF OSCILLATION OF ImF

A. Generalized Sum Rules

In order to illustrate the main idea of the proof of the sum rules stated in Proposition IV.A.2 below, we first derive a simpler set of sum rules.

Proposition IV.A.1

Let G be in $\mathcal{B}$, and let G_+ be its boundary value according to Theorem II.B.2. For $\delta>0$, η real, let the function $\Delta_{\delta,\eta}$ in $\mathcal{S}$ be defined by

$$\Delta_{\delta,\eta}(\rho) = \exp(-i\rho\eta)\exp(-\delta\rho^2).$$

Then

$$\lim_{\delta\to+0} G_+(\Delta_{\delta,\eta}) = 0 \tag{4.1}$$

for all $\eta<a$.

Proof: Define $\Theta_{\delta,\eta}$ by

$$\Theta_{\delta,\eta}(\xi) = \int_{-\infty}^{+\infty} \Delta_{\delta,\eta}(\rho)\exp(i\xi\rho)d\rho = (\pi/\delta)^{1/2}\exp[-(4\delta)^{-1}(\xi-\eta)^2].$$

Using the definition of the Fourier transform of a tempered distribution, we get

$$G_+(\Delta_{\delta,\eta}) = S(\Theta_{\delta,\eta})$$

[cf. Eq. (2.6)]. Let η be smaller than a. Put $\xi_0=\frac{1}{2}(a+\eta)$. Let $\alpha(\xi)$ be a C^∞ function equal to 0 for $\xi\le\xi_0$ and to 1 for $\xi\ge a$. According to the representation theorem, the support of S is contained in the set $\{\xi\,|\,\xi\ge a\}$. Therefore,

$$S(\Theta_{\delta,\eta}) = S(\alpha\Theta_{\delta,\eta}).$$

But, for all non-negative integers m and n,

$$\lim_{\delta\to+0} \xi^m \frac{d^n}{d\xi^n}[\alpha(\xi)\Theta_{\delta,\eta}(\xi)] = 0,$$

uniformly in ξ, i.e., $\alpha\Theta_{\delta,\eta}$ tends to 0 in $\mathcal{S}$ as δ tends to 0 through positive values. Relation (4.1) follows. Q.E.D.

If the form factor F belongs to $\mathcal{A}$, and if the function G in $\mathcal{B}$ associated with F is continuous in the closed upper half-plane Im$u\ge 0$, we obtain from this proposition the following "continuous infinity" of sum rules:

$$\lim_{\delta\to+0} \int_b^\infty \mathrm{Re}[F(\tau+i0)\exp(-i\eta(\tau-b)^{1/2})] \times\exp(-\delta\tau)d\tau = 0$$

for all $\eta<a$ (the integral converging in the sense of distributions).

We now derive a more refined class of sum rules.

Proposition IV.A.2

Let F be a function belonging to class $\mathcal{A}$. Let f be an entire function of order less than or equal to $\frac{1}{2}$ such that

$$\limsup_{r\to\infty} r^{-1/2}\ln|f(re^{i\theta})| \le a_0[\tfrac{1}{2}(1+|\cos\theta|)]^{1/2}, \tag{4.2}$$

where $a_0<a$. Then (symbolically)

$$\lim_{\delta\to+0} \int_{+\infty}^{+\infty} \mathrm{Im}F(\tau)f(\tau)\exp(-\delta\tau)d\tau = 0. \tag{4.3}$$

Remark that the choice $f(t)=t^N$ (N: non-negative integer) leads to the sum rules (1.7).

We shall derive this proposition from the following lemma, whose proof is based on the idea that was used in the proof of Proposition IV.A.1.

Lemma IV.A.3

Let T be a tempered distribution such that its Fourier transform $\tilde{T}$ has its support in $\{\xi\,|\,|\xi|\ge a\}$, where a is some positive number. Let g be an entire function of exponential type such that its indicator diagram D [30] *is contained in the open square $\{\rho+i\sigma\,|\,|\rho|+|\sigma|<a\}$. Define g_δ by*

$$g_\delta(\rho) = \exp(-\delta\rho^2)g(\rho). \tag{4.4}$$

Then

$$\lim_{\delta\to+0} T(g_\delta) = 0. \tag{4.5}$$

Remark: Our normalization conventions are fixed by the following symbolic relation:

$$\tilde{T}(\xi) = (2\pi)^{-1}\int_{-\infty}^{+\infty} T(\rho)e^{-i\xi\rho}d\rho.$$

Proof of Lemma IV.A.3: Let

$$a_0 = \max\{|\rho|+|\sigma|\,|\,\rho+i\sigma\epsilon D\}$$

(remember that the indicator diagram D is a compact set). Let c_0, c_1 and c_2 be real numbers with

$$(0\le)a_0<c_0<c_1<c_2<a.$$

Let Γ_0 be the closed curve $\{\rho+i\sigma\,|\,|\rho|+|\sigma|=c_0\}$. Γ_0 is a square with its vertices at c_0, ic_0, $-c_0$, and $-ic_0$. Let D_1 be the set $\{\rho+i\sigma\,|\,|\rho|+|\sigma|\le c_1\}$.

The function h, defined by

$$h(z)=\int_0^\infty g(\rho)\exp(-\rho z)d\rho\,, \tag{4.6}$$

is obviously holomorphic in $\mathrm{Re}z>a$. It is well known (cf. Ref. 30) that h can be analytically continued to a function which is holomorphic in the complement of D. One has the formula

$$g(u)=(2\pi i)^{-1}\oint_\Gamma \exp(uz)h(z)dz\,, \tag{4.7}$$

where Γ is any rectifiable curve, homeomorphic to a circle, lying in the complement of D, having D in its interior, and positively oriented. We shall choose Γ to be the contour Γ_0 defined above.

Let us define $\tilde{g}_\delta$ by

$$\tilde{g}_\delta(\xi)=\int_{-\infty}^{+\infty} g_\delta(\rho)\exp(i\xi\rho)d\rho\,. \tag{4.8}$$

The function $\tilde{g}_\delta$ is in $\mathcal{S}$ for $\delta>0$. Let us show that for any non-negative integers n and m

$$\lim_{\delta\to+0}\xi^m\frac{d^n}{d\xi^n}[\tilde{g}_\delta(\xi)]=0 \tag{4.9}$$

uniformly in ξ for $|\xi|\geq c_2$. From (4.4), (4.7), and (4.8), we obtain

$$2\pi i\frac{d^n}{d\xi^n}[\tilde{g}_\delta(\xi)]=(\pi/\delta)^{1/2}\oint_{\Gamma_0}dz\,h(z)\times\frac{d^n}{d\xi^n}\{\exp[(4\delta)^{-1}(z+i\xi)^2]\} \tag{4.10}$$

(the interchange of limiting processes is easily justified). To find a bound for the exponential and its derivatives on Γ_0, we proceed as follows. First, we remark that for all z in D_1 and for all ξ with $|\xi|>c_1$,

$$|\exp[(4\delta)^{-1}(z+i\xi)^2]|\leq\exp[-(4\delta)^{-1}(|\xi|-c_1)^2].$$

This follows from the inequality

$$\mathrm{Re}[(z+i\xi)^2]\leq-(|\xi|-c_1)^2,$$

which is valid under the same conditions. {*Proof:* Let

$$|\mathrm{Re}z|+|\mathrm{Im}z|\leq c_1$$

and

$$|\xi|>c_1.$$

Then

$$\begin{aligned}\mathrm{Re}[(z+i\xi)^2]&=(\mathrm{Re}z)^2-(\mathrm{Im}z+\xi)^2\\&\leq(\mathrm{Re}z)^2-(|\mathrm{Im}z|-|\xi|)^2\leq(c_1-|\mathrm{Im}z|)^2\\&\quad-(|\mathrm{Im}z|-|\xi|)^2\leq c_1^2+2|\mathrm{Im}z|(|\xi|-c_1)-|\xi|^2\\&\leq-(|\xi|-c_1)^2,\end{aligned}$$

since $|\mathrm{Im}z|\leq c_1$ and $|\xi|-c_1>0$.} Second, given any z on Γ_0 as center, we draw a circle tangential to the boundary of D_1. This circle is entirely contained in D_1, and its radius is $R=\frac{1}{2}\sqrt{2}(c_1-c_0)$. Cauchy's inequalities supply now the required bounds: For z on Γ_0 and for $|\xi|>c_1$,

$$\left|\frac{d^n}{d\xi^n}\{\exp[(4\delta)^{-1}(z+i\xi)^2]\}\right|=\left|\frac{d^n}{dz^n}\{\exp[(4\delta)^{-1}(z+i\xi)^2]\}\right|\leq n!R^{-n}\exp[-(4\delta)^{-1}(|\xi|-c_1)^2].$$

Therefore, it follows from (4.10) that

$$\left|\frac{d^n}{d\xi^n}[\tilde{g}_\delta(\xi)]\right|\leq n!R^{-n}L(\pi/\delta)^{1/2}\exp[-(4\delta)^{-1}(|\xi|-c_1)^2]$$

for $|\xi|>c_1$. We have put

$$L=(2\pi)^{-1}\oint_{\Gamma_0}|h(z)|\,|dz|\,.$$

Thus, (4.9) is verified. Now,

$$T(g_\delta)=\tilde{T}(\tilde{g}_\delta)\,.$$

Because of the support properties of $\tilde{T}$,

$$\tilde{T}(\tilde{g}_\delta)=\tilde{T}(\alpha\tilde{g}_\delta)\,,$$

where α is any C^∞ function equal to 1 for $|\xi|\geq a$ and to 0 for $|\xi|\leq c_2$. Since (4.9) implies that

$$\lim_{\delta\to+0}\alpha\tilde{g}_\delta=0$$

in $\mathcal{S}$, relation (4.5) follows. Q.E.D.

Proof of Proposition IV.A.2: Let G be the function in $\mathcal{B}$ associated with F. Using Proposition II.D.1, we get symbolically

$$\int_{-\infty}^{+\infty}\mathrm{Im}F(\tau)f(\tau)\exp(-\delta\tau)d\tau=\exp(-\delta b)\int_{-\infty}^{+\infty}G_i(\rho)g(\rho)\exp(-\delta\rho)^2d\rho\,, \tag{4.11}$$

where g is defined by

$$g(\rho)=\rho f(b+\rho^2)\,.$$

Putting $G_i=T$, $S_i=\tilde{T}$, we can apply Lemma 2 to the right-hand side of (4.11). This is so because condition (4.2) implies that the indicator diagram of g has the required properties, and because the representation theorem implies that the support of S_i is contained in $\{\xi\,|\,|\xi|\geq a\}$. Q.E.D.

[30] For the results on entire functions of exponential type used here, see, for example, Ref. 18, Chap. 5.

If ImF turns out to be a measurable function such that

$$\int_b^\infty |\mathrm{Im}F(\tau)|(1+|\tau|^N)d\tau<\infty \qquad (4.12)$$

for some positive integer N, it follows from (4.3) with $f(t)=t^n$, $0\leq n\leq N$, that

$$\int_b^\infty \mathrm{Im}F(\tau)\tau^n d\tau=0\,.$$

If ImF is measurable, and if condition (4.12) is valid for all positive integers N, or, more generally, if t^N ImF is a bounded distribution[31] for all positive integers N, then the Fourier transform S_i of the corresponding G_i is a C^∞ function of moderate growth.[32] Therefore S itself is a C^∞ function of moderate growth. Using standard methods,[33] we arrive via Eq. (2.7) at the following strengthening of condition (A3):

$$\lim_{t\to-\infty} |t|^N F(t)\exp(a|t|^{1/2})=0$$

for all positive integers N.

B. The Number of Changes of Sign of ImF

It is generally known that if F is a function of class $\mathfrak{A}$, then ImF has to change sign infinitely often (see Ref. 10). We propose below a precise meaning for such a statement even when ImF is not a continuous function with isolated zeroes. We shall also give an asymptotic lower bound on the number of changes of sign of ImF at large positive values of t.

Let $\mathfrak{C}$ be the set of those C^∞ functions g defined on the real line which have only zeroes of finite multiplicity [i.e., if $g(\tau)=0$, then there is a positive integer m such that $d^m g(\tau)/d\tau^m\neq 0$; the multiplicity of τ is the smallest of such integers]. If g in $\mathfrak{C}$, then the set of its zeroes has a finite number of points in each compact interval. Let us denote by $n(\tau)$ the number of zeroes of g (counted according to their multiplicity) in the interval $\{\sigma||\sigma|\geq\tau\}$. The product of a function g in $\mathfrak{C}$ with the tempered distribution ImF is a distribution in $\mathfrak{D}'$ (see Ref. 26).

Proposiiton IV.B.1

Let F be in $\mathfrak{A}$. If there is a real number ρ_0 and a function g in $\mathfrak{C}$ with

$$\limsup_{\tau\to+\infty}\ \tau^{-1/2}n(\tau)<\pi^{-1}a\,, \qquad (4.13)$$

such that g ImF *is a positive distribution on* $\Delta_0=\{\tau|\tau>\rho_0\}$,[34] *then F is identically zero.*

Remark: The example $F(t)=\exp[-a(b-t)^{1/2}]$ shows that the right-hand member of (4.13) cannot be made larger. See also Sec. VA.

Proof: We first construct an entire function f in the following way. Consider the set of the zeros of g which are in $\Delta_1=\{\tau|\tau>\rho_1=\max\{0,\rho_0,\ -b\}\}$. If this set is finite, we can as well assume that it is empty, by taking ρ_0 large enough. We then put $f=\pm1$, choosing the sign of f in such a way that f ImF be positive on Δ_1. If on the other hand this set is not finite, it is countable. We can enumerate it in such a way that

$$\rho_1<\tau_1<\tau_2<\cdots.$$

Because of (4.13), we can apply the Lemma shown in the Appendix. According to this Lemma, the infinite product

$$\pm\prod_{m=1}^{\infty}[1-(t/\tau_m)]^{\nu m}\,,$$

where ν_m denotes the multiplicity of the zero τ_m, converges to an entire function f of order smaller than or equal to $\frac{1}{2}$ satisfying

$$\limsup_{r\to\infty}\ r^{-1/2}\ln|f(re^{i\theta})|\leq a_0[\tfrac{1}{2}(1+|\cos\theta|)]^{1/2}, \quad (4.14)$$

with $a_0<a$. The sign is chosen in such a way that the C^∞ function $g(\tau)/f(\tau)$ is positive on Δ_1. The distribution f ImF is then again positive on Δ_1.

In both cases, the function $f(t)$ so constructed, as well as the functions $t^n f(t)$ for all positive integers n, satisfy the conditions of Proposition IV.A.2. Therefore, we have (symbolically)

$$\lim_{\delta\to+0}\int \tau^n f(\tau)\ \mathrm{Im}F(\tau)\exp(-\delta\tau)d\tau=0 \quad (4.15)$$

for all non-negative integers n.

We now use (4.15) together with the fact that f ImF is a positive distribution on Δ_1 to show that f ImF is actually zero on Δ_1 (see Ref. 22). Let ρ_2, ρ_3, ρ_4, and ρ_5 be numbers with

$$\rho_1<\rho_2<\rho_3<\rho_4<\rho_5\,,$$

and let φ be a non-negative function of $\mathfrak{D}$ with its support contained in the interval $\{\tau|\rho_4\leq\tau\leq\rho_5\}$. Let α be a C^∞ function which is equal to 1 for $\tau\geq\rho_2$ and to 0 for $\tau\leq\rho_1$. Then, for all non-negative integers n and all positive δ, we find

$$\begin{aligned}0&\leq f\,\mathrm{Im}F(\varphi)=f\,\mathrm{Im}F(\alpha\varphi)\leq f\,\mathrm{Im}F((\tau/\rho_3)^n\alpha\varphi)\\&=f\,\mathrm{Im}F((\tau/\rho_3)^n\alpha\exp(-\delta\tau)\varphi\exp(\delta\tau))\leq\exp(\delta\rho_5)\\&\qquad\times\max\{\varphi(\tau)\}f\,\mathrm{Im}F((\tau/\rho_3)^n\alpha\exp(-\delta\tau))\,,\end{aligned}$$

using the positivity of f ImF on Δ_1. We now show that we can make the last line as small as we please, thanks

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[31] Cf. Ref. 13, Chap. VI, Sec. 8.

[32] Cf. Ref. 13, Chap. VII, Sec. 5 and Théorème XV.

[33] See for example A. Erdélyi, *Asymptotic Expansions* (California Institute of Technology, Pasadena, Calif., 1955), p. 29.

[34] That is, such that g Im$F(\varphi)=$Im$F(g\varphi)\geq0$ for any non-negative function φ in $\mathfrak{D}$ which has its support in Δ_0. See Ref. 13, Chap. I, Sec. 4.

to (4.15). We have (symbolically)

$$\int (\tau/\rho_3)^n[1-\alpha(\tau)]f(\tau)\,\mathrm{Im}F(\tau)\,\exp(-\delta\tau)d\tau + \int (\tau/\rho_3)^n\alpha(\tau)f(\tau)\,\mathrm{Im}F(\tau)\,\exp(-\delta\tau)d\tau = E(\delta,n)\,,$$

where $E(\delta,n)$ tends to 0 as δ tends to 0 through positive values for each fixed value of n. Given ϵ, it is now sufficient to choose n so large that the absolute value of the first integral is less than $\frac{1}{2}\epsilon$ for all δ with $0\leq\delta<1$, and then to choose δ so small that $|E(\delta,n)|<\frac{1}{2}\epsilon$. The interval $\{\tau|\rho_4\leq\tau\leq\rho_5\}$ being an arbitrary interval contained in Δ_1, we have thus shown that f ImF is zero on Δ_1.

We next show that the vanishing of f ImF on Δ_1 implies that ImF has compact support. This is obvious in the case when $f=\pm 1$. In the other case, we argue in the following way. Since f ImF vanishes on Δ_1, the intersection of the support of ImF with Δ_1 is contained in the set of the zeroes of f. In other words, we can write

$$\mathrm{Im}F = T + \sum_{m=1}^{\infty}\sum_{\nu=0}^{\nu_m-1} c(m,\nu)\delta^{(\nu)}(\tau-\tau_m)\,,$$

where T has compact support and vanishes on Δ_1, the $c(m,\nu)$'s are real numbers, and $\delta^{(\nu)}$ denotes the νth derivative of the δ function. We want to show that the $c(m,\nu)$'s are all equal to 0. We limit ourselves to show that for any m, $c(m,\nu_m-1)$ is equal to 0. The method used can then be applied to show that $c(m,\nu_m-2)$, $c(m,\nu_m-3)$, $\cdots$ are also equal to 0. Consider the entire function f_0 defined by

$$f_0(t)=[1-(t/\tau_m)]^{-1}f(t)\,.$$

We have

$$f_0(t)=[1-(t/\tau_m)]^{\nu_m-1}\varphi(t)\,,$$

where φ is the entire function defined by

$$\varphi(t)=[1-(t/\tau_m)]^{-\nu_m}f(t)\,.$$

According to the definition of ν_m,

$$\varphi(\tau_m)\neq 0.$$

One computes f_0 ImF easily and finds

$$f_0\,\mathrm{Im}F = f_0T+(\nu_m-1)!\,\tau_m^{-\nu_m+1}\varphi(\tau_m)\times c(m,\nu_m-1)\delta(\tau-\tau_m)\,.$$

Thus, since T vanishes on Δ_1, and since δ is a positive distribution, f_0 ImF is a distribution with a definite sign on Δ_1. Since f_0 satisfies the conditions of Proposition IV.A.2, it follows from what we already showed that f_0 ImF is zero on Δ_1. This implies $c(m,\nu_m-1)=0$. We conclude that all the $c(m,\nu)$'s are equal to 0. We have thus shown that ImF has compact support.

To show that the last fact implies $F\equiv 0$, it suffices to use Proposition III.B.1. Q.E.D.

V. POLE MODELS

A. Examples

It is not difficult to give examples of functions F of class $\mathfrak{A}$ which are meromorphic. One such example is[35]

$$F(t)=[(-t)^{1/2}\sinh(a(-t)^{1/2})]^{-1}.$$

Here b can be taken to be zero. The poles of F are at

$$\tau_n=(\pi n/a)^2\,,$$

where $n=0, 1, 2, \cdots$. They are simple. ImF is found to be given by

$$\mathrm{Im}F(\tau)=a^{-1}\pi[\delta(\tau)+2\sum_{n=1}^{\infty}(-)^n\delta(\tau-\tau_n)].$$

In view of Proposition IV.B, one can make the following remark. The entire function g defined by

$$g(t)=\cos(a(t)^{1/2})$$

belongs to class $\mathfrak{C}$. The corresponding function $n(\tau)$ can be seen to satisfy

$$n(\tau)=\tau^{1/2}\pi^{-1}a+O(1)$$

as τ goes to $+\infty$. Also, g ImF is a positive distribution. This shows again that the right-hand member of the inequality (4.13) cannot be replaced by anything larger. Let us compute the tempered distribution S of Eq. (2.8), One way to do this is to make use of Proposition II.D.2 to compute G_i in terms of ImF, and then to calculate its Fourier transform S_i. S is then obtained from S_i by Eq. (3.1). G_i is found to be given by

$$G_i(\rho)=-a^{-1}\pi\delta'(\rho)+\sum_{n=1}^{\infty}(-)^n n^{-1}\times[\delta(\rho-\pi n/a)-\delta(\rho+\pi n/a)]\,,$$

and S_i turns out to be [use Eq. (2.6) for normalization]

$$S_i(\xi)=-\tfrac{1}{2}i(\xi/a)-\pi^{-1}i\sum_{n=1}^{\infty}(-)^n n^{-1}\sin((\pi n/a)\xi)\,.$$

But

$$-\pi^{-1}2a\sum_{n=1}^{\infty}(-)^n n^{-1}\sin((\pi n/a)\xi)=s(\xi)\,,$$

where s is the periodic function with period $2a$ defined by

$$s(\xi)=\xi-2na \quad\text{for}\quad (2n-1)a<\xi<(2n+1)a \quad (n=0,\pm 1,\pm 2,\cdots).$$

Therefore, S_i is in the present case a measurable function given by

$$S_i(\xi)=-in \text{ for } (2n-1)a<\xi<(2n+1)a\,.$$

Remark that it vanishes as it should between $-a$ and

[35] A similar example was shown to us by Stack (see Ref. 12).

$+a$. From Eq. (3.1), we get simply

$$S(\xi)=2i\theta(\xi)S_i(\xi).$$

It is easy to verify that Eq. (2.8) is indeed fulfilled.

Other examples of meromorphic functions of class $\mathfrak{A}$ with simple poles only may be constructed as follows. Without loss of generality, we choose $b=0$. Then we take S_i to be a locally integrable measurable function with the following properties: (a) It is periodic with period $2c(c>0)$; (b) it vanishes on $\{\xi \mid |\xi|\leq a\}$; (c) it satisfies $S_i(-\xi)=-S_i(\xi)$; and (d) it takes on only imaginary values. The its Fourier transform G_i is of the form[36]

$$G_i(\rho)=\sum_{n=-\infty}^{\infty}\lambda_n\delta(\rho-\pi n/c),$$

where the λ_n's are given by

$$\lambda_n=\frac{\pi}{c}\int_{-c}^{+c}S_i(\xi)\exp(in\pi\xi/c)d\xi.$$

They are real and tend to zero as $|n|$ tends to ∞; furthermore,

$$\lambda_{-n}=-\lambda_n.$$

Using Proposition II.D.1 to find $\mathrm{Im}F$, we obtain

$$\mathrm{Im}F(\tau)=\frac{2\pi}{c}\sum_{n=1}^{\infty}n\lambda_n\delta(\tau-(\pi n/c)^2).$$

From this formula, we guess that F will be a meromorphic function with simple poles located at the points $(\pi n/c)^2$, with residues equal to $-(2/c)n\lambda_n$, $(n=1, 2, \ldots)$. This guess is in fact correct. The distribution S of Eq. (2.8) is just $\theta(\xi)S_i(\xi)$. Remark that S_i, G_i, $\cdots$ vanish identically if $c<a$; this illustrates the content of Proposition IV.B since then the positions $(\pi n/c)^2$ of the poles would be spaced too far apart.

B. Number of Poles

We now make an application of Proposition IV.B.1 to functions F of class $\mathfrak{A}$ whose only singularities are simple poles. For such functions, $\mathrm{Im}F$ is of the form[37]

$$\mathrm{Im}F=\sum_m c_m\delta(\tau-\tau_m),$$

where the sequence τ_m, which we assume to satisfy $b\leq\tau_1<\tau_2<\cdots$, has no accumulation point. The c_m's are real. We shall assume that they are different from zero.

If the sequence τ_m had a finite number of terms only, then $\mathrm{Im}F$ would be of compact support, and it would follow from Proposition III.B.1 that it would be identically zero. We shall therefore assume that the sequence τ_m is infinite.

Let $n_0(\tau)$ denote the number of terms of the sequence τ_m which satisfy $|\tau_m|\leq\tau$. We shall show that

$$\limsup_{\tau\to+\infty}\ \tau^{-1/2}n_0(\tau)\geq\pi^{-1}a. \tag{5.1}$$

We first remark that the sequence c_m must show an infinite number of changes of sign. In other words, the number of pairs (c_m,c_{m+1}) of consecutive elements with $c_mc_{m+1}<0$ is infinite. For if it were not the case, it would follow from Proposition IV.B (with $g=+1$ or $g=-1$) that $F\equiv0$. Thus, the set formed by the numbers $\frac{1}{2}(\tau_m+\tau_{m+1})$, m being such that $c_mc_{m+1}<0$, is infinite. Let σ_m be an enumeration of this set such that $b<\sigma_1<\sigma_2<\cdots$. Denote by $n(\tau)$ the number of elements of the sequence σ_m such that $|\sigma_m|\leq\tau$. We have obviously

$$n_0(\tau)\geq n(\tau)-1. \tag{5.2}$$

We next remark that it is easy to construct a function g of class $\mathfrak{C}$ whose only zeros, all simple, coincide with the σ_m's. Then $g\,\mathrm{Im}F$ or $-g\,\mathrm{Im}F$ will turn out to be a positive distribution. Since we have assumed that $\mathrm{Im}F$ is not the distribution zero, it follows from Proposition IV.B.1 that

$$\limsup_{\tau\to+\infty}\ \tau^{-1/2}n(\tau)\geq\pi^{-1}a \tag{5.3}$$

Inequality (5.1) follows from (5.2) and (5.3).

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APPENDIX

Let $0<\tau_1\leq\tau_2\cdots$ be a sequence of real numbers. Let $n(\tau)$ be the number of elements of this sequence which satisfy $\tau_m\leq\tau$.

Lemma: Assume that there is a real number a such that

$$\limsup_{\tau\to\infty}\ \tau^{-1/2}n(\tau)\leq\pi^{-1}a. \tag{A1}$$

[36] Cf. Ref. 13, Chap. VII, Sec. 1.

[37] The sequences τ_m and c_m must obey certain restrictions in order that $\sum c_m\delta(\tau-\tau_m)$ be a tempered distribution. It is, for example, sufficient that there exists an integer k such that

$$\sum_{m=1}^{\infty}|c_m|(1+\tau_m^2)^{-\frac{1}{2}k}<\infty.$$

This condition is not necessary. This can be seen from the following example. Set

$$\tau_{2m-1}=m,\quad \tau_{2m}=m+\exp(-m),$$
$$c_{2m-1}=\exp(m),\quad c_{2m}=-\exp(m),\quad m=1, 2, 3, \cdots.$$

These sequences do not satisfy the condition just given. However we have, in the sense of distributions

$$\sum_{m=1}^{\infty}c_m\delta(\tau-\tau_m)=\frac{d^2}{d\tau^2}g(\tau),$$

where $g(\tau)$ is a continuous function with $|g(\tau)|<|\tau|$. It follows that $\sum c_m\delta(\tau-\tau_m)$ is a tempered distribution.

Then the product

$$\prod_{m=1}^{\infty}[1-(t/\tau_m)] \qquad \text{(A2)}$$

converges for all complex t to an entire function f of order smaller than or equal to $\frac{1}{2}$, *whose zeros coincide with the* τ_m*'s, and which satisfies*

$$\limsup_{r\to\infty} r^{-1/2}\ln|f(re^{i\theta})|\leq a[\tfrac{1}{2}(1+|\cos\theta|)]^{1/2}. \qquad \text{(A3)}$$

Proof: It follows from (A1) that

$$\int_0^{\infty}\tau^{-1-\alpha}n(\tau)d\tau<\infty$$

for all α with $\alpha>\frac{1}{2}$. Therefore, the product (A2) converges for all complex t to an entire function f of order smaller than or equal to $\frac{1}{2}$, and whose zeroes coincide with the τ_m's.[38] We have $f(t)>0$ for $t<0$. Let $g(t)$ be that branch of $\log f(t)$ which is real for $t<0$. The function g is holomorphic in $\{t|0<\arg t<2\pi\}$. In this domain, we have the formula[39]

$$g(t)=-t\int_0^{\infty}[\tau(\tau-t)]^{-1}n(\tau)d\tau.$$

Put $t=ir$. We have for r real

$$\mathrm{Re}g(ir)=r^2\int_0^{\infty}[\tau(\tau^2+r^2)]^{-1}n(\tau)d\tau.$$

According to (A1), for any positive ϵ, there is a number T such that $\tau\geq T$ implies

$$n(\tau)<\pi^{-1}(a+\epsilon)\tau^{1/2}.$$

Therefore

$$\mathrm{Re}g(ir)<r^2\int_0^{T}[\tau(\tau^2+r^2)]^{-1}n(\tau)d\tau+\pi^{-1}(a+\epsilon)r^2\int_T^{\infty}\tau^{-1/2}(\tau^2+r^2)^{-1}d\tau. \qquad \text{(A4)}$$

[38] See Ref. 18, Lemma 2.5.5 and Theorem 2.6.5.

[39] See Ref. 18, Eq. (4.1.4).

Using Lebesgue's bounded convergence theorem, we see that the first term on the right-hand side of (A4) is a continuous function of r which tends to

$$\int_0^{T}\tau^{-1}n(\tau)d\tau$$

as $|r|$ tends to ∞. The integral in the second term is majorized by

$$\int_0^{\infty}\tau^{-1/2}(\tau^2+r^2)^{-1}d\tau=\tfrac{1}{2}\sqrt{2}\pi|r|^{-3/2}.$$

Thus, for any positive ϵ, there is a real number C such that

$$\mathrm{Re}g(ir)<C+\tfrac{1}{2}\sqrt{2}(a+\epsilon)|r|^{1/2}.$$

Therefore, since

$$\ln|f(t)|=\mathrm{Re}g(t),$$

we find (r real)

$$\limsup_{|r|\to\infty}|r|^{-1/2}\ln|f(ir)|\leq\tfrac{1}{2}\sqrt{2}a.$$

Thus, for any positive ϵ, the function h, defined by

$$h(t)=f(t)\exp(-(a+\epsilon)t^{1/2})$$

($|\arg t|\leq\frac{1}{2}\pi$) is holomorphic for $\mathrm{Re}t>0$ and continuous for $\mathrm{Re}t\geq 0$. Furthermore, it is bounded on $\mathrm{Re}t=0$, and since it is the product of two functions of order smaller than or equal to $\frac{1}{2}$,

$$\sup\{|h(re^{i\theta})|\,|\,|\theta|\leq\tfrac{1}{2}\pi\}=O(\exp(r^{\frac{1}{2}+\eta}))$$

for any positive η as r goes to $+\infty$. Using Phragmén-Lindelöf's theorem,[40] we see that h is bounded in $\mathrm{Re}t\geq 0$. Thus, letting ϵ go to zero, we obtain

$$\limsup_{r\to\infty} r^{-1/2}\ln|f(re^{i\theta})|\leq a\cos(\tfrac{1}{2}\theta)$$

for $|\theta|\leq\frac{1}{2}\pi$. A similar argument yields

$$\limsup_{r\to\infty} r^{-1/2}\ln|f(re^{i\theta})|\leq a\cos(\tfrac{1}{2}(\theta-\pi))$$

for $|\theta-\pi|\leq\frac{1}{2}\pi$. Taken together, these estimates give (A3). Q.E.D.

[40] See Ref. 18, Theorem 1.4.1. Our argument is modelled on the proof of Theorem 5.1.2 of Ref. 18.