the same energy has been performed also by I. J. Bloodworth [Oxford thesis, 1967 (unpublished)]}. Of particular interest are the values of the total cross sections for the Σ channels: $\sigma(\Sigma^+K^0p)=26\pm4$, $\sigma(\Sigma^0K^+p)=17_{-2}^{+4}$, $\sigma(\Sigma^+K^+n)=57\pm7$ (in μb). The latter value disagrees with our prediction by a factor of 2. However, the other two values tend to remove the difference in magnitude which seemed to exist between the cross sections for the corresponding channels. Thus the model is still working satisfactorily, provided a stronger cutoff ($\alpha_\pi\sim20$–$22m_\pi^2$) is chosen for pion exchange in the case of Σ production. However, this value of the cutoff cannot be used also for the Λ, as is apparent from the analysis performed in this paper. Therefore the model must contain three cutoff parameters instead of two. Note that the fits to the spectra presented in this work (which refer all to the Λ) are not affected by the above considerations. A more detailed discussion of Σ production will be given in a forthcoming paper.

ACKNOWLEDGMENTS

One of us (E. F.) thanks Professor Y. Nambu, for the kind hospitality in the theoretical group of The Enrico Fermi Institute for Nuclear Studies in Chicago, and for the use of the computer facilities. We acknowledge interesting discussions with Professor P. G. O. Freund and Professor J. D. Jackson, and with Dr. A. C. Melissinos and Dr. M. Firebaugh who communicated to us the results of their experiments prior to publication. We thank also Professor R. E. Marshak and Professor S. Okubo for their interest in this work.

PHYSICAL REVIEW VOLUME 167, NUMBER 5 25 MARCH 1968

Muon Tridents

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(Received 25 October 1967)

Under the assumption that muons are heavy electrons, the total cross section for muon tridents on lead is calculated for 12.0-BeV incident muons including the effect of exchange for identical particles in the final state; various differential cross sections are presented. The positron spectrum for 31.5-MeV electron tridents on copper is also calculated and found to agree with Criegee's experimental results. It is found that the entire effect of statistics is confined to a region of phase space where the two leptons of like charge in the final state have an invariant mass of less than 3.5 times their rest mass.

THE muon is a very well measured[1] but very poorly understood particle. In every conceivable property observed to date, muons have behaved like heavy electrons.[2]

Yet, somehow, this is unreasonable. How can two particles, identical in all other respects, still manage to have unequal masses? There must exist a property, some *other* property, no matter how seemingly insignificant or obscure, that is different for muons and electrons, and as such gives rise to their difference in mass. Thirty-five years of searching for this elusive property have given not even a hint of success. This discourages some people, but spurs others on to look for still another unmeasured property, the one perhaps that might at long last be different.

One outstanding property that is very well known for electrons but not at all known for muons is their statistics.[3] States in which two identical electrons exist must be totally antisymmetric to the exchange of these particles. Although one fervently believes that this is also the case for muons,[4] there is a complete lack of experimental information on the subject because no one has yet been able to obtain a final state in which two identical muons were present.[5]

One reaction in which two identical muons can occur in the final state is the direct production of a muon pair by a muon in the field of a nucleus, or muon tridents. The trident process is also interesting from the point of view of checking quantum electrodynamics at small distances, so it has attracted considerable attention over the years.

The first calculation of direct pair production by electrons was performed almost immediately after the

* Alfred P. Sloan Research Fellow.
† Work partially supported by U. S. Atomic Energy Commission Contract No. AT(30-1) 2752.
[1] G. Feinberg and L. M. Lederman, Ann. Rev. Nucl. Sci. 13, 431 (1963); see, also, F. J. M. Farley, Progr. Nucl. Phys. 9, 259 (1964).
[2] New York Times, Jan. 22, 1961 (p. 33).
[3] W. Pauli, Z. Physik 31, 765 (1925).
[4] G. F. Dell'Antonio, Ann. Phys. (N. Y.) 16, 153 (1961); see, however, S. Kamefuchi and Y. Takahashi, Nucl. Phys. 36, 177 (1962).
[5] See, however, R. O. Stenerson, Bull. Am. Phys. Soc. 12, 31 (1967); and M. L. Morris and R. O. Stenerson, Nuovo Cimento (to be published).

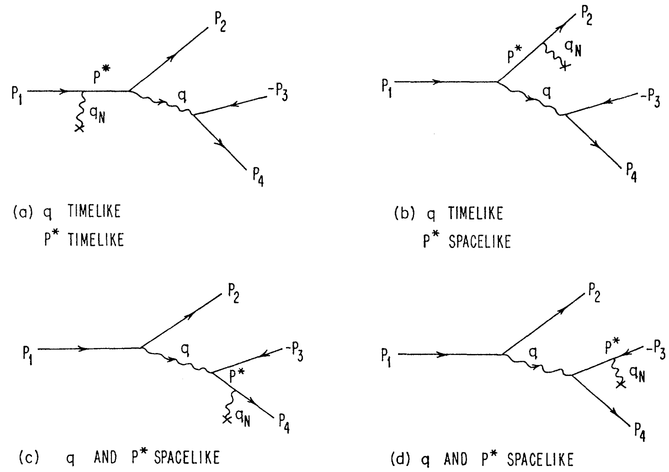

FIG. 1. The four Feynman diagrams for muon tridents (a) and (b) have timelike virtual photons while in (c) and (d) the virtual photon is spacelike. p^* is the four-momentum of the virtual muon; q is the four-momentum of the virtual photon and q_N is the nuclear recoil four-momentum.

discovery of the positron.[6] Subsequent authors[7] have repeated and improved the calculation to various degrees of approximation; but it was not until recently that the complete differential trident cross section was calculated.[8,9]

For a lepton of incident four-momentum p_1 which produces a pair of leptons with four-momenta p_3 (opposite charge) and p_4 (like charge) in the field of a heavy spin-zero nucleus [total charge Z, form factor $F(q_N^2)$], leaving a scattered lepton of four-momentum p_2, the differential cross section is given by the expression[10]

$$\frac{d^5\sigma}{dE_2dE_3d\Omega_1d\Omega_2d\Omega_3}=SZ^2F^2(q_N^2)\frac{\alpha^4}{2\pi^4}\frac{|\mathbf{p}_2||\mathbf{p}_3||\mathbf{p}_4|}{|\mathbf{p}_1|}\frac{1}{q_N^4}\sum_{\text{spins}}|M|^2, \quad (1)$$

where $q_N=p_1-p_2-p_3-p_4$ is the four-momentum transferred to the nucleus. The incident lepton is assumed to be unpolarized and the spins of the final particles are not observed. If the leptons produced in the pair are different from the incident one, then $S=1$ and M is a sum of the four Feynman amplitudes for tridents[11] (Fig. 1).

$$M=M_a+M_b+M_c+M_d.$$

If the initial and pair of leptons are identical, then $S=\frac{1}{2}$ and M is a sum of the four amplitudes of Fig. 1 *minus* these four amplitudes with p_2 and p_4 exchanged.[12]

The amplitudes can be reduced to traces by standard techniques,[13] but the algebra involved is very complicated because each reduced Feynman amplitude contributes a term like M_d, which we give for reference:

$$M_d\sim\frac{1}{q^2}\frac{1}{p^{*2}+m^2}U_H,$$

where U_H is a very complicated sum over products of various four-momenta.[14] Eight terms like this must be summed and then squared to find the trident cross section.

Such calculations have actually been performed by Bjorken and Chen and by Henry.[9] However, in spite of the fact that these authors give algebraic expressions for the differential trident cross section, these expressions are complicated so that they tend to obscure the physics involved. For a physical understanding of tridents, it is much better to separate completely the quantitative and qualitative aspects of the problem.

Quantitatively, since all calculations are performed on a computer in any case, there are two reasons why it is advantageous to calculate the amplitudes directly, as suggested by Brodsky,[8] instead of by the reduction of spin sums into traces. Firstly, by directly evaluating the products of complex matrices in a specific Dirac representation, the long strings of large but kinematically cancelling terms generated by the trace method no longer appear. Secondly, the lepton masses

[6] W. H. Furry and J. F. Carlson, Phys. Rev. 44, 237 (1933).

[7] H. J. Bhabha, Proc. Roy. Soc. (London) A152, 559 (1935); T. Murota, A. Ueda, and H. Tanaka, Progr. Theoret. Phys. (Kyoto) 16, 482 (1956); F. F. Ternovskii, Zh. Eksperim. i Teor. Fiz. 37, 739 (1959) [English transl.: Soviet Phys.-JETP 10, 565 (1960)]; J. D. Bjorken and S. D. Drell, Phys. Rev. 114, 1368 (1959); T. Yamamoto, Progr. Theoret. Phys. (Kyoto) 27, 233 (1962).

[8] S. J. Brodsky and Sam C. C. Ting, Phys. Rev. 145, 1018 (1966); S. J. Brodsky (unpublished).

[9] J. D. Bjorken and M. C. Chen, Phys. Rev. 154, 1335 (1967); G. Reading Henry, *ibid.* 154, 1534 (1967); see, also, B. Carlin, G. Frielingsdorf, and G. Peressutti (unpublished).

[10] We use notation in which a four-momentum $p=(\mathbf{p},iE)$ such that $p\cdot p=\mathbf{p}\cdot\mathbf{p}-E^2=-m^2$ for a particle of mass m. The four-momentum of the virtual photon is denoted as q; of the virtual lepton as p^*; and of the nuclear recoil as q_N (Fig. 1).

[11] The explicit expressions for the four amplitudes of Fig. 1 are given in Ref. 8, Eqs. (2)–(7). Compton terms are not considered. We are indebted to Professor Brodsky for pointing out to us the importance of the statistical factor S.

[12] Note that the amplitudes with exchanged momenta are subtracted so as to keep the final state antisymmetric in the two identical particles; i.e., we assume that the muons are fermions.

[13] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964), Chap. 7.

[14] Readers who wish to see U_H in full are referred to G. Reading Henry (Ref. 9, Appendix).

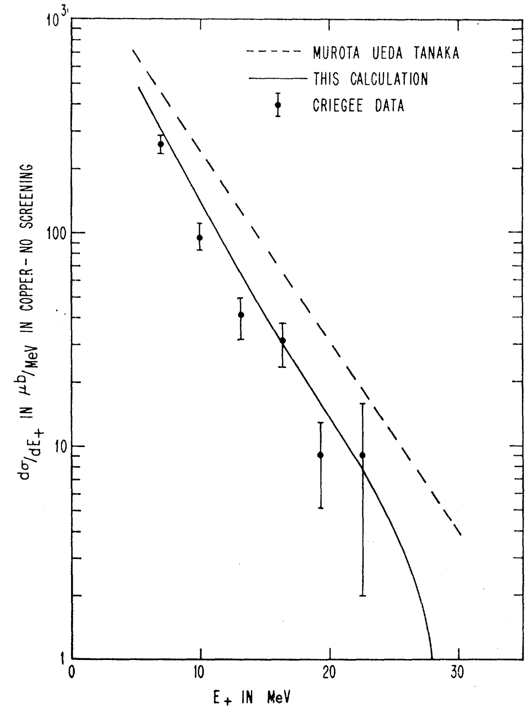

FIG. 2. Positron energy spectrum for 31.5-MeV electron tridents on copper. The experimental data points of Criegee (Ref. 19) are also shown along with the calculation of Murota, Ueda, and Tanaka (Ref. 7).

can be easily included in the calculation since all manipulations are done numerically so that algebraic simplifications are unnecessary.

For a qualitative understanding of tridents, we can simply ignore all the complicated terms in the numerator and just consider the effects of the denominator. Then the square of each amplitude makes a contribution to the cross section that looks like

$$d\sigma_d \sim \frac{Z^2F^2(q_N^2)}{q_N^4}\frac{1}{q^4}\frac{1}{(p^{*2}+m^2)^2}.$$

Clearly the trident differential cross section is biggest in the regions of phase space where a term in any of the denominators of the eight matrix elements gets very small. This means that there are essentially only four regions of phase space where the trident differential cross section is not vanishingly small:

(1) A lepton is collinear with the virtual photon $\mathbf{q}$. In this case, $p^{*2}+m^2$ nearly vanishes for some diagram.

(2) A lepton of like charge is collinear with the incident particle. In this case, the spacelike virtual photon four-momentum squared is small (M_c and M_d).

(3) Two leptons of opposite charge are collinear in the final state. This means that the timelike virtual four-momentum squared is minimized (M_a and M_b) and is especially important for electron tridents.

(4) $\mathbf{q}_N$, the nuclear recoil, is collinear with the incident lepton. This is the configuration in which q_N^2 takes on its minimum value when one of the final-state lepton momenta is varied while the other two lepton momenta remain fixed.[15,16]

Actually, the trident differential cross sections by themselves are not very useful because even in the regions of phase space where they are largest, they are still quite small ($<10^{-31}$ cm^2/MeV^2 sr^3). Coupled with the fact that existing muon beams[17] are of relatively low intensity, this means that any muon trident experiment foreseeable in the near future will require detectors of very large solid-angle acceptance to see any events at all, so that measurements of the differential cross section will be essentially impossible. Thus, any practical discussion of muon tridents can be made only in terms of various *integrals* of the trident cross section. In fact, the problem of finding the total trident cross section is closely related to the problem of finding the effect of muon statistics on trident production.

The integration of the trident differential cross section is a tricky business because there is no simple analytical expression to use as a guide. The integration must be done entirely numerically; and one must search a vast phase space for the four tiny regions in which the contributions to the integral are not negligible.[18]

The effect of muon statistics will be most evident in regions of phase space where two muons of the same charge overlap. This is most probable when all of the three final leptons have small transverse momenta with respect to the incident direction. In this configuration, however, the four conditions for large differential cross sections are all satisfied at once. Thus, the region of small-angle tridents, which has maximum sensitivity to the effect of statistics, is also the region of the maximum contribution to the integrated cross section.

We have integrated the trident cross section with no constraints on any of the final-state particles so as to obtain the total cross section for muon trident production. A novel Monte Carlo integration method was used. Events were generated by a special procedure (Appendix A) only in regions of phase space where the differential cross section is largest ($q_N^2 \sim m^4/E^2$, $q^2 \sim m^4/E^2$, and $p^{*2}+m^2 \sim m^4/E^2$). It was assumed that there were no contributions to the integrated cross sections from outside these regions. Each event was weighted by its differential cross section [Eq. (1)] times the volume of phase space in which it was generated and the integral was obtained by making a

[15] Note that the configuration with $\mathbf{q}_N$ exactly collinear with the incident direction is the region of the quantum-electrodynamical dip where the cross sections are forced to be small by current conservation. However, if $\mathbf{q}_N$ acquires a transverse component equal in magnitude to its minimum longitudinal value, then the cross section indeed becomes a maximum (see Ref. 16).

[16] G. Reading Henry, Phys. Rev. **153**, 1649 (1967).

[17] L. M. Lederman and M. J. Tannenbaum, in *Advances in High Energy Physics*, edited by R. E. Marshak and R. L. Cool (Interscience Publishers, Inc., New York, 1967).

[18] For an idea of how difficult this can be, consult Johnson's thesis, Harvard University, 1962 (unpublished); also, see E. Johnson, Jr., Phys. Rev. **140**, B1005 (1965).

weighted histogram of the generated events. This method had the virtue that the trident cross-section differential in any variable could be obtained with just the single set of generated events simply by making a weighted histogram in that variable.

Of course, nobody would believe any Monte Carlo calculation, this one in particular, unless there was some impeccable way of positively checking it. Fortunately, for muon tridents there exists a convenient check because the calculation assumes that muons are just heavy electrons; and all properties of electrons are well known, even their trident cross sections. Thus, the muon calculation can be checked by using it to compute electron trident cross sections which can be compared to existing experimental data.

The experiment of Criegee[19] in which 31.5-MeV electron tridents were observed in copper is particularly good for this purpose. First of all, Criegee measured the energy spectrum of the unlike particle, a cross section differential in one variable. Secondly, electron tridents at 31.5 MeV ($\gamma \equiv E/m = 61.6$) should be analogous to muon tridents at 6.5 BeV, an energy that is compatible with existing muon beams. Thirdly, the data have never been successfully fitted by a calculation involving no free parameters.

Our results for the positron energy spectrum for 31.5-MeV electron tridents in copper are presented in Table I along with the calculation of Muorta, Ueda, and Tanaka[7] and the data of Criegee.[19] These same data are also shown in Fig. 2.

Screening has not been included in our calculation. The copper form factor was taken as

$$F(q_N^2) = (1 + q_N^2 \times 32.9)^{-2},$$

where q_N^2 is in BeV^2, which corresponds[20] to an exponential charge density of rms radius 3.92 F.

The agreement is quite satisfactory, especially considering the fact that only the Born approximation was used and that screening was neglected. These effects, if properly included, would *each* reduce our calculated cross section by about 10%.[21,22]

Having thus established some degree of credibility for our calculation, we can now proceed to the main problem of how muon tridents can be used to measure the statistics of muons.

In any final state the effect of statistics is essentially a limitation on phase space,[23] and for this reason is essentially the same for both tridents and triplets.[24] In

[19] L. Criegee, Z. Physik **158**, 433 (1960).

[20] R. Herman and R. Hofstadter, *High Energy Electron Scattering Tables* (Stanford University Press, Stanford, Calif., 1960).

[21] H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. **93**, 788 (1953); W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1954), 3rd ed., pp. 249–50 and 254.

[22] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934); Ann. Phys. **5**, 325 (1930) (see p. 385).

[23] J. Joseph and F. Rohrlich, Rev. Mod. Phys. **30**, 354 (1958); see, also, D. Benaksas and R. Morrison, Phy Rev. **160**, 1245 (1967).

[24] Note that a trident is the electroproduction of an electron-positron pair in the field of a nucleus. A triplet is the photoproduction of an electron-positron pair in the field of an electron.

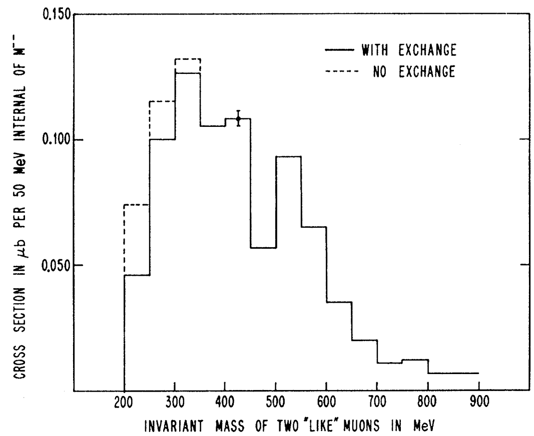

FIG. 3. Distribution in invariant mass of the two muons of like charge for 12.0-BeV muon tridents on lead. The effect of exchange is important only in the first three bins. The error flag represents the error in the absolute cross section. The result shown is the cross section integrated over the 50-MeV interval of m^{--}.

regions of phase space in which two identical particles overlap, the states of parallel spin will be excluded by Fermi-Dirac statistics, thus depressing the cross section. This is most easily seen by considering the two identical muons in their center-of-mass system. In this system each muon will have equal and opposite momentum P^{--}.

For configurations in which the two identical muons have very little relative motion, P^{--} will be small; and the entire phase space will be a region in which the two like muons overlap so that the cross section will be greatly depressed by Fermi statistics. As P^{--} increases, there will be less overlap of the like muons so that the effect of statistics will be reduced.

This is strikingly illustrated by a plot of the differential cross section, $d\sigma/dm^{--}$, for 12.0-BeV muons incident on lead (Fig. 3); where m^{--} is the invariant mass of the two like muons:

$$\begin{aligned}(m^{--})^2 &\equiv -(p_2+p_4)^2\\ &= 2(E_2E_4 - \mathbf{p}_2\cdot\mathbf{p}_4) + 2m^2\\ &= 4[(P^{--})^2 + m^2].\end{aligned}$$

The results are given both for the case where the ex-

TABLE I. Positron energy distribution for 31.5-MeV electron tridents on copper.

E_+ (MeV)	Criegee $d\sigma/dE_+$ (μb/MeV)	This calculation $d\sigma/dE_+$ (μb/MeV)	Ratio (this calculation/ Criegee)	MUT[a] $d\sigma/dE_+$ (μb/MeV)
7.0	260 ±20	305	1.17±0.09	475
10.1	98 ±15	140	1.43±0.22	253
13.2	41.5± 9	60	1.45±0.31	130
16.3	31.5± 8	29	0.92±0.23	68
19.4	9.1± 4	15	1.65±0.72	36
22.6	9 ± 7	7.5	0.83±0.65	18

[a] See Ref. 19.

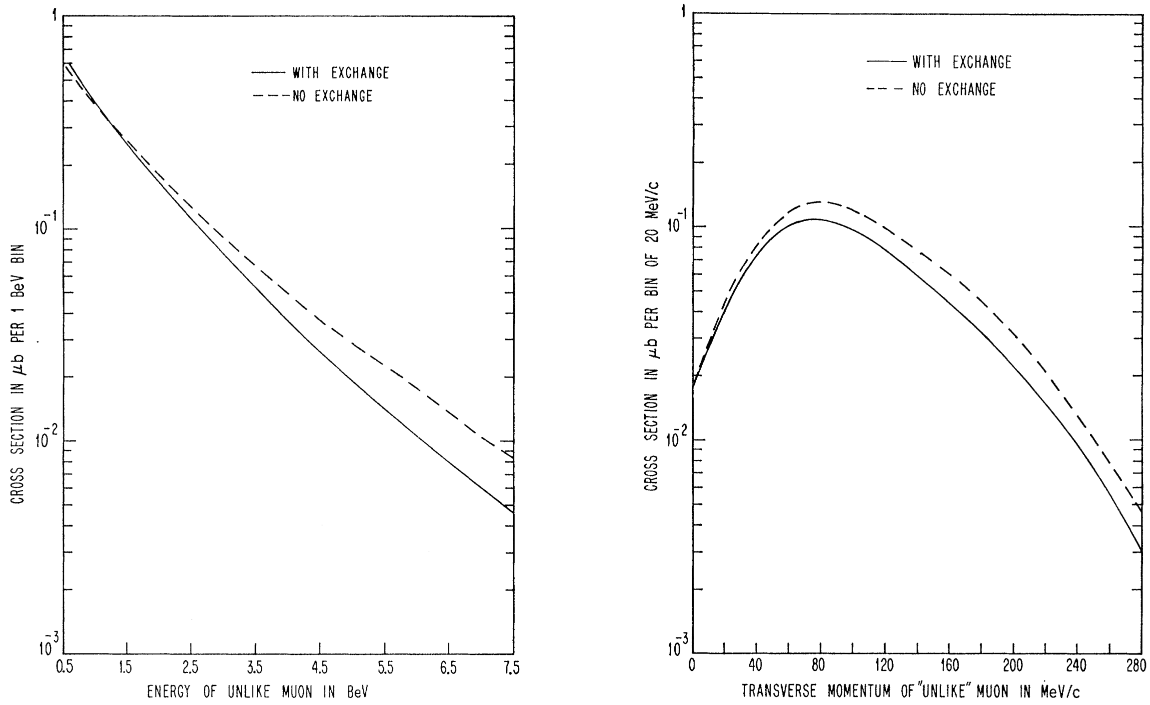

FIG. 4. (a) Energy and (b) transverse momentum distributions of the muon of unlike charge for 12.0-BeV muon tridents on lead. The result presented is the differential cross section integrated over the bin of 1.0 BeV (a) or 20 MeV/c (b).

change diagrams have been included and for the case where they have been left out. Screening has been neglected but is a much smaller effect for muon tridents than for electrons. The lead form factor was taken as

$$F(q_N^2)=(1+q_N^2\times 63.0)^{-2} \quad \text{for} \quad q_N^2 \text{ in BeV}^2,$$

corresponding to an exponential charge distribution of rms radius 5.42 F. Only the elastic form factor was used.

Clearly, the entire effect of statistics occurs for invariant masses m^{--} less than $3.5m$, i.e., $P^{--}<1.5m$, where m is the muon mass.[25] The muon trident cross section for invariant mass m^{--} less than 350 MeV is 275 nb with exchange, compared to 325 nb if exchange is ignored. For m^{--} greater than 350 MeV, exchange has essentially no effect and the cross section is 600 nb in both cases.[26] These results along with those for 17.0-BeV muons incident on lead are summarized in Table II.

TABLE II. Total cross sections for 12.0- and 17.0-BeV muon tridents on lead.

E_1 (BeV)	Cross section, m^{--} <350 MeV (μb)	Cross section, m^{--} >350 MeV (μb)	Total cross section (μb)
12.0 with exchange	0.275	0.600	0.875
no exchange	0.325	0.600	0.925
17.0 with exchange	0.400	0.960	1.36
no exchange	0.465	0.970	1.44

Although the effect of statistics is primarily an effect on the invariant mass spectrum of the two like particles, it also has some secondary effects. The invariant mass of the two like muons can be written in terms of their energies and opening angle:

$$(m^{--})^2=4m^2+4E_2E_4\sin^2(\tfrac{1}{2}\theta_{24})+m^2[(E_2-E_4)^2/E_2E_4].$$

From this formula we see that m^{--} will tend to be small when θ_{24} is small or when E_2-E_4 is small or when both E_2 and E_4 are small. Thus, we would expect distributions in these quantities to also be sensitive to the effects of statistics.

The distribution of the energy of the unlike muon for 12.0-BeV muon tridents on lead is shown in Fig. 4. The transverse momentum distribution for this particle is also shown. The effect of statistics is again clear. States of low invariant mass are most probable when the unlike muon has most of the trident energy because then very little energy remains for the two like muons. This is why the effect of statistics increases with increasing E_3. The slopes of the curves with and without exchange are not very different for E_3 greater than 3

[25] Note that the same conditions also apply to electron tridents if the electron mass is used.

[26] Note that all our calculated cross section have a ±6% absolute uncertainty due to the statistical nature of the Monte Carlo method; however, the ratio of exchange to no-exchange cross sections is good to a much higher precision because both cross sections are computed at the same set of random phase-space points.

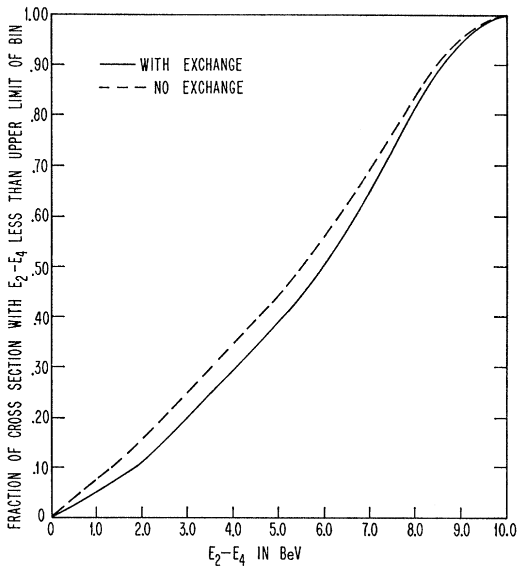

FIG. 5. Integrated and normalized distribution of the difference in energy of the two muons of like charge for 12.0-BeV muon tridents on lead. For this figure the energy of the muon of unlike charge has been restricted to be greater than 1.5 BeV.

BeV; however, the absolute cross sections differ by a factor of about 1.5.

The distribution of the energy of the unlike muon is especially interesting from an experimental viewpoint because it is a clear signature of a trident when a muon of one charge enters a target and a muon of opposite charge emerges.

Another distribution that shows the effect of statistics is the distribution in the difference in energies of the two like particles, $d\sigma/d(E_2-E_4)$, Fig. 5. The distributions with and without exchange have both been normalized to unity so that the effect of a difference in shape would stand out. Cutting the E_2-E_4 distribution to require that $E_2-E_4>6.0$ BeV would eliminate 50.5% of the events for Fermi statistics compared to 56.0% if exchange were ignored.

We have presented results for various integrals of the muon trident cross section to illustrate that effects of statistics for identical particles are significant when the two identical particles have an invariant mass of less than 3.5 times their rest mass. In particular, we have seen that the total muon trident cross section for 12-BeV muons incident on lead is 0.875 μb, if the effect of Fermi-Dirac statistics is included, compared to 0.925 μb if the exchange effect is ignored. As a check, we have also calculated the positron spectrum for 31.5-MeV electron tridents on copper and found it to be in agreement with the experimental results of Criegee.[19]

ACKNOWLEDGMENTS

I would like to thank Professor George Reading Henry for many stimulating the enlightening discussions. I am greatly indebted to Dr. Stanley Brodsky who kindly gave me his computer code for calculating the trident matrix elements.

APPENDIX: DETAILS OF THE MONTE CARLO CALCULATION

We only consider events in which two muons each have transverse momentum up to $2m$ (m is the muon mass) with respect to the incident particle and the momentum of the third muon is such that the total transverse momentum of the three final-state muons never exceeds $2m$. We estimate that the events excluded by this procedure contribute less than 0.5% to the total cross section.

The energies of two of the final-state muons are chosen at random so that the energies of the three muons in the final state add up to the incident energy (i.e., the kinetic energy of the recoiling nucleus is neglected). Directions are defined for two of the final-state muons by assigning them transverse momenta between 0 and $2m$, uniformly distributed at random in polar angle θ and azimuthal angle ϕ with respect to the incident direction. The events are generated uniformly in their polar angle, rather than uniformly in solid angle, in order to give relatively more events at small angles where the cross section is largest. The energies of the particles differ from event to event so that the solid-angle intervals in which the particles are generated also differ. Both these effects are taken into account in the weighting procedure.

For the third muon, the direction is found such that the total transverse momentum of the other two muons is exactly cancelled. The third muon is then assigned transverse momentum between 0 and $2m$ about this direction (again uniformly in θ and ϕ).

To complete the event, the recoiling nucleus is assigned a momentum so as to make the vector sum of the final-state momenta equal to the incident momentum.