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Kerr Metric, Rotating Sources, and Machian Effects

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In a recent paper, the author gave a recipe for constructing *exact* interior solutions which might serve as sources of the Kerr metric. Here the recipe is given again, but in simpler form, and an example of an interior solution is given in detail. Some properties of these solutions are discussed, including the shape, mass density, and rotation rates of the bodies described. It is proven that solutions of this type exist which describe *uniformly* rotating bodies at least up to second order in the angular momentum parameter b. In the final section, a discussion is given which indicates that strong gravitational fields plus high-rotational velocities can lead to correspondingly high-rotational inertial effects on the observer who is near or inside the body. These inertial effects also give some direct information about the rotation of the body. This enables us to prove that *uniformly* rotating rings of mass could not serve as sources of the Kerr metric.

I. INTRODUCTION

The study, in the framework of general relativity, of finite bodies which possess angular momentum is, in general, extremely difficult. But this is exactly the problem set before us if, for example, we want the gravitational field associated with a rotating star. The need for understanding this problem assumes much greater importance when we are concerned with rotating super-massive or neutron stars where the gravitational fields may be quite strong. Most studies of the more difficult problems in relativity proceed by employing techniques such as perturbation theory, linearization, numerical integration, etc. Here we shall discuss a finite rotating body where the fields are known exactly.

A particular solution of the Einstein field equations of great interest is the Kerr solution[1] which is the only known exact solution describing the field exterior to some finite *rotating* body. In a recent paper[2] the author outlined a method for constructing *exact* interior solutions which might serve as sources of the Kerr field. These sources described *nonfluid*,[3] axially symmetric, rotating bodies which were obtained essentially by perturbing strongly the familiar Schwarzschild interior solution. The physical significance of these solutions might be questioned because the material described may be quite unfamiliar. However, little is known concerning general relativistic theories of matter, especially nonfluid forms, so, at present, we must be satisfied with any solutions which do not violate any principles of physics. Actually for weak enough fields one could construct models in a laboratory which are identical to these models to any given accuracy. These exact solutions serve a useful purpose in the same sense that an exact solution is useful in any area of physics. It provides a concrete example with which to work and test our ideas. In this case we may be led to a better understanding of the role of angular momentum in general relativity.

In this paper, we shall look more closely at the interior solutions constructed in the earlier work. The original method for constructing interior solutions used coordinates similar to those used in Kerr's original paper. Recently, however, the Kerr metric has been expressed in Schwarzschild-like coordinates[4] which in many ways are simpler than the Kerr coordinates. Hence, for the sake of completeness and because the resulting metric is easier to understand, we shall, in Sec. II, again give the method of construction of interior solutions, but this time using the Schwarzschild-like coordinates. We shall also give explicitly an interior solution thus constructed and the details involved. Section III is devoted to the investigation of the physical properties of the interior solutions. More specifically we discuss the shape of the body, its mass density and rotation. In Sec. IV we discuss the effects of this rotating body on the inertial frames inside and outside the body.

Throughout this work we use coordinates such that $c=G=1$. Greek letters refer to the four coordinates r, θ, ϕ, and t.

II. SCHWARZSCHILD COORDINATES AND INTERIOR SOLUTIONS

The Kerr metric can be written in Schwarzschild-like coordinates,[4] i.e., coordinates such that when the angular momentum parameter b is zero one gets the Schwarzschild metric in its familiar form. These new coordinates also simplify the metric by eliminating un-

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[1] R. P. Kerr, Phys. Rev. Letters, **11**, 327 (1963).
[2] W. C. Hernandez, Phys. Rev. **159**, 1070 (1967).
[3] In our earlier work (Ref. 1) we showed that our constructed solutions could not describe fluid bodies because our arbitrary boundary ($r=r_1$) could not be the boundary of a fluid. Moreover, we also gave arguments which indicated that fluid sources for the Kerr metric do not exist.
[4] H. Boyer and R. Lindquist, J. Math Phys. **8**, 265 (1967).

necessary g_{rt}, $g_{r\phi}$ cross terms. Thus we have

$$ds^2=[(r^2+b^2\cos^2\theta)/(r^2-2mr+b^2)]dr^2+(r^2+b^2\cos^2\theta)d\theta^2+[r^2+b^2+2mrb^2\sin^2\theta/(r^2+b^2\cos^2\theta)]\times\sin^2\theta d\phi^2+[4mbr\sin^2\theta/(r^2+b^2\cos^2\theta)]dtd\phi-[1-2mr/(r^2+b^2\cos^2\theta)]dt^2. \quad (1)$$

Next we rewrite this as

$$ds^2=dr^2/(1-2m/r)+r^2d\theta^2+r^2\sin^2\theta d\phi^2-(1-2m/r)dt^2-(b^2/r^2)1/(1-2m/r+b^2/r^2)[1/(1-2m/r)-\cos^2\theta]dr^2+b^2\cos^2\theta d\theta^2+\{1+(2m/r)\sin^2\theta/[1+(b^2/r^2)\cos^2\theta]\}b^2\sin^2\theta d\phi^2-(2m/r^3)\{b^2\cos^2\theta/[1+(b^2/r^2)\cos^2\theta]\}dt^2+(4bm/r)\{\sin^2\theta/[1+(b^2/r^2)\cos^2\theta]\}d\phi dt. \quad (2)$$

This has the form

$$g_{\mu\nu}=S_{\mu\nu}+bA_{\mu\nu}, \quad (3)$$

where $S_{\mu\nu}$ is the Schwartzschild metric and $A_{\mu\nu}$ is that part of the metric which has b as a factor.

The requirements we impose on our interior solution are that it have (1) non-negative energy density everywhere, (2) non-negative principal stresses everywhere,[5] (3) energy density greater than the stresses,[6] and (4) that it satisfies the proper boundary conditions at the surface separating the interior and exterior solutions. This last condition is met if we demand that the first and second fundamental forms be continuous across the surface.[7] We shall choose the boundary as $r=r_1$ for simplicity. Then for a metric having the components g_{rr}, $g_{\theta\theta}$, $g_{\phi\phi}$, $g_{\phi t}$, and g_{tt} it is easily shown that these continuity conditions are satisfied if we choose the metric components such that all the $g_{\mu\nu}$, $g_{\mu\nu,\theta}$,and $g_{\mu\nu,r}$ except $g_{rr,r}$ are continuous at the surface. A finite discontinuity in $g_{rr,r}$ at the surface is allowed. For the interior solution we choose

$$ds^2=dr^2/(1-r^2/R^2)+r^2d\theta^2+r^2\sin^2\theta d\phi^2-[\tfrac{3}{2}(1-r_1^2/R^2)^{1/2}-\tfrac{1}{2}(1-r^2/R^2)^{1/2}]^2dt^2-b^2A(r)\{1/[1-2mh(r)+b^2g(r)]\}\times\{1/[1-2mh(r)]-\cos^2\theta\}dr^2+b^2B(r)\cos^2\theta d\theta^2+b^2B(r)\{1+2mf(r)\sin^2\theta/[1+b^2g(r)\cos^2\theta]\}\sin^2\theta d\phi^2+\{4bmC(r)\sin^2\theta/[1+b^2k(r)\cos^2\theta]\}d\phi dt-\{2b^2mF(r)\cos^2\theta/[1+b^2g(r)\cos^2\theta]\}dt^2. \quad (4)$$

Thus to the $S_{\mu\nu}$ part of the metric we have simply matched the familiar Schwarzschild interior solution (with R^2 defined by $r_1^2/R^2=2m/r_1$). The remainder of the interior metric has a form which makes it easy to match it to the $bA_{\mu\nu}$ part of the metric. The various functions $A(r)$, $f(r)$, etc., are somewhat arbitrary analytic functions chosen so that the boundary conditions at $r=r_1$ are satisfied and such that the resulting metric and its inverse $g^{\mu\nu}$ are analytic everywhere. As an example we shall choose for the functions the simple polynomials:

$$\begin{aligned} h(r)&=r^2/2mR^2\geq 0,\\ f(r)&=(3/2r_1)-(r^2/2r_1^3)>0,\\ g(r)&=(2/r_1^2)-(r^2/r_1^4)>0,\\ A(r)&=r^2/r_1^4\geq 0,\\ B(r)&=(5r^4/r_1^4)-(4r^5/r_1^5)\geq 0,\\ C(r)&=(5r^2/2r_1^3)-(3r^4/2r_1^5)\geq 0,\\ k(r)&=(4r/r_1^3)-(3r^2/r_1^4)\geq 0,\\ F(r)&=(6r^2/r_1^5)-(5r^2/r_1^6)\geq 0, \end{aligned} \quad (5)$$

where r has the range $0\leq r\leq r_1$. It is obvious that the metric is analytic in this range if $1-2mh$ remains positive. For the choice of h above this implies that we need $r_1>2m$. The inverse $g^{\mu\nu}$ is also analytic everywhere except at points where

$$\det g_{\mu\nu}=g_{rr}g_{\theta\theta}(g_{\phi\phi}g_{tt}-g_{\phi t}^2)=0. \quad (6)$$

The first factor g_{rr} has the form

$$(1-r^2/R^2)^{-1}-b^2X(r,\theta), \quad (7)$$

where $X(r,\theta)$ is a bounded function. Hence g_{rr} is never zero *for sufficiently small b*. The second factor $g_{\theta\theta}$ is zero only at the origin $r=0$. The third factor can be expressed as

$$-r^2\sin^2\theta\{[\tfrac{3}{2}(1-r_1^2/R^2)^{1/2}-\tfrac{1}{2}(1-r^2/R^2)^{1/2}]^2+b^2Y(r,\theta)\}, \quad (8)$$

where $Y(r,\theta)$ is a bounded function. Hence *for sufficiently small b* this factor is zero only on the z axis ($\theta=0$) which includes the origin $r=0$. If one so desired, $X(r,\theta)$ and $Y(r,\theta)$ could be calculated for the example of Eqs. (5) and the exact limit on b obtained. For this example it turns out that the singularity along the z axis is a coordinate effect only. We prove this in Appendix A by transforming to another coordinate system which removes the singularity. It follows that for sufficiently small b our interior solution describes a "smooth" geometry. Since the metric is analytic in the parameter b at $b=0$ we know that the calculation of the stress-energy tensor by direct substitution into the Einstein equations

$$G_\nu{}^\mu=-8\pi T_\nu{}^\mu \quad (9)$$

will take the form

$$T_\nu{}^\mu=t_\nu{}^\mu+bl_\nu{}^\mu, \quad (10)$$

[5] This gives a material under pressure rather than tension and hence tends to make the body more stable.

[6] The velocity of sound in a *fluid* is given by $v_s^2=dp/d\rho$. Causality requires that $v_s\leq c=1$. This leads us to demand that the physical components of the stress of our material be small enough so as to allow an interpretation where the velocity of sound will not be greater than the velocity of light.

[7] W. C. Hernandez, Phys. Rev. **153**, 1359 (1967).

where $t_\nu{}^\mu$ is the Schwarzschild stress-energy tensor. Furthermore, for small enough b the functions $l_\nu{}^\mu$ will be finite. Since $t_\nu{}^\mu$ satisfies the requirements imposed earlier then for sufficiently small b we have $T_\nu{}^\mu$ satisfying the requirements also.

For the ordinary Schwarzschild interior solution there is a restriction $2r_1 \leq 8m/9$ which insures finite pressure at the origin. In our interior solutions there are analogous restrictions imposed on the parameters m, r_1, and b. These can be obtained by calculating the $T_\nu{}^\mu$ explicitly, but we shall not do this.

III. PROPERTIES

In general, our constructed solution will describe a rotating, axially symmetric body whose surface has the topology of a 2-sphere and composed of material which possess some solidlike (shear sustaining) properties. For very small b it will closely resemble the Schwarzschild interior solution which simply describes a static uniform density sphere with isotropic pressures.

A. Shape

The element of spatial distance dl defined in terms of the three space coordinate elements is given by[8]

$$dl^2 = \gamma_{ij} dx^i dx^j , \qquad (11)$$

where

$$\gamma_{ij} = g_{ij} - g_{0i} g_{0j} / g_{00} . \qquad (12)$$

Evaluating this for the surface of the body ($r = r_1$) we get

$$\gamma_{\theta\theta} = r_1{}^2 + b^2 \cos^2\theta ,$$
$$\gamma_{\phi\phi} = (r_1{}^2 + b^2) \sin^2\theta + 2mr_1 b^2 \sin^4\theta / (r_1{}^2 - 2mr_1 + b^2 \cos^2\theta) . \qquad (13)$$

Thus the circumference of the body at the equator ($\theta = \frac{1}{2}\pi$) is given by

$$2\pi [r_1{}^2 + b^2 r_1 / (r_1 - 2m)]^{1/2} . \qquad (14)$$

The polar circumference ($\phi = \text{const}$) is given by

$$\int_0^{2\pi} (r_1{}^2 + b^2 \cos^2\theta)^{1/2} d\theta < 2\pi (r_1{}^2 + b^2)^{1/2} . \qquad (15)$$

Thus the equatorial circumference is larger as one might desire intuitively for a rotating body, but remember that the boundary choice was arbitrary. One should also note that the measurements we described are those that would be made by an observer on the surface of the body who is at rest with respect to a distant observer. A moving observer would obtain different values for these measurements because of relativistic contraction and Machian effects. For example, one can show that these effects single out a preferred observer revolving around the body at the equator who will measure the smallest equatorial circumference.

B. Mass Density and Angular Velocity

Consider the eigenvalue equations

$$T_\nu{}^\mu V_{(i)}{}^\nu = \theta_i V_{(i)}{}^\mu . \qquad (16)$$

The four eigenvalues are the three principal stresses and the negative of the energy density. In general, a calculation of the Einstein tensor $G_{\mu\nu}$ for a metric of the form of Eq. (4) will give as nonzero components $G_{rr}, G_{r\theta}, G_{\theta\theta}, G_{\phi\phi}, G_{\phi t}, G_{tt}$. From Eq. (9) these are also the nonzero components of $T_{\mu\nu}$. Because many cross terms are zero, the solutions of Eqs. (16) are simple. However, we shall limit our discussion to the timelike eigenvector equation

$$G_\nu{}^\mu V_{(4)}{}^\nu = 8\pi\rho V_{(4)}{}^\mu , \qquad (17)$$

where we have substituted for $T_\nu{}^\mu$ by use of Eq. (9). Here ρ is the energy density and $V_{(4)}{}^\mu$ a multiple of the material 4-velocity U^μ. It is obvious that a form $V_{(4)}{}^\nu = \Omega\delta_\phi{}^\nu + \delta_t{}^\nu$ will work. Substituting this into Eq. (17) and simplifying, we get

$$\Omega^2 R_\phi{}^t + \Omega(R_t{}^t - R_\phi{}^\phi) - R_t{}^\phi = 0 , \qquad (18)$$

$$\Omega G_\phi{}^t + G_t{}^t + \rho = 0 , \qquad (19)$$

where $R_j{}^i$ is the Ricci tensor. Now the coordinates ϕ and t are defined as parameters along the paths of motions. They are uniquely defined by requiring that at spatial infinity the line element reduces to the flat-space form expressed in the familiar spherical coordinates. It then follows that Ω is simply the angular velocity of the material as seen by a distant observer.

A particular interior solution which would be obviously highly desirable would be one for which Ω is a constant. It is easily verified that $\Omega = \text{const}$ is necessary and sufficient for Born-type rigid motion. However, the discussion of such a solution is rather difficult and is currently being studied by the author. So at present, as a simplification we shall look for a rigidly rotating source up to second order in b only (but to all order in the mass m). Let e^α and $-e^\gamma$ represent, respectively, the g_{rr} and g_{tt} components of the Schwarzschild interior metric. Then to second order in b we find that $R_{\phi t}$ is dependent only on the unknown function C and is given by

$$R_{\phi t} = bm\, e^{-\alpha} \sin^2\theta [C'' + \tfrac{1}{2}(\gamma' - \alpha')C' + (2\gamma'/r)C] - (2bm \sin^2\theta / r^2) C + O(b^3) , \qquad (20)$$

where the primes indicate differentiation with respect to the variable r. Note that the terms of order b^2 vanish. The other components of the Ricci tensor have the form

$$R_{\mu\nu} = {}_s R_{\mu\nu} + O(b^2) , \qquad (21)$$

where the subscript s' indicates the Schwarzschild value. The terms of order b vanish in these components and we shall not need to calculate explicitly the terms of

[8] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1951), Sec. 10-4.

order b^2. Assuming Ω is of order b and substituting Eqs. (20) and (21) into Eq. (18) we finally get to second order in b

$$e^{-\alpha}[C''+\tfrac{1}{2}(\gamma'-\alpha')C'+(2\gamma'/r)C]-(2/r^2)C \\ -8\pi(3p+\epsilon)C=(8\pi\Omega/bm)(p+\epsilon)r^2. \quad (22)$$

We have made use of the field equations for the Schwarzschild solution ${}_sG_t{}^t=8\pi\epsilon$ and ${}_sG_r{}^r={}_sG_\theta{}^\theta={}_sG_\phi{}^\phi=p$. The mass density $\epsilon=3/8\pi R^2$ is a constant, and the pressure p is a well-known function of r. Thus $C(r)$ must satisfy the differential equation above and also the boundary conditions at the surface, $C(r_1)=1$, $C'(r_1)=-1/r_1$. We have not been able to solve this equation. However, a solution does exist as we prove in Appendix B. Thus, at least to second order in b, a rigidly rotating source does exist.

Before discussing Eq. (22) further let us look at Eq. (19). Substituting our expression for $R_{\mu\nu}$ into this equation gives

$$\rho=\epsilon+O(b^2), \quad (23)$$

where terms first order in b vanish. The fact that Ω has terms of first order in b and ρ has not terms of first order in b is not surprising. If we let $\phi\to-\phi$ and $b\to-b$ then the Kerr metric is unchanged. Thus letting $b\to-b$ changes only the direction of rotation of the body. So in general we have $\Omega(-b)=-\Omega(b)$ and $\rho(-b)=\rho(b)$.

Returning to Eq. (22), we shall look for a solution when we retain terms to zeroth order in m only (Newtonian limit). Equation (22) becomes

$$C''-(2/r^2)C=(8\pi\Omega\epsilon/bm)r^2. \quad (24)$$

Requiring that the solution be regular at the origin and satisfy the boundary conditions at $r=r_1$, we easily get

$$C=(5r^2/2r_1{}^3)-(3r^4/2r_1{}^5), \quad (25)$$

with Ω given by[9]

$$\Omega=-\tfrac{5}{2}b/r_1{}^2. \quad (26)$$

In the Newtonian limit we must have $b^2\ll 1$ (actually b^2 is of order m).[2] Thus in the Newtonian limit our source is a uniform density sphere with consequently a moment of inertia $I=\frac{2}{5}mr_1{}^2$. It follows that the angular momentum is given by

$$L=I\Omega=-mb, \quad (27)$$

in agreement with the value[1] as determined by the asymptotic expansion of the Kerr exterior solution. Any other choice of $C(r)$ (which naturally would not be a solution of the differential equation) leads to a nonuniform rotation. For example,

$$C=(6r^4/r_1{}^5)-(5r^5/r_1{}^6) \quad (28)$$

yields a rotation rate

$$\Omega=(10b/r_1{}^2)-(15b/r_1{}^3)r. \quad (29)$$

But this still gives a net angular momentum of $L=-mb$. Finally one might notice that the $C(r)$ of Eq. (25) is identical to the $C(r)$ of Eq. (5). This just means the solution given by Eqs. (5) is rotating rigidly in its *Newtonian limit*.

IV. MACH EFFECTS

Any induced rotation in local inertial frames caused by mass possessing angular momentum is generally viewed as a Machian effect.[10] We shall identify and discuss this effect in both the Kerr (exterior) metric and our interior metric.

A. Kerr Metric

Consider a stationary observer at some point on the z axis ($r=r_0$, $\theta=0$) who is "rotating" in such a manner such that he feels no centrifugal forces. He can set up, at least locally, a "cylindrical" type coordinate system with himself as the origin with a metric of the form

$$ds^2=(1+a)dz^2+(1+b)d\rho^2+\rho^2(1+C)d\phi^2 \\ +2\omega_l\rho^2(1+d)d\phi d\tau-(1+e)d\tau^2, \quad (30)$$

where a, b, c, d, and e are analytic functions which vanish at the origin. Here τ is the observer's proper time, but ϕ is still the unique ϕ defined earlier. Thus the $\phi=$const lines may be rotating with respect to the observer at some angular velocity ω_l so we have included the necessary $g_{\phi\tau}$ term in the metric. By comparing the Kerr metric of Eq. (1) with the metric of Eq. (30) at the point $r=r_0$, $\theta=0$ we have

$$[1-2mr_0/(r_0{}^2+b^2)]^{1/2}dt=d\tau. \quad (31)$$

Expanding the Kerr metric component $g_{\phi\phi}$ around the point $r=r_0$, $\theta=0$ and comparing its lowest-order term with the lowest-order term of $g_{\phi\phi}$ in Eq. (30), we get

$$\rho^2=\theta^2(r_0{}^2+b^2)+\text{higher-order terms}. \quad (32)$$

Finally we expand the Kerr $g_{\phi t}d\phi dt$ term at the point $r=r_0$, $\theta=0$, substitute in Eqs. (31) and (32), and compare it, to lowest order with the $g_{\phi t}d\phi dt$ term of Eq. (30). This gives

$$\omega_l=+\frac{2mr_0b}{(r_0{}^2+b^2)^2}\left(1-\frac{2mr_0}{r_0{}^2+b^2}\right)^{1/2}. \quad (33)$$

Thus the observer on the z axis sees the $\phi=$const lines rotating with an angular frequency ω_l with respect to his "inertial frame."[11] It easily follows that a distant observer sees the inertial frames on the z axis rotating with an angular frequency

$$\omega_\infty=-2mr_0b/(r_0{}^2+b^2)^2. \quad (34)$$

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[9] Notice that the constant Ω is itself determined just as we expect it should be on purely physical grounds. In Appendix B we see that Ω is also determined in the solution of the original equation [Eq. (21)].

[10] D. Brill and J. Cohen, Phys. Rev. **143**, 1011 (1966).

[11] This is not truly an inertial frame since the observer will feel a gravitational force in the z direction.

Notice that when $r_0=\infty$ or $b=0$ then $\omega_\infty=0$, as is desirable. Suppose next that a possible source of the Kerr metric were a doughnut or ring-shaped object which has been suggested as a possibility.[12] Then the Kerr metric describes the vacuum region all along the z axis. But notice that for $r_0=0$, $b\neq 0$ we have $\omega_\infty=0$. We interpret this as saying that some of the material in the ring is rotating in one direction and some in the opposite direction such that the net Machian effect at the center is zero. It follows that *rigidly* rotating rings of mass could not serve as sources of the Kerr metric. A closer investigation of the magnitude of the Mach effect might lead to a proof that any kind of rotating ring is not allowed as a source of the Kerr metric.

B. Interior Solution

The calculation of the induced rotations in the inertial frames on the z axis for the interior metric is done in exactly the same manner as for the exterior metric. We shall simply state the result

$$\omega_\infty=-2bmC(r_0)/[1+b^2k(r_0)][r_0^2+b^2B(r_0)]. \quad (35)$$

If we choose the $C(r)$ of Eq. (5) and evaluate ω_∞ at $r_0=0$, we get

$$\omega_\infty=-(2m/r_1)(5b/2r_1^2). \quad (36)$$

The Newtonian result of Eq. (26) indicates that b/r_1^2 is a measure of the angular velocity of the body while m/r_1 is a measure of the strength of the gravitational field. Thus it would appear that a large rotation rate for a body with a strong gravitational field would produce correspondingly large rotation rates in the inertial frames within the body. If we choose the $C(r)$ given by Eq. (28) and evaluate ω_∞ at $r_0=0$, we get

$$\omega_\infty=0. \quad (37)$$

As in the case of the ring source this indicates that the body is rotating with positive angular velocities at some points and negative angular velocities at others such that the total angular momentum is finite, but the net induced rotation in the inertial frame at the center of the body is zero. This interpretation is proven by the fact that this $C(r)$ does yield such a differential rotation in the Newtonian limit as given by Eq. (29).

As a final point of interest we show that the weak-field Mach effects as presented here are consistent with the results of Thirring.[13] We consider the weak-field case with $C(r)$ given by Eq. (25). As mentioned before this corresponds to a uniform-density sphere rotating uniformly with an angular velocity $\Omega=-5b/2r_1^2$. The Thirring result says that a thin spherical shell of mass M, radius R, and angular velocity ω_s contributes an induced rotation within the shell of

$$\omega'=\omega_s 4M/3R. \quad (38)$$

[12] For example, see E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965).

[13] H. Thirring, Z. Physik **19**, 33 (1918); **22**, 29 (1921).

Integrating the effects due to the "many" shells of our rotating sphere we have

$$\omega_\infty=\int_0^{r_1}\left(\frac{-5b}{r_1^2}\right)\frac{4}{3r}(4\pi r^2\rho)dr=\frac{-5mb}{r_1^3}, \quad (39)$$

which agrees with our result Eq. (36). Using the differential rotation velocity given by Eq. (29) and performing the integration we get zero Machian effect at the center which agrees with our result of Eq. (37).

APPENDIX A: NONSINGULAR COORDINATES ON Z AXIS

In order to show that the singularities along the z axis of the metric of Eq. (4) are coordinate effects only, we shall exhibit another coordinate system which makes both the metric and its inverse analytic at any point on the z axis. Consider the new coordinates (x,y,z) given by the following set of transformations:

$$\begin{aligned} x&=\rho\cos\Phi,\\ y&=\rho\sin\Phi,\\ z&=k_3 r,\\ \rho^2&=k_1^2\sin^2\theta,\quad \rho\leq k_1\\ \tau&=k_2 t,\\ \Phi&=\phi+\{2bm/k_1^2[1+b^2k(r_0)]\}t. \end{aligned} \quad (A1)$$

The constants k_1, k_2, and k_3 are defined in terms of the metric components of Eq. (4) by $(g_{\theta\theta})^{1/2}$, $(-g_{tt})^{1/2}$, and $(g_{rr})^{1/2}$ evaluated at a point ($\theta=0$, $r=r_0\neq 0$) along the z axis. Then by direct substitution one can show that the metric of Eq. (4) becomes the quasi-Cartesian form:

$$ds^2=dx^2+dy^2+dz^2+O[(x^2+y^2+z^2)^{1/2}]dx^\mu dx^\nu, \quad (A2)$$

with $r=r_0$, $\theta=0$ as the origin. It is analytic, has a nonvanishing det $(g_{\mu\nu})$ at the origin and hence has an analytic inverse at the origin. The point $\theta=0$, $r=0$ requires special consideration because in this case Eqs. (A1) give $\sin\theta=\rho/k_1$ which is no longer analytic. If we choose the functions A, B, C, F, and k such that they behave like

$$A(r)\sim r^2,\ B(r)\sim r^4,\ C(r)\sim r^2,\ F(r)\sim r^2,\ k(r)\sim r \quad (A3)$$

for small r, then the metric of Eq. (6) can immediately be expanded about the origin as

$$ds^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\Phi^2-d\tau^2+O(r^2)dx^\mu dx^\nu, \quad (A4)$$

where

$$\begin{aligned} \Phi&=\phi-2bm(C/r^2)_{r=0}\,t,\\ \tau&=[\tfrac{3}{2}(1-r_1^2/R^2)^{1/2}-\tfrac{1}{2}]t. \end{aligned} \quad (A5)$$

This is analytic with nonvanishing determinant at the origin. Thus we see the reason for the particular choice of functions of Eq. (5). Note that the metric of Eq. (A4) says that a particle at the center of the body feels no forces, which we know is necessary on physical grounds.

APPENDIX B: EXISTENCE OF SOLUTIONS FOR EQ. (21)

Our Eq. (22) can be expressed in the form

$$C''(r)+P_1(r)C'(r)+P_2(r)C(r)=\Omega Q(r)\,, \qquad \text{(B1)}$$

where r has the range, $0\leq r\leq r_1$, and $C(r)$ has the boundary conditions $C(r_1)=1$, $C'(r_1)=-1/r_1$. The coefficients P_1, P_2, and Q are analytic everywhere in the interval except for P_2 at the point $r=0$; but the function

$$r^2P_2 \qquad \text{(B2)}$$

is analytic at $r=0$, and so $r=0$ is a *regular singular point*.[14]

Consider the homogeneous equation

$$C''(r)+P_1(r)C'(r)+P_2(r)C(r)=0\,. \qquad \text{(B3)}$$

Expand the function C in powers of r as

$$C=\sum_{n=0}^{\infty} c_n r^{n+m}\,. \qquad \text{(B4)}$$

Next expand the known functions P_1 and P_2 in power of r also and substitute all this into Eq. (B3). Equation (B3) can then be simplified to

$$K_0r^m+K_1r^{m+1}+K_2r^{m+2}+\cdots=0\,, \qquad \text{(B5)}$$

where each K_i must be zero. The equation $K_0=0$ is called the indicial equation and in this example has the form

$$(m-2)(m+1)=0\,, \qquad \text{(B6)}$$

with solutions $m=2$, -1. A well-known theorem in differential equations[15] says that the Eq. (B3) has two independent solutions given by

$$C_A=r^2\sum_{n=0}^{\infty} c_n r^n, \quad c_0\neq 0 \qquad \text{(B7)}$$

$$C_B=r^{-1}\sum_{n=0}^{\infty} c_n^* r^n+kC_A\ln r, \quad c_0^*\neq 0\,, \qquad \text{(B8)}$$

where k is a constant and where the solution is valid over some deleted interval $0<r<r_0$. Another theorem[16] enables us to extend the range over which we know these solutions are valid to the interval $0<r\leq r_1$. The special solution of the nonhomogeneous equation can be found by the method of variation of parameters.[17] Assuming a special solution of the form

$$C_s=\nu_AC_A+\nu_BC_B\,, \qquad \text{(B9)}$$

Eqs. (4.40) of Ross give

$$\nu_A=-\int^r\frac{\Omega Q(t)C_B(t)dt}{W(t)}\,, \qquad \text{(B10)}$$

$$\nu_B=\int^r\frac{\Omega Q(t)C_A(t)dt}{W(t)}\,, \qquad \text{(B11)}$$

where $W(t)$ is the Wronskian

$$W(t)=C_A(t)C_B'(t)-C_A'(t)C_B(t)\,. \qquad \text{(B12)}$$

Thus the general solution of Eq. (B1) is given by

$$C=AC_A+BC_B+\Omega C_s\,, \qquad \text{(B13)}$$

where A and B are constants and Ω has been factored out of C_s. The solution C_B is obviously singular at $r=0$. An investigation of C_s reveals that it is not singular at $r=0$. Hence, choosing $B=0$, we have a solution which is regular at the origin and hence valid over the required closed interval $0\leq r\leq r_1$. The other two constants A and Ω are to be determined by the boundary equations

$$AC_A(r_1)+\Omega C_s(r_1)=1\,, \qquad \text{(B14)}$$

$$AC_A'(r_1)+\Omega C_s'(r_1)=-1/r_1\,. \qquad \text{(B15)}$$

This pair of equations have a solution if the determinant

$$D=C_A(r_1)C_s'(r_1)-C_A'(r_1)C_s(r_1) \qquad \text{(B16)}$$

is nonzero. By use of Eqs. (B9)–(B11) it is easy to transform the right-hand side of Eq. (B16) so that it becomes

$$D=\nu_B(r_1)W(r_1)\,.$$

Since the Wronskian W of the two independent solutions C_A and C_B is never zero, one must simply choose the integration constant of Eq. (B11) so that $\nu_B(r_1)\neq 0$ in order to guarantee that D is nonzero. Note (1) that the angular velocity Ω is determined uniquely and (2) that the solution $C(r)$ has the required r^2 behavior for small r as discussed in Appendix A.

[14] For example, see S. L. Ross, *Differential Equations* (Blaisdell Publishing Co., Inc., New York, 1964), p. 190.
[15] See Ref. 14, p. 196.
[16] For example, see W. Kaplan, *Ordinary Differential Equations* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1958), p. 115.
[17] See Ref. 14, p. 120.