Equation (14) is derivable from the variational principle

$$\delta\int \operatorname{csch}^2(\operatorname{Re}\nu)\nabla\nu\cdot\nabla\nu^* dv=0. \quad (31)$$

If one performs the nonanalytic transformation

$$\operatorname{Im}\nu=\alpha',$$
$$\operatorname{Re}\nu=\ln\cosh(\beta'/2), \quad (32)$$

the variational principle assumes a form quite similar to that of Matzner and Misner; namely,

$$\delta\int [|\nabla\beta'|^2+\sinh^2\beta'|\nabla\alpha'|^2]dv=0. \quad (33)$$

This Lagrangian is related to the metric for a timelike hyperboloid in a Lorentz 3-space with polar angles β', α'. For a nearly spherically symmetric situation, α' and β' are much simpler than α and β (although even α' and β' are more complicated than $\mathcal{E}$, ξ, or ν). It would be very interesting if this simple geometrical picture could be used effectively to deduce solutions of the vacuum field equations other than those discussed in Sec. III.

VI. CONCLUSIONS

The reformulation of the axially symmetric gravitational field problem in terms of the $\mathcal{E}$ equation facilitates an intensive investigation of solutions corresponding to uniformly rotating sources. In particular, it permits a simple derivation of the Kerr metric which proceeds from the assumption of axial symmetry rather than the assumption that the metric is algebraically special.

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Conservation of Momentum and Angular Momentum in Relativistic Classical Particle Mechanics

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For a classical-mechanical system of two particles, the conditions for Lorentz-invariant equations of motion are expressed in terms of relativistic momentum variables, and are shown to imply that neither the conventional total kinematic particle momentum nor the conventional total kinematic particle angular momentum is a constant of the motion unless the accelerations are zero. This is compared with a theorem of Van Dam and Wigner.

In a classical-mechanical system of two particles, interactions can be described by relativistically invariant equations of motion which specify the accelerations as functions of the positions and velocities at one time.[1–4] As yet, their properties are mostly unexplored. Here we show that their constants of the motion include neither the conventional total momentum nor the conventional total angular momentum as defined for free particles or for particles in a field, i.e., the kinematic particle momentum and angular momentum. These quantities could have the same values before and after a collision by being asymptotic limits of constants of the motion. The constants of the motion would depend on the interaction. They could be momentum and angular momentum which correspond to the generators of space translations and rotations.

Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be the positions of the particles, and let $\mathbf{v}^1$ and $\mathbf{v}^2$ be their velocities. (These are three-vectors.) Let

$$\mathbf{u}^n=m_n\mathbf{v}^n[1-(\mathbf{v}^n)^2]^{-1/2}$$

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[1] D. G. Currie, Phys. Rev. **142**, 817 (1966).
[2] R. N. Hill, J. Math. Phys. **8**, 201 (1967).
[3] D. G. Currie (to be published).
[4] D. G. Currie and T. F. Jordan, 1967 Summer Institute for Theoretical Physics, University of Colorado (to be published).

for $n=1,2$, with m_1 and m_2 positive numbers. Let $\mathbf{x}=\mathbf{x}^1-\mathbf{x}^2$ and $\mathbf{v}=\mathbf{v}^1-\mathbf{v}^2$. Consider equations of motion

$$d\mathbf{u}^n/dt=\mathbf{f}^n(\mathbf{u}^1,\mathbf{u}^2,\mathbf{x})$$

for $n=1,2$. Translation invariance implies that $\mathbf{f}^1$ and $\mathbf{f}^2$ depend on the positions of the particles only as functions of the relative position $\mathbf{x}$. Rotation invariance implies that $\mathbf{f}^1$ and $\mathbf{f}^2$ rotate as vectors when $\mathbf{u}^1$, $\mathbf{u}^2$, and $\mathbf{x}$ are rotated.

The conditions for Lorentz invariance are

$$\begin{aligned}(-1)^n x_j f^{n'}{}_l \partial f^n{}_k/\partial u^{n'}{}_l + \delta_{jk}(\mathbf{u}^n\cdot\mathbf{f}^n)[(\mathbf{u}^n)^2+m_n{}^2]^{-1/2} \\ -u^n{}_j[(\mathbf{u}^n)^2+m_n{}^2]^{-1/2}f^n{}_k \\ +x_j[(\mathbf{u}^{n'})^2+m_{n'}{}^2]^{-1/2}u^{n'}{}_l\partial f^n{}_k/\partial x_l \\ -[(\mathbf{u}^n)^2+m_n{}^2]^{1/2}\partial f^n{}_k/\partial u^n{}_j \\ -[(\mathbf{u}^{n'})^2+m_{n'}{}^2]^{1/2}\partial f^n{}_k/\partial u^{n'}{}_j=0\end{aligned}$$

for $j,k,l=1,2,3$ and $n,n'=1,2$ with n' different from n (if $n=1$ then $n'=2$ and if $n=2$ then $n'=1$); the repeated index l implies a sum. These conditions are derived from the usual Lorentz transformation of space-time coordinates and from the requirement that for an infinitesimal Lorentz transformation the change of $d\mathbf{u}^n/dt$ is the same as the change of $\mathbf{f}^n$ as a function of $\mathbf{u}^1$, $\mathbf{u}^2$, and $\mathbf{x}$.[1,2,4] We use these conditions to obtain the following.

Theorem 1: The conventional total kinematic particle momentum $\mathbf{u}^1+\mathbf{u}^2$ is not a constant of the motion unless both $\mathbf{f}^1$ and $\mathbf{f}^2$ are zero.[5]

Proof: Suppose $\mathbf{u}^1+\mathbf{u}^2$ is a constant of the motion. Then $\mathbf{f}^1+\mathbf{f}^2=0$. Let $\mathbf{e}$ be any three-vector orthogonal to $\mathbf{x}$. Multiplying the conditions for Lorentz invariance by e_j, summing for $j=1,2,3$, and adding the result for $n=1,2$ yield

$$\begin{aligned}[(\mathbf{u}^1)^2+m_1{}^2]^{-1/2}[(\mathbf{u}^1\cdot\mathbf{f}^1)\mathbf{e}-(\mathbf{u}^1\cdot\mathbf{e})\mathbf{f}^1] \\ +[(\mathbf{u}^2)^2+m_2{}^2]^{-1/2}[(\mathbf{u}^2\cdot\mathbf{f}^2)\mathbf{e}-(\mathbf{u}^2\cdot\mathbf{e})\mathbf{f}^2]=0\end{aligned}$$

or

$$\{[(\mathbf{u}^1)^2+m_1{}^2]^{-1/2}\mathbf{u}^1-[(\mathbf{u}^2)^2+m_2{}^2]^{-1/2}\mathbf{u}^2\}\times(\mathbf{e}\times\mathbf{f}^1)=0,$$

which means that $\mathbf{e}\times\mathbf{f}^1$ is collinear with the relative velocity

$$\mathbf{v}=\mathbf{v}^1-\mathbf{v}^2=[(\mathbf{u}^1)^2+m_1{}^2]^{-1/2}\mathbf{u}^1-[(\mathbf{u}^2)^2+m_2{}^2]^{-1/2}\mathbf{u}^2.$$

If $\mathbf{f}^1$ is not zero, then $\mathbf{e}$ is orthogonal to $\mathbf{v}$, which means that $\mathbf{x}$ and $\mathbf{v}$ are collinear, and $\mathbf{f}^1$ is orthogonal to $\mathbf{v}$.

Theorem 2: The conventional total kinematic particle angular momentum $\mathbf{x}^1\times\mathbf{u}^1+\mathbf{x}^2\times\mathbf{u}^2$ is not a constant of the motion unless both $\mathbf{f}^1$ and $\mathbf{f}^2$ are zero.[5]

Proof: Suppose $\mathbf{x}^1\times\mathbf{u}^1+\mathbf{x}^2\times\mathbf{u}^2$ is a constant of the motion. Then $\mathbf{x}^1\times\mathbf{f}^1+\mathbf{x}^2\times\mathbf{f}^2=0$. Let $\mathbf{e}$ be any three-vector orthogonal to $\mathbf{x}$. Multiplying the conditions for Lorentz invariance by $e_j\epsilon_{klm}x^n{}_l$, summing for $j,k,l=1,2,3$, using $\mathbf{v}^n=\mathbf{u}^n[(\mathbf{u}^n)^2+m_n{}^2]^{-1/2}$, and adding the results for $n=1,2$ yield the mth component of

$$\begin{aligned}(\mathbf{v}^1\cdot\mathbf{f}^1)\mathbf{e}\times\mathbf{x}^1+(\mathbf{v}^2\cdot\mathbf{f}^2)\mathbf{e}\times\mathbf{x}^2 \\ -(\mathbf{e}\cdot\mathbf{v}^1)\mathbf{f}^1\times\mathbf{x}^1-(\mathbf{e}\cdot\mathbf{v}^2)\mathbf{f}^2\times\mathbf{x}^2=0.\end{aligned}$$

Taking the scalar product of this with $\mathbf{e}$, we get

$$(\mathbf{e}\cdot\mathbf{v})\mathbf{x}^1\times\mathbf{f}^1\cdot\mathbf{e}=0.$$

If $\mathbf{f}^1$ is not zero, then $\mathbf{e}\cdot\mathbf{v}$ is zero, which means that $\mathbf{x}$ and $\mathbf{v}$ are collinear.

These techniques were used previously to show that $m_1\mathbf{v}^1+m_2\mathbf{v}^2$ is not a constant of the motion unless the accelerations are zero.[6]

Van Dam and Wigner[7] prove a similar theorem that states that there is no interaction if the conventional total kinematic particle momentum and the total particle kinetic energy are constants of the motion and the particles are free asymptotically. Their proof is for two, three, or four particles. It does not use equations of motion. We do not assume that the total kinetic energy is a constant of the motion. Since we do not use an asymptotic condition, our result holds for bound systems as well as for collisions. Van Dam and Wigner[7] obtain constants of the motion by adding interaction terms to the conventional kinematic particle momentum and angular momentum. For this they use their equations of motion, which are manifestly invariant but involve positions and velocities at more than one time.[8] They find that the interaction term of the momentum vanishes asymptotically.

[5] We assume that $\mathbf{f}^1$ and $\mathbf{f}^2$ are functions which are differentiable enough for the Lorentz-invariance conditions to be meaningful. We do not consider singular functions which are zero for almost all values of $\mathbf{u}^1$, $\mathbf{u}^2$, and $\mathbf{x}$. Thus, if $\mathbf{f}^1$ and $\mathbf{f}^2$ are nonzero only when the relative position $\mathbf{x}$ is collinear with the relative velocity $\mathbf{v}$, we say that $\mathbf{f}^1$ and $\mathbf{f}^2$ are zero.

[6] D. G. Currie and T. F. Jordan, Phys. Rev. Letters **16**, 1210 (1966).

[7] H. Van Dam and E. P. Wigner, Phys. Rev. **142**, 838 (1966).

[8] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, B1576 (1965).