$M'<10\,M_\odot$. In fact, the situation is worse than this: The mass of an eventual condensation is $M<M'\Delta B/N$, or $M<10^{-8}\,M_\odot$. We therefore come to the conclusion that the time available is too short for the formation of baryon inhomogeneities, and that if inhomogeneities are the explanation of large-scale aggregations of matter, then they are an integral part of the universe from the beginning of its expansion.

The case for density fluctuations is certainly no better. For a galactic mass of $M=10^{10}\,M_\odot$, we have from (45), $\rho\ll 10^{-2}$ g cm^{-3} (since $M'\gg M$), and the instability hypothesis leads to the improbable conclusion that the foundations of galactic structure are laid down in the radiation era. If we accept the primordial-structure hypothesis and assume that density inhomogeneities exist from the earliest moments, we are still in difficulty because small amplitude fluctuations are not amplified,[30] and the density inhomogeneity is of the order 10^9 times greater than the required compositional inhomogeneity.

[30] Nor are they dissipated. Any dissipation mechanism [such as Misner's (Ref. 4)] in the lepton or hadron eras causes only the very shortwavelengths of $\lambda<ct$ to decay.

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New Formulation of the Axially Symmetric Gravitational Field Problem

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The field equations governing the gravitational field of a uniformly rotating axially symmetric source are reformulated in terms of a simple variational principle. The new formalism affords a concise unified derivation of the solutions discovered by Weyl and Papapetrou, and permits a simple derivation of the Kerr metric in terms of prolate spheroidal coordinates. More complex solutions are identified by applying perturbation theory.

I. INTRODUCTION

Of considerable current interest is the problem of finding the gravitational field of a uniformly rotating body. Although a possible exterior field has been found by Kerr, who investigated algebraically special metrics, attempts to generalize the Kerr solution of the vacuum field equations have not been marked by success.[1] In the present paper the problem is reformulated in terms of a complex function $\mathcal{E}$ independent of azimuth, which must be chosen in accordance with the variational principle

$$\delta\int\frac{\nabla\mathcal{E}\cdot\nabla\mathcal{E}^*}{(\mathrm{Re}\mathcal{E})^2}dv=0\,,\tag{1}$$

where dv is the three-dimensional Euclidean volume element. When such a complex $\mathcal{E}$ function is found, a corresponding axially symmetric solution of Einstein's vacuum field equations may be constructed.

This formulation of the axially symmetric gravitational field problem has a number of nice features. Neither the variational principle nor the corresponding field equation

$$(\mathrm{Re}\mathcal{E})\nabla^2\mathcal{E}=\nabla\mathcal{E}\cdot\nabla\mathcal{E}\tag{2}$$

makes reference to a particular coordinate system. According to one's desires, one may work with the equations in an abstract manner, or express them in terms of cylindrical, prolate spheroidal, or any other coordinates. Furthermore, the field equation is homogeneous quadratic, and it serves as an excellent vehicle for the application of perturbation theory. Finally, all the known axially symmetric solutions can be expressed simply in terms of the $\mathcal{E}$ function.

[1] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963). A detailed discussion has been given by F. J. Ernst, in Proceedings of the Relativity Seminar of the Illinois Institute of Technology (unpublished).

II. DERIVATION OF THE $\mathcal{E}$ EQUATION

Following Papapetrou we express the line element in the form

$$ds^2=f^{-1}[e^{2\gamma}(dz^2+d\rho^2)+\rho^2d\phi^2]-f(dt-\omega d\phi)^2,\tag{3}$$

where f, ω, and γ are functions of z and ρ only.[2] The complete set of field equations may be derived either using traditional tensor methods or the currently fashionable methods of exterior calculus. However, we are presently interested in the equations governing f and ω only, and these may be obtained from the Lagrangian density

$$\mathfrak{L}=-\tfrac{1}{2}\rho f^{-2}\nabla f\cdot\nabla f+\tfrac{1}{2}\rho^{-1}f^2\nabla\omega\cdot\nabla\omega\,.$$

Varying the functions f and ω we obtain the field equations

$$f\nabla^2 f=\nabla f\cdot\nabla f-\rho^{-2}f^4\nabla\omega\cdot\nabla\omega\,,\tag{4}$$

$$\nabla\cdot(\rho^{-2}f^2\nabla\omega)=0\,.\tag{5}$$

[2] A. Papapetrou, Ann. Physik **12**, 309 (1953).

In both equations the *three*-dimensional divergence operator is to be understood. On the other hand, if $\hat{n}$ is a unit vector in the azimuthal direction and φ is any reasonable function independent of azimuth, one has the identity

$$\nabla\cdot(\rho^{-1}\hat{n}\times\nabla\varphi)=0. \qquad (6)$$

Equation (5) may be regarded as the integrability condition for the existence of the function φ defined by

$$\rho^{-1}f^2\nabla\omega=\hat{n}\times\nabla\varphi. \qquad (7)$$

Since this relation is equivalent to

$$f^{-2}\nabla\varphi=-\rho^{-1}\hat{n}\times\nabla\omega,$$

the identity (6) implies the field equation

$$\nabla\cdot(f^{-2}\nabla\varphi)=0 \qquad (8)$$

for the new potential φ. When Eq. (4) is expressed in terms of the function φ and compared with Eq. (8), one sees that the complex function

$$\mathcal{E}=f+i\varphi \qquad (9)$$

satisfies the simple homogeneous quadratic differential equation (2).

There are a number of simple modifications of the $\mathcal{E}$ equation which are also of use. For example, if one substitutes

$$\mathcal{E}=(\xi-1)/(\xi+1), \qquad (10)$$

one obtains the differential equation

$$(\xi\xi^*-1)\nabla^2\xi=2\xi^*\nabla\xi\cdot\nabla\xi, \qquad (11)$$

which is derivable from the variational principle

$$\delta\int\frac{\nabla\xi\cdot\nabla\xi^*}{(\xi\xi^*-1)^2}dv=0. \qquad (12)$$

Furthermore, it is sometimes convenient to write either $\mathcal{E}$ or ξ in exponential form. In this case one has the following results:

$$\mathcal{E}=e^{\mu};\quad \nabla^2\mu=i\tan(\mathrm{Im}\mu)\nabla\mu\cdot\nabla\mu, \qquad (13)$$

$$\xi=e^{\nu};\quad \nabla^2\nu=\coth(\mathrm{Re}\nu)\nabla\nu\cdot\nabla\nu. \qquad (14)$$

III. KNOWN EXACT SOLUTIONS OF THE $\mathcal{E}$ EQUATION

It is obvious from Eq. (11) that if ξ is a solution, then so is $e^{i\alpha}\xi$, where α is any real constant. In the case of a constant-phase solution, we may introduce a new potential ψ such that

$$\xi=-e^{i\alpha}\coth\psi. \qquad (15)$$

The real function ψ then satisfies the Laplace equation

$$\nabla^2\psi=0. \qquad (16)$$

Consequently we can express ψ in terms of a "multipole expansion." The solutions found in this way are, of course, the well-known fields studied by Weyl[3] in 1917 ($\alpha=0$) and by Papapetrou[2] in 1953 ($\alpha=\frac{1}{2}\pi$). Unfortunately, whenever $\alpha\neq0$ (modπ) one must exclude the monopole contribution from ψ if the space is to become flat at infinity. Thus, the latter solutions do not seem to possess great physical significance.

As has been emphasized by Zipoy, there is merit in separating Eq. (16) in prolate spheroidal coordinates rather than cylindrical coordinates.[4] If we set

$$\rho=(x^2-1)^{1/2}(1-y^2)^{1/2},$$
$$z=xy, \qquad (17)$$

the Laplacian operator assumes the form

$$\nabla^2=\frac{1}{x^2-y^2}\left[\frac{\partial}{\partial x}(x^2-1)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}(1-y^2)\frac{\partial}{\partial y}\right]. \qquad (18)$$

It is also worthwhile to note that

$$\nabla A\cdot\nabla B=\frac{1}{x^2-y^2}\left[(x^2-1)\frac{\partial A}{\partial x}\frac{\partial B}{\partial x}+(1-y^2)\frac{\partial A}{\partial y}\frac{\partial B}{\partial y}\right] \qquad (19)$$

in the event that one wishes to write various abstract equations in terms of prolate spheroidal coordinates.

From the form of expression (18) it is clear that the solution ψ of Eq. (16) can be expressed as a linear superposition

$$\psi=\sum_l \alpha_l Q_l(x)P_l(y) \qquad (20)$$

of Legendre functions. In the case of pure $l=0$ we have

$$\psi=\tfrac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\quad\text{or}\quad \xi=x. \qquad (21)$$

It may be verified that this case corresponds to the Schwarzschild field. Choosing the unit of length to be m, we may indentify the Schwarzschild radial coordinate $r=x+1$ and the angular coordinate $\cos\theta=y$.

An attractive feature of prolate spheroidal coordinates is that both of the operators (18) and (19) are symmetric under the interchange of x and y. Consequently, if $\xi(x,y)$ is a solution of Eq. (11), then so is $\xi(y,x)$. When this transformation is applied to the Schwarzschild solution (21), we arrive at a new solution corresponding to $\xi=y$. If one then looks for a linear combination of the solutions $\xi=x$ and $\xi=y$ with the property that it also satisfies Eq. (11), one arrives at the following solution:

$$\xi=x\cos\lambda+iy\sin\lambda. \qquad (22)$$

The parameter λ may assume any real value.

The solution (22) is completely equivalent to the metric discovered by Kerr while he was searching for algebraically special metrics. To facilitate comparison

[3] H. Weyl, Ann. Physik **54**, 117 (1917).
[4] D. Zipoy, J. Math. Phys. **7**, 1137 (1966).

we introduce $\tan\lambda = a$ and $\sec\lambda = m$, noting that distances are measured in units of $(m^2-a^2)^{1/2}$. When the entire metric is constructed, one obtains

$$ds^2=\left[(r^2+a^2\cos^2\theta)\left(d\theta^2+\frac{dr^2}{r^2+a^2-2mr}\right)\right]+\left[(r^2+a^2)\sin^2\theta d\phi^2-dt^2+\frac{2mr}{r^2+a^2\cos^2\theta}(dt+a\sin^2\theta d\phi)^2\right], \quad (23)$$

where the coordinates are defined by

$$r=x(m^2-a^2)^{1/2}+m, \quad \cos\theta=y. \quad (24)$$

The result may be cast into Papapetrou's canonical form (3) by the transformation

$$\rho=(r^2+a^2-2mr)^{1/2}\sin\theta,$$
$$z=(r-m)\cos\theta. \quad (25)$$

To achieve the form published by Kerr, it is only necessary to introduce the coordinates

$$u=t+\int\left(1-\frac{2mr}{r^2+a^2}\right)^{-1}dr,$$
$$\phi'=\phi-\int\frac{adr}{r^2+a^2-2mr}. \quad (26)$$

IV. PERTURBATION SOLUTIONS

The $\mathcal{E}$ equation, either in the form (2) or in the modified form (11), provides a suitable basis for a perturbation treatment of the field equations. In the zeroth order we assume $\xi=x$. The imaginary first-order correction satisfies the linear partial differential equation

$$\frac{\partial}{\partial x}(x^2-1)\frac{\partial\xi_1}{\partial x}+\frac{\partial}{\partial y}(1-y^2)\frac{\partial\xi_1}{\partial y}=4x\frac{\partial\xi_1}{\partial x}-2\xi_1, \quad (27)$$

from which a Laplace equation

$$\nabla^2(\partial^2\xi_1/\partial x^2)=0$$

can be deduced. Thus we conclude that

$$\frac{\partial^2\xi_1}{\partial x^2}=i\sum_{l=2}^{\infty}\alpha_l Q_l(x)P_l(y).$$

The two integrations may be performed by using the identity

$$\int Q_l(x)dx=(2l+1)^{-1}(Q_{l+1}-Q_{l-1}).$$

The result,

$$\xi_1=i\sum_{l=2}^{\infty}\alpha_l\left(\int\int Q_l(x)\right)P_l(y)+iay, \quad (28)$$

which we have written in a somewhat symbolic fashion, can readily be shown to constitute a solution of Eq. (27). The Kerr metric corresponds to the case in which all $\alpha_l=0$. Thus, if perturbation theory is a reliable guide, it would seem that the Kerr metric constitutes a particularly trivial special case within a class of much more complicated solutions.

Currently we are studying the convergence of the perturbation series in an attempt to develop a more convincing proof of the existence of corresponding exact solutions. In addition, we are trying to determine what characteristics of a uniformly rotating source give rise to contributions of the type $l>1$.

V. COMPARISON WITH EARLIER FORMULATIONS

The fundamental distinction between the present approach and previous formulations of the axially symmetric field problem is that ours is the first canonical description which employs a field $\mathcal{E}$ not expressible in terms of the metric tensor $g_{\mu\nu}$ by a point transformation.[5] The extreme importance of this complex field $\mathcal{E}$ is indicated by the simplicity of all known solutions, when they are expressed in terms of this formalism. Thus, for pragmatic reasons we are very reluctant to employ nonanalytic functions of $\mathcal{E}$ as the basic field variables.

The formulation of the axially symmetric field problem published by Matzner and Misner[6] differs radically in point of view. In the first place their field variables α and β are obtained from the metric tensor $g_{\mu\nu}$ by a point transformation; viz.,

$$g_{tt}=-\rho(\cos\alpha\cosh\beta+\sinh\beta)=-f,$$
$$g_{\phi\phi}=\rho(\cos\alpha\cosh\beta-\sinh\beta), \quad (29)$$
$$g_{\phi t}=\rho\sin\alpha\cosh\beta=f\omega.$$

Although this transformation leads to a simple variational principle,

$$\delta\int[|\nabla\beta|^2-\cosh^2\beta|\nabla\alpha|^2]dv=0, \quad (30)$$

the appearance of ρ in the transformation makes α and β assume complicated forms for a problem involving nearly spherical symmetry.

Nevertheless, the Matzner-Misner formulation is attractive in that their Lagrangian is related to the metric for a spacelike hyperboloid in a Lorentz 3-space. Such a picturesque description is likely to be much more fruitful when reformulated in terms of the $\mathcal{E}$ field as follows:

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[5] We are using the term "point transformation" in the sense explained by H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1950), p. 238.

[6] R. A. Matzner and C. W. Misner, Phys. Rev. **154**, 1229 (1967).

Equation (14) is derivable from the variational principle

$$\delta\int \mathrm{csch}^2(\mathrm{Re}\nu)\nabla\nu\cdot\nabla\nu^* dv=0. \quad (31)$$

If one performs the nonanalytic transformation

$$\mathrm{Im}\nu=\alpha',$$
$$\mathrm{Re}\nu=\ln\cosh(\beta'/2), \quad (32)$$

the variational principle assumes a form quite similar to that of Matzner and Misner; namely,

$$\delta\int [|\nabla\beta'|^2+\sinh^2\beta'|\nabla\alpha'|^2]dv=0. \quad (33)$$

This Lagrangian is related to the metric for a timelike hyperboloid in a Lorentz 3-space with polar angles β', α'. For a nearly spherically symmetric situation, α' and β' are much simpler than α and β (although even α' and β' are more complicated than $\mathcal{E}$, ξ, or ν). It would be very interesting if this simple geometrical picture could be used effectively to deduce solutions of the vacuum field equations other than those discussed in Sec. III.

VI. CONCLUSIONS

The reformulation of the axially symmetric gravitational field problem in terms of the $\mathcal{E}$ equation facilitates an intensive investigation of solutions corresponding to uniformly rotating sources. In particular, it permits a simple derivation of the Kerr metric which proceeds from the assumption of axial symmetry rather than the assumption that the metric is algebraically special.

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Conservation of Momentum and Angular Momentum in Relativistic Classical Particle Mechanics

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For a classical-mechanical system of two particles, the conditions for Lorentz-invariant equations of motion are expressed in terms of relativistic momentum variables, and are shown to imply that neither the conventional total kinematic particle momentum nor the conventional total kinematic particle angular momentum is a constant of the motion unless the accelerations are zero. This is compared with a theorem of Van Dam and Wigner.

IN a classical-mechanical system of two particles, interactions can be described by relativistically invariant equations of motion which specify the accelerations as functions of the positions and velocities at one time.[1-4] As yet, their properties are mostly unexplored. Here we show that their constants of the motion include neither the conventional total momentum nor the conventional total angular momentum as defined for free particles or for particles in a field, i.e., the kinematic particle momentum and angular momentum. These quantities could have the same values before and after a collision by being asymptotic limits of constants of the motion. The constants of the motion would depend on the interaction. They could be momentum and angular momentum which correspond to the generators of space translations and rotations.

Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be the positions of the particles, and let $\mathbf{v}^1$ and $\mathbf{v}^2$ be their velocities. (These are three-vectors.) Let

$$\mathbf{u}^n=m_n\mathbf{v}^n[1-(\mathbf{v}^n)^2]^{-1/2}$$

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[1] D. G. Currie, Phys. Rev. **142**, 817 (1966).
[2] R. N. Hill, J. Math. Phys. **8**, 201 (1967).
[3] D. G. Currie (to be published).
[4] D. G. Currie and T. F. Jordan, 1967 Summer Institute for Theoretical Physics, University of Colorado (to be published).