Baryon Inhomogeneity in the Early Universe

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In the early high-density high-temperature universe, matter and antimatter are in equilibrium and exist in almost equal quantities. In a comoving element of volume the ratio of the baryon number ΔB to the number of baryons B is small. We must explain why initially there is a slight excess of matter over antimatter and why finally, after annihilation, matter is concentrated into objects of galactic size. By proposing that the initial conditions consist of spatial variations in the baryon number rather than variations in density (i.e., the number of baryons), we can link together the survival and aggregation of matter or antimatter as closely related subjects. Various topics are discussed, such as: the amplification of baryon compositional inhomogeneity $\Delta B/B$ in an expanding universe; the prestellar hadron, lepton, and radiation eras; the effect of inhomogeneity on final composition; the approximate constancy of the total number of particles (including photons) in a comoving element of volume; the muon-neutrino and electron-neutrino temperatures; and the required amount of initial baryon inhomogeneity.

1. INTRODUCTION

THE existence of matter in the universe, and its concentration into astrophysical structures, raises problems of great importance in cosmology. It can be shown that the existence of matter and its aggregation into large structures are the result of compositional inhomogeneity in an early era of the expanding universe.[1]

Initially an expanding universe contains energy at high density, and equal or almost equal amounts of matter and antimatter are in equilibrium. Subsequently, the energy density diminishes, and pair annihilation occurs at various stages of expansion. When matter and antimatter are present in equal amounts, the eventual products are principally photons and neutrinos, as shown by Zeldovich[2] and Chiu.[3] The survival of matter implies that initially in the dense universe there is a slight excess of matter over antimatter. The problem is to determine why there is this slight initial excess and why it consists of matter rather than antimatter.

Also puzzling is the aggregation of matter into objects as large as galaxies. The relatively simple and featureless cosmological environment does not appear to favor the formation of large condensations.[4,5] If $\mu=\delta\rho/\rho$ is the contrast density, where ρ is the density and $\delta\rho$ a space-varying perturbation, linearized gravitational theory shows that under optimum conditions,

$$\mu=\mu_\alpha(\rho_\alpha/\rho)^{1/3}, \tag{1}$$

where μ_α is the initial value at density ρ_α. (This result ignores thermal instabilities and assumes that the curvature constant is of negligible effect.) According to (1), the contrast density is amplified during expansion; but unfortunately μ/μ_α cannot be made arbitrarily large by supposing that the fluctuations evolve from a sufficiently early epoch. When $\rho_\alpha>\rho_c\sim10^{-21}$ g cm^{-3}, the universal background radiation[6–8] has a density greater than the density of matter, and radiative drag[9] effectively couples matter to the radiation field and impedes the formation of condensations. Furthermore, the pressure is close to one-third the energy density, and as a consequence, μ oscillates at constant amplitude.[10] Hence in (1), amplification begins at $\rho_\alpha\sim\rho_c$, and since $\rho\sim10^{-24}$ g cm^{-3} is a typical mean value for the galaxies, it follows that the final value is $\mu\sim10\mu_\alpha$. It is unlikely that thermodynamic instabilities[11–13] will appreciably alter this result. As emphasized by Lifshitz[10] and Bonnor,[14] density fluctuations do not possess pronounced instability, and we are forced to the conclusion that when $\rho\sim\rho_c$, there already exist large-amplitude density variations of matter (but not necessarily of radiation) deriving from an earlier epoch. Evidently the existence of matter and its fragmentation into galaxies are as yet unresolved problems.[12]

It is suggested by Goldhaber[15] that the symmetry of particles and antiparticles is reflected in a particle-antiparticle population symmetry in the universe. Alpher and Herman[16] have discussed this proposal, and it was shown earlier[17] that statistical fluctuations cannot account for appreciable separation in a particle-anti-

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[6] A. A. Penzias and R. W. Wilson, Astrophys. J. 142, 419 (1965).
[7] R. H. Dicke, P. J. E. Peebles, P. G. Roll, and D. T. Wilkinson, Astrophys. J. 142, 414 (1965).
[8] For earlier work on radiation in the "hot Gamow model," see G. Gamow, Nature 162, 680 (1948); Rev. Mod. Phys. 21, 367 (1949); Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 27, No. 10 (1953). For further references, see R. A. Alpher and R. C. Herman, Rev. Mod. Phys. 22, 153 (1950).
[9] P. J. E. Peebles, Astrophys. J. 142, 1317 (1965).
[10] E. M. Lifshitz, J. Phys. USSR 10, 116 (1946); R. K. Sachs and A. M. Wolfe, Astrophys. J. 147, 73 (1967).
[11] G. B. Field, Astrophys. J. 142, 531 (1965).
[12] E. R. Harrison, Mem. Soc. Roy. Sci. Liege 14, 15 (1967).
[13] W. C. Saslaw, Monthly Notices Roy. Astron. Soc. 136, 39 (1967).
[14] W. B. Bonnor, Z. Astrophys. 39, 143 (1956); Monthly Notices Roy. Astron. Soc. 117, 104 (1957).
[15] M. Goldhaber, Science 124, 218 (1958).
[16] R. A. Alpher and R. C. Herman, Science 128, 904 (1958).
[17] R. A. Alpher, J. W. Follin, and R. C. Herman, Phys. Rev. 92, 1347 (1953).

particle medium. The idea of population symmetry is pursued further by Alfvén and Klein,[18] and it is apparent that the main problem is to discover an effective mechanism which separates matter and antimatter into discrete regions. Attempts to solve the first problem (the existence of matter) by postulating population symmetry have so far succeeded only in making more difficult the solution of the second problem (the aggregation of matter).

Although there is little to be said at present on why the primordial universe contains a foundation of rudimentary structure, we can at least show that the survival and the aggregation of matter are intimately related subjects. We break away from the notion that the initial structure consists of density variations, and propose instead that it consists of spatial variations in the slight difference between matter and antimatter. Complete annihilation is impossible with this type of initial condition, and there remains eventually a fragmented universe consisting of separated regions of matter and antimatter. This proposal has the advantage that fluctuations of composition are amplified far more efficiently than density fluctuations in a fluid of uniform composition.

2. HOMOGENEOUS COMPOSITION

The *number of baryons*[19] per unit volume is

$$b=\sum_i b_i, \tag{2}$$

where b_i is the number of the ith kind. The *baryon number* per unit volume, however, is

$$\Delta b=\sum_i \epsilon_i b_i, \tag{3}$$

where $\epsilon_i=+1$ for baryons, and $\epsilon_i=-1$ for antibaryons. If b^+ (b^-) is the number density of baryons (antibaryons), then $b=b^++b^-$, $\Delta b=b^+-b^-$. Let V be a comoving element of volume; we can then say that $Vb=B=B^++B^-$ and $\Delta B=B^+-B^-$, and that ΔB is a conserved baryon number, whereas the number of baryons B is not a conserved quantity. In a fluid of homogeneous composition, particle motions will not affect the conservation of ΔB.

At an early epoch, the density and temperature are large, and the ratio

$$\frac{\text{baryon number}}{\text{number of baryons}}=\frac{\Delta B}{B} \tag{4}$$

is small. (The amounts of matter and antimatter are closely equal.) As the universe expands, the density and temperature drop, and baryon-pair annihilation causes $B\rightarrow|B^+-B^-|=|\Delta B|$, or

$$\Delta B/B\rightarrow\pm1, \tag{5}$$

and the number of baryons $=\pm$ the baryon number. Of course complete annihilation never occurs, but the difference from unity in (5) at later epochs is small.[2,3]

We now suppose that the b_i are space-varying:

$$b_i=b_{i0}+\delta b_i, \tag{6}$$

where the b_{i0} are the unperturbed densities. In a fluid of homogeneous composition, all $\delta b_i/b_i$ are identical (and similarly for all nonbaryons), and hence

$$\delta b_i/b_{i0}=\delta b/b_0=\delta\rho/\rho_0, \tag{7}$$

where ρ is the mass density. Therefore

$$b=b_0(1+\delta b/b_0), \tag{8}$$

$$\Delta b=\Delta b_0(1+\delta b/b_0)=\Delta b_0+\delta\Delta b, \tag{9}$$

and introducing V, it follows that

$$\Delta B_0/B_0=\Delta B/B, \tag{10}$$

and the ratio (4) is unaltered. Also,

$$\delta\Delta B/\Delta B=\delta B/B=\delta\rho/\rho, \tag{11}$$

and because ΔB is conserved, pair annihilation leaves $\delta\rho/\rho$ unchanged.

3. INHOMOGENEOUS COMPOSITION

It is generally assumed in cosmology that structure emerges from density fluctuations in a fluid of homogeneous composition. We now propose instead that the initial state consists of fluctuations in composition. Thus the δb_i in (6) are not necessarily in phase, nor are the relative amplitudes $\delta b_i/b_i$ identical. We suppose that the baryon number of the unperturbed state is zero, and for simplicity assume that the fluid density is initially uniform: $\sum\epsilon_i b_{i0}=0$, $\sum\delta b_i=0$. The fluctuating quantity is therefore

$$\Delta b=\sum_i \epsilon_i\delta b_i. \tag{12}$$

For a comoving element of volume, $B=Vb$, $\Delta B=V\Delta b$, and

$$\Delta B=\delta B^+-\delta B^- \tag{13}$$

is a conserved quantity provided diffusion is negligible. Possibly diffusion can be avoided if the joint concentration gradients of each kind are zero, i.e., $\delta n_i^++\delta n_i^-=0$. For large wavelengths, the diffusion in any case is presumably small (see Sec. 4.6).

As the universe expands, pair anihilation occurs, and

$$\Delta B/B\rightarrow\epsilon, \tag{14}$$

$\epsilon=\pm1$, as before in (5). But now ϵ is not a universal constant, and instead $\epsilon=+1$ in those regions of $\Delta B>0$, and $\epsilon=-1$ for regions of $\Delta B<0$. The universe consists of discrete regions of matter and antimatter, as illus-

[18] H. Alfvén and O. Klein, Arkiv Fysik **23**, 187 (1962); O. Klein, Astrophys. Norveg. **9**, 161 (1964); H. Alfvén, Rev. Mod. Phys. **37**, 652 (1965).

[19] Where there is no possibility of confusion, baryons and antibaryons are referred to collectively as baryons. Similarly, leptons and matter are used in this collective sense.

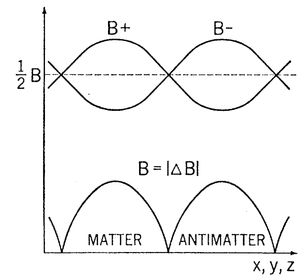

FIG. 1. Illustration of baryon inhomogeneity. The number of baryons (antibaryons) in a comoving element of volume V is denoted by B^+ (B^-). The conserved baryon number in V is $\Delta B = B^+ - B^-$, and the number of baryons in V is $B = B^+ + B^-$. The compositional inhomogeneity $\Delta B/B$ is a space-varying quantity, as shown in the upper curves. As the universe expands, pair annihilation occurs, and $\Delta B/B \to \pm 1$, as shown in the lower curve. The universe is thus fragmented into regions of matter and antimatter.

trated in Fig. 1. Interdiffusion and mass motions may cause additional annihilation in the interface zones and tend to enhance fragmentation.

The inhomogeneous-composition hypothesis offers an explanation of a fragmented universe and avoids any necessity for explaining why matter is in excess of antimatter. The average baryon number is zero, and the total amounts of matter and antimatter are equal; the population symmetry is broken locally but not globally.

4. DISCUSSION

A. Composition

If the temperature is uniform, all mesons densities are uniform, and only leptons and baryons contribute to the inhomogeneity. Initially there are baryons, mesons, leptons, photons, and gravitons (which we neglect), and when annihilation and decay processes cease, there remain p, n, e^-, ν, $\bar{\nu}$, γ or $\bar{p}$, $\bar{n}$, e^+, ν, $\bar{\nu}$, γ. Charge neutrality ensures $N_p = N_e$ and removes any arbitrariness in the relative numbers of protons and electrons.[20] The final products are always the same, and their proportions are affected only by fireball[8,21] and stellar nucleosynthesis. The extent of the thermonuclear burnup in the fireball is regulated by the initial magnitude of ΔB, i.e., the amount of matter that survives.

B. Hadron, Lepton, Radiation, and Stellar Eras

A composition history is shown schematically in a log-log plot of ρ versus T in Fig. 2. Apart from irreversible processes in the steller era, the sequence of events in the prestellar period is the same for either an expanding or a contracting universe. The following outlines roughly what happens as we go back in time.

Assuming that the black-body temperature is at present $T_0 \sim 3$°K and that the mass density of neutrinos

[20] $N_i = V n_i$, where n_i is the number per unit volume of the ith kind.

[21] P. J. E. Peebles, Astrophys. J. **146**, 542 (1966); R.V. Wagoner, W. A. Fowler, and F. Hoyle, *ibid.* **148**, 3 (1967).

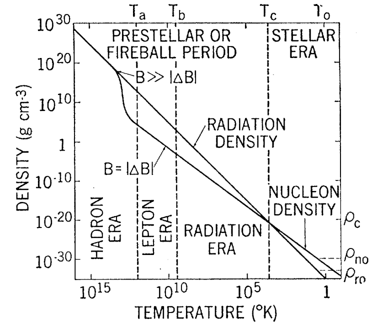

FIG. 2. History of the composition of the universe displayed in the log-log plot of ρ versus T. It is assumed that the background-radiation temperature is $T_0 = 3$°K and the nucleon density is 10^{-6} cm^{-3}. In the stellar era of $T < T_c$, where $T_c \simeq T_0 \rho_{n0}/\rho_{r0}$, matter predominates over radiation (photons and neutrinos), and the pressure is small compared with the energy density. In the prestellar or fireball period of $T > T_c$ there are three eras, and the pressure is close to one third the energy density. In the radiation era there is a deluge of photons and neutrinos whose density rises (as we go back in time) until it is approximately 10^6 times that of matter. As $T \to 5\times10^9$ °K, electron pairs appear, and we enter the lepton era consisting predominantly of μ, e, ν, and γ. The density of matter now rises close to the radiation density. In the hadron era of $\infty > T > 10^{12}$ °K the universe is flooded with strongly interacting particles. In this era the nucleon density rises to the radiation density, and the number of baryons is large compared with the baryon number.

is not larger than that of photons (see below), the radiation density of photons and neutrinos is $\rho_{r0} \sim 10^{-33}$ g cm^{-3}. Also, the nucleon density at present is $\rho_{n0} \sim 10^{-30}$ g cm^{-3}. To a first approximation, $T \propto V^{-1/3}$, and therefore $\rho_r \propto T^4$, $\rho_n \propto T^3$. While $T < T_c$, where

$$T_c = T_0 \rho_{n0}/\rho_{r0} \sim 3\times10^3 \text{ °K}, \qquad (15)$$

the density of matter predominates, and we have the present stellar era of astrophysical structures. But when $T > T_c$, we enter the prestellar period aptly referred to as the primordial fireball,[7] and radiation at first is more dense than matter. The prestellar period subdivides into three eras: radiation, lepton, and hadron.

Radiation era. ($T_b > T > T_c$, $T_b \sim 5\times10^9$ °K, $T_c \sim 3\times10^3$ °K) A radiation flood of photons and neutrinos of $\rho_r \sim T\rho_m/T_c$ rises until its density is approximately 10^6 times that of matter. It is inconceivable that any structure except the most rudimentary can survive this devastating experience.

Lepton era. ($T_a > T > T_b$, $T_a \sim 10^{12}$ °K) As $kT \to m_e c^2$, $m_\mu c^2$, there occur deluges of electron and muon pairs, and the density of matter rises close to the radiation density and varies as T^4. The nucleon density continues to vary as T^3.

Hadron era. ($\infty > T > T_a$) The universe is flooded with strongly interacting particles—first the π, κ, η and then the N, Λ, Σ, Δ, Ξ, Ω, $\cdots$ baryons with the meson multiplets of higher mass. The nucleon density at last rises to the radiation density, and the number of baryons becomes large compared with the baryon number.

C. Constancy of N

For the ith particle state,[22] we have

$$dE_i = T_i dS_i - p_i dV + \mu_i dN_i, \tag{16}$$

$$d\Phi_i = -S_i dT_i + V dp_i + \mu_i dN_i, \tag{17}$$

where μ_i and $\Phi_i = \mu_i N_i$ are the chemical and thermodynamic potentials, p_i are the pressures, and $E_i = \rho_i c^2 V$. If the baryon and lepton numbers are relatively small, as in the prestellar period, then from the chemical-equilibria equations,[23] $\mu_i = 0$ and

$$dW_i = T_i dS_i + V dp_i, \tag{18}$$

$$W_i = T_i S_i, \tag{19}$$

where $W_i = (\rho_i c^2 + p_i)V$. From (17) and (19),

$$\rho_i c^2 + p_i = T_i dp_i / dT_i, \tag{20}$$

and for constant entropy $dS = \sum dS_i = 0$, we get from (18)

$$d(\rho c^2 V) + p dV = 0, \tag{21}$$

and from (19)

$$\sum_i W_i / T_i = S = \text{const.} \tag{22}$$

We now show that the total number N of all particles in a comoving element of volume of a uniform universe is approximately constant and independent of time.[24] [From (22) and it can be seen roughly that if the mean energy per particle is approximately kT, then $N \sim S/k$ is a constant in time.] We have

$$n_i = (g_i / 2\pi^2)(kT/\hbar c)^3 I_i^{11}(\pm), \tag{23}$$

$$\rho_i c^2 = (g_i kT / 2\pi^2)(kT/\hbar c)^3 I_i^{21}(\pm), \tag{24}$$

$$p_i = (g_i kT / 6\pi^2)(kT/\hbar c)^3 I_i^{03}(\pm), \tag{25}$$

where

$$I_i^{mn}(\pm) = \int_{c_i}^{\infty} x^m (x^2 - c_i^2)^{n/2} (e^x \pm 1)^{-1} dx, \tag{26}$$

$c_i = m_i c^2 / kT$, and the g_i are the number of spin states ($g_i = 1$ for neutrinos, and other zero-mass particles have $g_i = 2$). Equation (22) can now be expressed in the form

$$S = k \sum_i N_i (I_i^{21} + \tfrac{1}{3} I_i^{03}) / I_i^{11}. \tag{27}$$

The integrals (26) are exponentially small when $kT < m_i c^2$, and when $kT > m_i c^2$, they rapidly approach the values

$$I^{11}(+) = \tfrac{3}{2}\zeta(3), \quad I^{21}(+) = I^{03}(+) = 7\pi^4/120 \tag{28}$$

for fermions, and

$$I^{11}(-) = 2\zeta(3), \quad I^{21}(-) = I^{03}(-) = \pi^4/15 \tag{29}$$

for bosons, where $\zeta(3) = 1.202$.

As the temperature rises, pair production populates in succession the particle states of higher mass. The production does not occur smoothly but in bursts as kT passes through successive values of $m_i c^2$. Let us select all pairs of adjacent particle states j, $j+1$ having m_j, m_{j+1} sufficiently separated so that in an intermediate temperature interval dT, production is negligible. In these intervals, T varies as $V^{-1/3}$. The integrals (26) are zero for $i \geq j+1$ and have the values (28) and (29) for $i \leq j$, and therefore

$$S/k = [2\pi^4 / 45\zeta(3)][N_{\text{bosons}} + (7/6) N_{\text{fermions}}], \tag{30}$$

and because $N = N_{\text{bosons}} + N_{\text{fermions}}$,

$$0.24 S k^{-1} < N < 0.28 S k^{-1}. \tag{31}$$

Thus as the temperature changes, the number of particles N fluctuates, but in those intervals of temperature in which production is small, N falls within the limits of (31). In spite of its approximate nature, this result is of considerable interest. If the baryons and mesons form a hierarchy of masses which increases indefinitely, their continuous production as $V \to 0$ means that $N_\gamma / N \to 0$ and that the relative concentration of photons goes to zero. Similarly, if the muon is the highest-mass lepton, the relative concentration of the leptons goes to zero as $V \to 0$.

Let us assume that the pressure and density vary smoothly and that

$$p = (\tfrac{1}{3} - \alpha) \rho c^2, \tag{32}$$

where α is constant. Then from (20) and (21)

$$\rho \propto T^{(4-3\alpha)/(1-3\alpha)} \propto V^{\alpha - 4/3}. \tag{33}$$

In general, α is small; for example, in the range 10^{12}–10^{13} °K, we find from (22) that α has a mean value of 0.05. Without more knowledge of the higher-mass particle states, we cannot estimate $T(V)$, nor can we determine $\rho(t)$ from general relativity theory with any precision.

D. Neutrino Temperatures

In an expanding universe the neutrinos are decoupled in the lepton era, and thereafter their interaction is small with rest of the contents of the universe. The decoupling occurs roughly when the interaction time τ equals the age t of the universe and τ increases faster that t. Let

$$t \sim (G\rho)^{-1/2} \sim (c\hbar / kT)^2 (c / G\hbar)^{1/2},$$

and for electron-neutrino scattering[25]

$$\tau_{e\nu} \sim (\hbar c / kT)^3 (m_e c^2 / kT)^2 / c \sigma_{0e\nu},$$

when $\sigma_{0e\nu} = 2 \times 10^{-44}$ cm^2. Hence

$$\tau_{e\nu} / t \sim (m_e c^2 / kT)^3, \tag{34}$$

and the electron neutrinos decouple prior to or at the time of electron pair annihilation. Because

$$\sigma_{0\mu\nu} = (m_\mu / m_e)^2 \sigma_{0e\nu},$$

[22] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press Inc., New York, 1958), Chap. 2.

[23] Landau and Lifshitz (Ref. 22), Chaps. 10 and 11.

[24] The remark in Ref. 1 to the effect that N may change appreciably is in error.

[25] J. N. Bahcall, Phys. Rev. **136**, B1164 (1964).

we have for muon-neutrino scattering $\tau_{\mu\nu}=\tau_{e\nu}$ (for equal electron and muon densities), and therefore the muon neutrinos decouple at the time of muon-pair annihilation. After the neutrinos decouple, their temperatures vary as $V^{-1/3}$.

The subsequent neutrino and photon temperatures are easily found using (22). At present

$$S=\tfrac{4}{3}V_0c^2\left(\frac{\rho_{\gamma 0}}{T_{\gamma 0}}+\frac{\rho_{\nu e0}}{T_{\nu e0}}+\frac{\rho_{\nu\mu 0}}{T_{\nu\mu 0}}\right), \tag{35}$$

where ρ_γ, $\rho_{\nu e}$, and $\rho_{\nu\mu}$ are the photon, electron-, and muon-neutrino densities, respectively. Prior to electron-pair annihilation

$$S=\tfrac{4}{3}V_1c^2\left(\frac{\rho_{\gamma 1}+\rho_{e1}+\rho_{\nu e1}}{T_1}+\frac{\rho_{\nu\mu 1}}{T_{\nu\mu 1}}\right), \tag{36}$$

and prior to muon-pair annihilation

$$S=\tfrac{4}{3}V_2c^2\left(\frac{\rho_{\gamma 2}+\rho_{\mu 2}+\rho_{e2}+\rho_{\nu\mu 2}+\rho_{\nu e2}}{T_2}\right), \tag{37}$$

and ρ_μ, ρ_e are the electron-pair and muon-pair densities, respectively. By using the relations

$$\rho_\gamma=(4/7)\rho_\mu=(4/7)\rho_e=(8/7)\rho_{\nu\mu}=(8/7)\rho_{\nu e},$$

$\rho_{\gamma x}\propto T_{\gamma x}{}^4$, $\rho_{\nu\mu x}V_x/T_{\nu\mu x}=\rho_{\nu\mu 2}V_2/T_2$, $\rho_{\nu e0}V_0/T_{\nu e0}=\rho_{\nu e1}V_1/T_1$, we find[26] from (35)-(37)

$$T_{\nu e0}=(4/11)^{1/3}T_{\gamma 0}=0.714T_{\gamma 0}, \tag{38}$$

$$T_{\nu\mu 0}=(116/473)^{1/3}T_{\gamma 0}=0.626T_{\gamma 0}. \tag{39}$$

When all pair annihilation has ceased, the radiation energy is

$$E=E_\gamma+E_{\nu e}+E_{\nu\mu}=1.36E_\gamma, \tag{40}$$

and the number of photons and neutrinos is

$$\begin{aligned}N&=N_\gamma+N_{\nu e}+N_{\nu\mu}\\&=N_\gamma(1+3/11+87/473)=1.457N_\gamma.\end{aligned} \tag{41}$$

E. Magnitude of $\Delta B/B$

If n_0 is the present mean density of nucleons, the conserved baryon number in V is $\Delta B=\pm V_0n_0$. Also, from (41), the approximately constant number N of particles in V is $N=0.356V_0(kT_0/\hbar c)^3$. Hence

$$|\Delta B|/N=2.8n_0(\hbar c/kT_0)^3=2.9\times 10^{-2}n_0/T_0{}^3 \tag{42}$$

is an approximate constant, and for $n_0\sim 10^{-6}$ cm^{-3}, $T_0=3$°K, we have $|\Delta B|/N\sim 10^{-9}$. This is the inhomogeneity required to produce the observed amount of fragmented matter. Since $B<N$, (42) sets a lower limit on $\Delta B/B$. If the baryon particle states predominate at high density, then $\Delta B/B\rightarrow\Delta B/N\sim 10^{-9}$ as $V\rightarrow 0$.

If the universe contained no matter ($\Delta B=0$), the effect on the radiation density would be very slight. Let T be the universal microwave temperature if all matter is annihilated, and let δT be the change in temperature when ΔB is not zero. Then, to first order,

$$\delta T/T\simeq -|\Delta B|/N, \tag{43}$$

and the material content of the universe lowers the temperature by only one part in 10^9. The universal radiation, which is so unimportant in the stellar era, is the relic of $1-|\Delta B|/N$ of the prestellar content of the universe, whereas matter, which is now of predominant importance, derives from only $\Delta B/N$ of the early universe.

F. Origin of Inhomogeneities

The subject of density or compositional variations in the early universe can be considered from two points of view[12]:

Primordial-structure hypothesis. This assumes that rudimentary structure is imprinted in the universe from its earliest moments and is an indispensable part of the cosmogenic design. Under this heading one might consider nonconservation of baryon number[27] related in some way with the breaking of population symmetry on a local scale.

Instability hypothesis. According to this hypothesis, structure evolves naturally from amorphous initial conditions.[28]

There are serious difficulties confronting structural growth in the early universe. Statistical fluctuations of $|\Delta B/B|\sim B^{-1/2}\sim 10^{-9}$ give eventual aggregations of only $|\Delta B|\sim 10^9$ nucleons (for both density and compositional fluctuations). Any attempt to increase $|\Delta B|$ with correlations and interaction mechanisms encounters causality limitations. Let $t\sim(\rho G)^{-1/2}$ be the age of the universe; then only particles at a distance $\lambda\lesssim ct$ have interacted with a given particle, and causality imposes a mass limit of $M'\lesssim\rho\lambda^3$, or

$$M'\lesssim(c^6/\rho G^3)^{1/2}, \tag{44}$$

for any kind of interaction.[29] Expressing M' in terms of solar masses $M_\odot$, we find

$$M'/M_\odot\lesssim 10^9\,\rho^{-1/2}. \tag{45}$$

In the hadron era, $\rho>10^{16}$ g cm^{-3}, and therefore

[26] The result (38) was previously obtained by Alpher, Follin, and Herman (Ref. 17). See also Ref. 21.

[27] J. A. Wheeler, *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach Science Publishers, Inc., New York, 1964), p. 501. See also A. D. Sakharov, Zh. Eksperim. i Teor. Fiz. Pis'ma. v, Redaktsiyu **5**, 32, 36 (1967) [English transl.: JETP Letters **5**, 24, 27 (1967)].

[28] It is often supposed by those favoring this hypothesis that all that is needed is an "instability criterion" showing that a range of wavelengths is time-growing. But instability also requires that disturbances grow sufficiently rapidly that a fundamental change in configuration is possible in the time available. It is then found that present theories demand initial conditions equivalent to well-developed structure and therefore in fact subscribe to the primordial-structure hypothesis.

[29] If the curvature term is zero, it can be shown that the mass enclosed by the particle horizon at $\lambda=2ct$ is equal to the Schwarzschild mass of $c^2\lambda/2G$.

$M' < 10\, M_\odot$. In fact, the situation is worse than this: The mass of an eventual condensation is $M < M'\Delta B/N$, or $M < 10^{-8}\, M_\odot$. We therefore come to the conclusion that the time available is too short for the formation of baryon inhomogeneities, and that if inhomogeneities are the explanation of large-scale aggregations of matter, then they are an integral part of the universe from the beginning of its expansion.

The case for density fluctuations is certainly no better. For a galactic mass of $M = 10^{10}\, M_\odot$, we have from (45), $\rho \ll 10^{-2}$ g cm^{-3} (since $M' \gg M$), and the instability hypothesis leads to the improbable conclusion that the foundations of galactic structure are laid down in the radiation era. If we accept the primordial-structure hypothesis and assume that density inhomogeneities exist from the earliest moments, we are still in difficulty because small amplitude fluctuations are not amplified,[30] and the density inhomogeneity is of the order 10^9 times greater than the required compositional inhomogeneity.

[30] Nor are they dissipated. Any dissipation mechanism [such as Misner's (Ref. 4)] in the lepton or hadron eras causes only the very shortwavelengths of $\lambda < ct$ to decay.

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New Formulation of the Axially Symmetric Gravitational Field Problem

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The field equations governing the gravitational field of a uniformly rotating axially symmetric source are reformulated in terms of a simple variational principle. The new formalism affords a concise unified derivation of the solutions discovered by Weyl and Papapetrou, and permits a simple derivation of the Kerr metric in terms of prolate spheroidal coordinates. More complex solutions are identified by applying perturbation theory.

I. INTRODUCTION

OF considerable current interest is the problem of finding the gravitational field of a uniformly rotating body. Although a possible exterior field has been found by Kerr, who investigated algebraically special metrics, attempts to generalize the Kerr solution of the vacuum field equations have not been marked by success.[1] In the present paper the problem is reformulated in terms of a complex function $\mathcal{E}$ independent of azimuth, which must be chosen in accordance with the variational principle

$$\delta \int \frac{\nabla\mathcal{E}\cdot\nabla\mathcal{E}^*}{(\mathrm{Re}\,\mathcal{E})^2}dv = 0\,, \tag{1}$$

where dv is the three-dimensional Euclidean volume element. When such a complex $\mathcal{E}$ function is found, a corresponding axially symmetric solution of Einstein's vacuum field equations may be constructed.

This formulation of the axially symmetric gravitational field problem has a number of nice features. Neither the variational principle nor the corresponding field equation

$$(\mathrm{Re}\,\mathcal{E})\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E} \tag{2}$$

makes reference to a particular coordinate system. According to one's desires, one may work with the equations in an abstract manner, or express them in terms of cylindrical, prolate spheroidal, or any other coordinates. Furthermore, the field equation is homogeneous quadratic, and it serves as an excellent vehicle for the application of perturbation theory. Finally, all the known axially symmetric solutions can be expressed simply in terms of the $\mathcal{E}$ function.

[1] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963). A detailed discussion has been given by F. J. Ernst, in Proceedings of the Relativity Seminar of the Illinois Institute of Technology (unpublished).

II. DERIVATION OF THE $\mathcal{E}$ EQUATION

Following Papapetrou we express the line element in the form

$$ds^2 = f^{-1}[e^{2\gamma}(dz^2 + d\rho^2) + \rho^2 d\phi^2] - f(dt - \omega d\phi)^2\,, \tag{3}$$

where f, ω, and γ are functions of z and ρ only.[2] The complete set of field equations may be derived either using traditional tensor methods or the currently fashionable methods of exterior calculus. However, we are presently interested in the equations governing f and ω only, and these may be obtained from the Lagrangian density

$$\mathfrak{L} = -\tfrac{1}{2}\rho f^{-2}\nabla f\cdot\nabla f + \tfrac{1}{2}\rho^{-1}f^2\nabla\omega\cdot\nabla\omega\,.$$

Varying the functions f and ω we obtain the field equations

$$f\nabla^2 f = \nabla f\cdot\nabla f - \rho^{-2}f^4\nabla\omega\cdot\nabla\omega\,, \tag{4}$$

$$\nabla\cdot(\rho^{-2}f^2\nabla\omega) = 0\,. \tag{5}$$

[2] A. Papapetrou, Ann. Physik **12**, 309 (1953).