Calculated Spectrum of Inverse-Compton-Scattered Photons

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We consider an electron of a given energy moving in a monoenergetic, isotropic radiation field. The energy spectrum of the photons that are scattered by the electron has been calculated both exactly and in a greatly simplied approximate form suitable for astrophysical calculations. The approximation may be derived either by expanding the exact solution in a small parameter and keeping only the leading terms or by employing a simplifying physical approximation at the beginning of the calculation. The approximate spectrum is similar to one previously derived by Ginzburg and Syrovatskii, the principal difference being that the present one does not break down if $\hbar\omega_1 E > (m_e c^2)^2$, where $\hbar\omega_1$ is the initial photon energy and E the electron energy. We indicate the astrophysical applications of our approximate spectrum by calculating the spectrum of photons scattered by electrons with an inverse-power-law energy distribution.

I. INTRODUCTION

In recent years the process referred to as inverse Compton scattering has had a revival of interest among astrophysicists. It was introduced in 1947 by Follin[1] as a mechanism for the loss of energy of cosmic-ray electrons and was investigated by Feenberg and Primakoff[2] and by Donahue[3] in this context. Since that time it has been employed in many treatments[4–8] of cosmic-ray electrons, and the process was investigated in some detail by the present author in an earlier paper.[9]

It was first suggested as a source of energetic photons by Savedoff[10] and by Felten and Morrison[11] and has since received considerable attention[12–19] from this point of view. Most of the calculations of photon spectra to date have been based on a rather simple approximation. It has been noted that the average energy transferred to a photon in a Compton collision is proportional to the initial energy of the photon and the square of the electron energy. This dependence on the square of the electron energy is reminiscent of the synchrotron process, and for this reason the radiated photon spectra for a single-electron energy is approximated by a δ-function spike at the average radiated energy. This spectrum is then folded into the distribution of electron energies to produce the resultant photon spectrum. Although this method is known to give satisfactory results for synchrotron spectra, it is now known[20] that the inverse Compton spectra are sufficiently different to raise some doubts as to its applicability in this area. However, it can be shown[21] that in the case of inverse power law distributions of electron energies, the method is applicable to both cases in spite of their differences.

In the present paper, we derive exact formulas for the scattered-photon energy distribution for the case of an electron of energy $\gamma = E/mc^2$ moving through a region of space filled with a unit density of photons distributed isotropically with initial energy $\alpha_1 = \hbar\omega_1/mc^2$. We shall also derive several approximate formulas and discuss their validity in the light of the exact formulas. Similar calculations have been recently published by Baylis *et al.*[22]; however, we find that our results disagree with theirs in several respects. In particular, we disagree with their conclusion that a particular approximate spectrum is of as wide a validity as they claim. On the contrary, we derive correction terms that become significant when certain conditions of validity first stated by Ginsburg and Syrovatskii[20] are violated.

In Sec. II, we derive an approximate spectrum based on a simplifying physical assumption. The breakdown of this approximation will also be discussed from a physical point of view. In Sec. III, the scattered spectrum will be calculated exactly and compared (as well as possible) with the approximate spectrum. We will see that the exact form is often not very useful for computation, and the reason for this will be discussed. In Sec. IV, we shall exhibit a method of expanding the exact formula in the small parameter that causes the

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trouble. This not only provides a method for computing the exact spectrum, but also provides a systematic way of rederiving our approximate formula along with some correction terms. In Sec. V, we will disucss some astrophysical implications of these results.

II. APPROXIMATE SPECTRUM

In this section, we shall derive an approximate spectrum by making a simplifying physical assumption concerning angles. Figure 1 illustrates the angles involved in a scattering problem as seen in the rest frame of the electron (E. R. frame). All quantities as measured in this frame will be primed, and energies of the photon before and after collision (α_1 and α, respectively), as well as the electron energy γ, are understood to be in units of the electron rest energy mc^2. The polar angles θ_1' and θ' are measured with respect to the electron velocity $\boldsymbol{\beta}=\mathbf{v}/c$ (strictly with respect to *minus* $\boldsymbol{\beta}$; $\boldsymbol{\alpha}_1\cdot\boldsymbol{\beta}=-\alpha_1\beta\cos\theta$). The scattering takes place through a polar angle χ' and an azimuthal angle ϕ', where the $\alpha_1'\beta$ plane is chosen as the $\phi'=0$ plane.

The presence of so many angles along with the constraining relations between them complicates the problem, as we shall see later. It would greatly simplify things if we could eliminate some of them. To this end, let us examine the angular distribution of the incoming photons in the E. R. frame.

For photons, isotropic and monoenergetic with energy α_1 in the lab frame, the angular distribution in the E. R. frame is given by

$$n'(\theta_1')d(\cos\theta_1')=\frac{d(\cos\theta_1')}{2\gamma^2(1-\beta\cos\theta_1')^2}. \qquad (1)$$

If $\beta\approx1$, half of the photons have polar angles within the range $0\leq\theta_1'\leq\theta_{1/2}'$, where $\theta_{1/2}'\approx1/\gamma$. In other words, as the electron becomes more and more energetic, the incoming photons appear to be more and more like a monodirectional beam, with $\theta_1'=0$.

The approximation to be made is now obvious; we shall consider the electron to be energetic enough so that we may take $\theta_1'=0$. A glance at Fig. 1 shows that in this case, $\chi'=\theta'$, and since the scattering cross section is independent of the azimuth ϕ', we really have only one angle left to worry about.

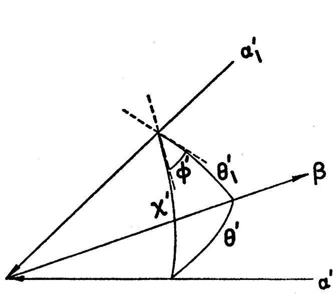

FIG. 1. Angles involved in the scattering process as viewed in the electron rest frame.

The energy distribution of the photons in the E. R. frame before collision is given by

$$n'(\alpha_1')d\alpha_1'=(\alpha_1'/2\gamma\alpha_1^2)S(\alpha_1';\,\alpha_1/2\gamma,\alpha_1 2\gamma)d\alpha_1' \qquad (2)$$

for $\gamma\gg1$, where $S(x;\,a,b)$ is the characteristic function of the interval a, b, i.e.,

$$\begin{aligned} S(x;\,a,b)&=1 \quad \text{for} \quad a\leq x\leq b\\ &=0 \quad \text{for} \quad x<a,\ b<x. \end{aligned}$$

The Klein-Nishina cross section for Compton scattering is given by

$$\sigma(\alpha',\alpha_1',y')=\frac{r_0^2(1+y'^2)}{2[1+\alpha_1'(1-y')]^2}\times\left\{1+\frac{\alpha_1'^2(1-y')^2}{(1+y'^2)[1+\alpha_1'(1-y')]}\right\}\delta(\alpha'-f(\alpha_1',y')), \qquad (3)$$

where

$$y'\equiv\cos\chi', \quad r_0\equiv e^2/mc^2,$$

and

$$f(\alpha_1',y')=\alpha_1'/[1+\alpha_1'(1-y')].$$

The number of collisions per unit time t' is just $n'c\sigma$, and since $dN/dt=\gamma^{-1}dN/dt'$, we have, after integrating over ϕ',

$$\frac{d^4N}{dtd\alpha_1'd\alpha'dy'}=\frac{\pi r_0^2c}{2\alpha_1^2\gamma^2}\left\{\frac{1+y'^2}{[1+\alpha_1'(1-y')]^2}\times\left(1+\frac{\alpha_1'^2(1-y')^2}{(1+y'^2)[1+\alpha_1'(1-y')]}\right)\times\alpha_1'\delta(\alpha'-f(\alpha_1',y'))S(\alpha_1';\,\alpha_1/2\gamma,\alpha_1 2\gamma)\right\}. \qquad (4)$$

Since $d\alpha'd\alpha_1'dy'=[1+\alpha_1'(1-y')]^2d\alpha'dy'df$, we may integrate over f immediately to obtain

$$\frac{d^3N}{dtd\alpha'dy'}=\frac{\pi r_0^2c}{2\alpha_1^2\gamma^2}\left[(1+y'^2)+\frac{\alpha'^2(1-y')^2}{1-\alpha'(1-y')}\right]\times\frac{\alpha'}{1-\alpha'(1-y')}S\left[\frac{\alpha'}{1-\alpha'(1-y')};\,\alpha_1/2\gamma,\alpha_1 2\gamma\right]. \qquad (5)$$

We may relate α' to the final lab-frame energy α by the Doppler-shift formula $\alpha'=\alpha/\gamma(1-\beta y')$. If we introduce the variable $\eta=(1-\beta y')$, we have

$$\frac{d^3N}{dtd\alpha d\eta}=\frac{\pi r_0^2c\alpha}{2\gamma^4\alpha_1^2(1-\alpha/\gamma)}\left[\eta^2-2\eta+2+\frac{(\alpha/\gamma)^2}{(1-\alpha/\gamma)}\right]\times\frac{S(\eta;\,\eta_1,\eta_2)}{\eta^2}, \qquad (6)$$

where we have assumed that $1-y'\approx\eta$, and where $\eta_1=\alpha/2\alpha_1\gamma^2(1-\alpha/\gamma)$ and $\eta_2=2\alpha/\alpha_1(1-\alpha/\gamma)$. The integral may be readily performed to give

$$\left[\eta-2\ln\eta-\frac{2}{\eta}-\frac{(\alpha/\gamma)^2}{(1-\alpha/\gamma)\eta}\right]_L^U, \tag{7}$$

where the upper and lower limits U and L depend on what part (if any) of the interval η_1, η_2 lies within the limits $1/2\gamma^2$ and 2.

For $\alpha_1/4\gamma^2\leq\alpha\leq\alpha_1$, we have $\eta_1\leq 1/2\gamma^2$ and $1/2\gamma^2\leq\eta_2\leq 2$. We then have, neglecting terms of order $1/\gamma^2$ or less when compared to unity,

$$\frac{d^2N}{dtd\alpha}\approx\frac{\pi r_0{}^2c}{2\gamma^4\alpha_1}\left(\frac{4\gamma^2\alpha}{\alpha_1}-1\right). \tag{8}$$

For $\alpha_1\leq\alpha\leq 4\alpha_1\gamma^2/(1+4\alpha_1\gamma)$, we have $\eta_2\geq 2$ and $1/2\gamma^2\leq\eta_1\leq 2$, and

$$\frac{d^2N}{dtd\alpha}\approx\frac{2\pi r_0{}^2c}{\alpha_1\gamma^2}\left[2q''\ln q''+(1+2q'')(1-q'')+\frac{1}{2}\frac{(4\alpha_1\gamma q'')^2}{(1+4\alpha_1\gamma q'')}(1-q'')\right], \tag{9}$$

where $q''=\alpha/4\alpha_1\gamma^2(1-\alpha/\gamma)$ and $1/4\gamma^2<q''\leq 1$.

In the above equations, we see that the maximum value that α/γ can have is $(\alpha/\gamma)_{\max}=4\alpha_1\gamma/(1+4\alpha_1\gamma)<1$. If $4\alpha_1\gamma\ll 1$, then $\alpha/\gamma\ll 1$, and we have $q''\approx\alpha/4\alpha_1\gamma^2$. The last term in the square brackets may be dropped, and we are left with the approximate spectrum of Ginzburg and Syrovatskii.[20]

Expression (9) is valid, however, no matter how large $4\alpha_1\gamma$ may become. However, we must not assume that this approximation is uniformly valid. In fact, it turns out that it is not a good approximation for $\alpha/\alpha_1\approx 1/4\gamma^2$.

To see the reason for this, let us consider what would happen if we took our assumption that $\theta_1'=0$ seriously and transformed our photon energy distribution, expression (2), back to the lab frame with *no* scattering at all. We would obtain a spectrum given by

$$n(\alpha)d\alpha=(\alpha/\alpha_1{}^2)S(\alpha;\,\alpha_1/4\gamma^2,\alpha_1)d\alpha. \tag{10}$$

We can see from this that the approximation alone tends to populate the region of the spectrum from $\alpha_1/4\gamma^2$ to α_1 with *no* scattering at all and that this region will be exaggerated for small-angle scattering as well. In other words, because of the large sensitivity of the Doppler-shift formula to slight changes in θ' for small θ', neglect of these small deviations of θ' from zero introduces considerable error for small-angle scattering.

Small-angle scattering would be, of course, angles of the order of $1/\gamma$ or smaller, or for $y'\gtrsim 1-1/2\gamma^2$. Since for most values of α/α_1 there is a contribution from a considerable range of y' other than the region $1\geq y'\gtrsim 1-1/2\gamma^2$, this error will be negligible. However, for the very bottom of the spectrum $\alpha/\alpha_1\approx 1/4\gamma^2$, the contribution is entirely from the region $\eta=1-\beta y'\approx 1/2\gamma^2$, and here we would expect a significant error. This will be borne out by the results of Sec. IV.

III. EXACT CALCULATION

In this section, we shall calculate in closed form the scattered-photon spectrum for the case of an electron of energy γ moving through a region of space filled with a unit density of isotropically distributed, monoenergetic photons of energy α_1. As in the last section, we will first transform the incident photon distribution to the E. R. frame and formulate the problem in this frame. The final energy will then be expressed in its lab-frame value α, and we will integrate over all available angles, holding α_1, γ, and α fixed.

In the E. R. frame the incident photon distribution has the form

$$n'(\alpha_1',x')d\alpha_1'd\Omega'=\frac{\alpha_1'\delta[\alpha_1'\gamma(1-\beta x')-\alpha_1]}{\alpha_1 4\pi\gamma(1-\beta x')}d\alpha_1'd\Omega', \tag{11}$$

where $x'\equiv\cos\theta'$ and $d\Omega'$ is the element of solid angle $2\pi dx'$. Expression (11) may be obtained by noting that $n(\alpha_1,x)d\alpha_1 d\Omega$ is a density and hence transforms like an energy. If we divide by the energy α_1, we obtain an invariant, and hence

$$(1/\alpha_1)n(\alpha_1,x)d\alpha_1 d\Omega=(1/\alpha_1')n'(\alpha_1',x')d\alpha_1'd\Omega'$$

and

$$n'(\alpha_1',x')d\alpha_1'd\Omega'=(\alpha_1'/\alpha_1)n(\alpha_1,x)d\alpha_1 d\Omega.$$

Expressing α_1 and x in terms of α_1' and x' completes the derivation. The cross section is given by expression (3'):

$$\sigma(\alpha_1',\alpha',y')d\Omega'(y')d\alpha'=\frac{r_0{}^2(1+y'^2)}{2[1+\alpha_1'(1-y')]^2}\times\left\{1+\frac{\alpha_1'^2(1-y')^2}{(1+y'^2)[1+\alpha_1'(1-y')]}\right\}\times\delta(\alpha'-f(\alpha_1',y'))d\phi'dy'd\alpha'. \tag{3'}$$

Since $dN/dt=n'c\sigma/\gamma$, we have

$$\frac{d^6N}{dtdx'dy'd\phi'd\alpha_1'd\alpha'}=\frac{r_0{}^2c\delta(\alpha_1'\gamma(1-\beta x')-\alpha_1)}{4\gamma^3(1-\beta x')^3}\delta(\alpha'-f(\alpha_1',y))\times\left\{\frac{1+y'^2}{[1+\alpha_1'(1-y')]^2}+\frac{\alpha_1'^2(1-y')^2}{[1+\alpha_1'(1-y')]^3}\right\}, \tag{12}$$

where we have used the relation $\alpha_1/\alpha_1'=\gamma(1-\beta x')$. Once again employing the relation $d\alpha_1'=[1+\alpha_1'(1-y')]^2df$, we may integrate over f immediately. We also note that since $z\equiv\cos\theta'=\cos\theta_1'\cos\chi'+\sin\theta_1'\sin\chi'\cos\phi'$, we have

$$d\phi'=2dz'/(1-x'^2-y'^2-z'^2+2x'y'z')^{1/2}.$$

With this substitution, we now have

$$\frac{d^5N}{dtdx'dy'dz'd\alpha'}=\tfrac{1}{2}cr_0^2\frac{1}{\gamma^3(1-\beta x')^2}\times\delta\left(\frac{\alpha'\gamma(1-\beta x')}{1-\alpha'(1-y')}-\alpha_1\right)\left(1+y'^2+\frac{\alpha'^2(1-y')^2}{[1-\alpha'(1-y')]}\right)\times(1-x'^2-y'^2-z'^2+2x'y'z')^{-1/2}. \quad (13)$$

Transforming α' to α by using the relation

$$\alpha'=\alpha/\gamma(1-\beta z'),$$

we have

$$\frac{d^5N}{dtd\alpha dx'dy'dz'}=\tfrac{1}{2}cr_0^2\frac{[1-\beta z'-(\alpha/\gamma)(1-y')]}{\gamma^4(1-\beta x')^2(1-\beta z')}\times\delta\left(\alpha(1-\beta x')-\alpha_1(1-\beta z')+\frac{\alpha_1\alpha}{\gamma}(1-y')\right)\times\left\{1+y'^2+\frac{\alpha^2(1-y')^2}{\gamma^2(1-\beta z')[1-\beta z'-(\alpha/\gamma)(1-y')]}\right\}J^{-1/2}, \quad (14)$$

where $J=1-(x')^2-(y')^2-(z')^2+2x'y'z'$, and where we have made use of

$$\delta(Ax-B)=\frac{1}{A}\delta\left(x-\frac{B}{A}\right).$$

From this point on, the object is to integrate over all possible values of x', y', and z', holding α, γ, and α_1 fixed. For a given set of values for the parameters α, γ, and α_1, only a certain volume of x', y', z' space (possibly zero) will be compatible kinematically with this particular choice. This requirement is expressed by the condition that the Jacobian of the transformation from ϕ' to z' be real, or that $J\geq 0$. Inspection of the form of J shows that the requirements that $|x'|$, $|y'|$, and $|z'|$ be ≤ 1 is automatically fulfilled by keeping $J\geq 0$, unless all three variables are simultaneously out of bounds in such a way that $(x'y'z')>0$. Therefore this requirement need be consciously enforced on only one of the three variables.

It is immaterial which order we choose in integrating the three variables, and we arbitrarily choose the order x', y', z'. The first integration is trivial because of the δ function, and we obtain

$$\frac{d^4N}{dtd\alpha dy'dz'}=\tfrac{1}{2}cr_0^2\frac{\alpha}{\gamma^4\alpha_1^2[1-\beta z'-(\alpha/\gamma)(1-y')](1-\beta z')}\left\{1+y'^2+\frac{(\alpha/\gamma)^2(1-y')^2}{(1-\beta z')[1-\beta z'-(\alpha/\gamma)(1-y')]}\right\}(\beta^2J)^{-1/2}, \quad (15)$$

where

$$\beta^2J=(\beta^2+\epsilon^2+2\beta\epsilon z')(y_2-y')(y'-y_1), \quad (16)$$

$$y_2=y_0+\delta, \quad y_1=y_0-\delta, \quad (17)$$

$$y_0=(\epsilon+\beta z')(\rho+\epsilon\rho-1+\beta z')/\rho(\beta^2+\epsilon^2+2\beta\epsilon z'), \quad (18)$$

$$\delta=\frac{\beta(1-z'^2)^{1/2}[\rho^2\beta^2+2\rho\epsilon(1-\rho)(1-\beta z')-(\rho-1+\beta z')^2]^{1/2}}{\rho(\beta^2+\epsilon^2+2\beta\epsilon z')}, \quad (19)$$

and $\rho=\alpha/\alpha_1$, $\epsilon=\alpha_1/\gamma$.

The integration over y' may be facilitated by the transformation $y'=y_0+\delta\eta$, where $-1\leq\eta\leq 1$. We then have

$$\frac{d^4N}{dtd\alpha dz'd\eta}=\frac{cr_0^2\alpha}{2\gamma^4\alpha_1^2(\beta^2+\epsilon^2+2\beta\epsilon z')^{1/2}}\left[\frac{1}{(1-\beta z')}+\frac{y_0^2+2y_0\delta\eta+\delta^2\eta^2}{(a+b\eta)}+\frac{(\alpha/\gamma)(1-y_0-\delta\eta)}{(a+b\eta)^2}\right]\frac{1}{(1-\eta^2)^{1/2}(1-\beta z')}, \quad (20)$$

where $a=1-\beta z'-(\alpha/\gamma)(1-y_0)$, $b=\alpha\delta/\gamma$.

Integration of η from -1 to 1 gives

$$\frac{d^3N}{dtd\alpha dz'}=\frac{\pi r_0^2c\alpha}{2\gamma^4\alpha_1^2(\beta^2+\epsilon^2+2\beta\epsilon z')^{1/2}}\left[\frac{1}{(1-\beta z')^2}+\frac{y_0^2}{(1-\beta z')(a^2-b^2)^{1/2}}+\frac{2y_0\gamma}{\alpha(1-\beta z')}-\frac{2y_0\gamma a}{\alpha(1-\beta z')(a^2-b^2)^{1/2}}-\frac{a\gamma^2}{\alpha^2(1-\beta z')}+\frac{a^2\gamma^2}{\alpha^2(1-\beta z')(a^2-b^2)^{1/2}}+\frac{\alpha(1-y_0)}{\gamma(1-\beta z')(a^2-b^2)^{3/2}}+\frac{\alpha^2\delta^2}{\gamma^2(1-\beta z')(a^2-b^2)^{3/2}}\right]. \quad (21)$$

After some manipulation, we have

$$a^2-b^2=\frac{[(1-\alpha/\gamma)^2-1/\gamma^2](1-\beta z')^2+[2\alpha/\gamma^3](1-\beta z')}{(\beta^2+\epsilon^2+2\beta\epsilon z')}=\frac{[\gamma^2(1-\epsilon\rho)^2-1](1-\beta z')^2+2\epsilon\rho(1-\beta z')}{\gamma^2(\beta^2+\epsilon^2+2\beta\epsilon z')}. \quad (22)$$

If we introduce the variable $\zeta = 1-\beta z'$ and the quantities

$$E_1 = (\beta^2+\epsilon^2+2\epsilon)-2\epsilon\zeta = (1+\epsilon)^2-(1/\gamma^2)-2\epsilon\zeta\,,$$
$$E_2 = \gamma^2[(1-\epsilon\rho)^2-1/\gamma^2]\zeta^2+2\epsilon\rho\zeta\,,$$

the final integration over z' may be performed in a straightforward manner, and after some rearrangement of terms, we obtain

$$\frac{d^2N}{dtd\alpha} = \frac{\pi r_0{}^2 c\alpha}{2\gamma^4\beta\alpha_1{}^2}[F(\zeta_+)-F(\zeta_-)]\,, \tag{23}$$

where $\zeta_\pm$ are the upper and lower limits of the integration in ζ, and the function F is given by

$$F(\zeta) = f_1(\zeta)+f_2(\zeta)+f_3(\zeta)+f_4(\zeta)\,,$$

$$f_1(\zeta) = E_1{}^{-1/2}\left\{\frac{\gamma}{\alpha}\left(1+\frac{2(1+\alpha\alpha_1)}{(\gamma+\alpha_1)^2-1}\right)+\frac{\gamma^2}{\alpha^2}\left(\frac{\gamma^2-1}{\gamma\alpha_1}+\frac{\alpha_1}{\gamma}+3-\frac{\alpha}{\alpha_1}\right)-\frac{3\gamma^2}{\alpha^2}\zeta-\frac{1}{\zeta}\right\}, \tag{24}$$

$$f_2(\zeta) = E_2{}^{-1/2}\left[\frac{\gamma^3}{\alpha^2}\zeta^2+\left(\frac{1+\alpha\alpha_1}{\alpha[(1-\alpha/\gamma)^2-1/\gamma^2]}+\frac{\alpha_1-\alpha}{\gamma}+1+\frac{\alpha}{\alpha_2}\right)\zeta-\gamma\right], \tag{25}$$

$$f_3(\zeta) = \frac{2\gamma^2}{\alpha[(\gamma+\alpha_1)^2-1]^{1/2}}\left[1+\frac{(\gamma+\alpha_1)^2+\alpha\alpha_1}{(\gamma+\alpha_1)^2-1}\right]\cosh^{-1}\left[\frac{(\gamma+\alpha_1)^2-1}{2\alpha_1\gamma\zeta}\right]^{1/2}, \tag{26}$$

$$f_4(\zeta) = \frac{-2\gamma^2}{\alpha[(\gamma-\alpha)^2-1]^{1/2}}\left[1+\frac{(\gamma-\alpha)^2+\alpha\alpha_1}{(\gamma-\alpha)^2-1}\right]\sinh^{-1}\left\{\frac{[(\gamma-\alpha)^2-1]\gamma\zeta}{2\alpha}\right\}^{1/2} \quad \text{for} \quad \gamma-\alpha>1\,,$$
$$= \frac{-2\gamma^2}{\alpha[1-(\gamma-\alpha)^2]^{1/2}}\left(1-\frac{(\gamma-\alpha)^2+\alpha\alpha_1}{1-(\gamma-\alpha)^2}\right)\sin^{-1}\left\{\frac{[1-(\gamma-\alpha)^2]\gamma\zeta}{2\alpha}\right\}^{1/2} \quad \text{for} \quad \gamma-\alpha<1\,. \tag{27}$$

We now turn to the question of determining the limits of integration $\zeta_\pm$. These are determined by the requirements that the quantity δ be real and, in addition, that $|z'|\leq 1$, for we see upon inspection of the expression (19) that δ may be real for certain values of z' that violate this condition. These two requirements are fulfilled if the quantity ζ lies between the values $1\pm\beta$, called the boundary lines, and simultaneously lies between the values

$$\zeta_\pm(\rho) = \rho\{(1+\epsilon-\epsilon\rho)\pm[(1+\epsilon-\epsilon\rho)^2-1/\gamma^2]^{1/2}\}\,, \tag{28}$$

called the boundary curve. It is easy to see that at $\rho=1$ the boundary curve intersects the boundary lines. At $\rho_s = 1+(\gamma-1)/\gamma\epsilon = 1+(\gamma-1)/\alpha_1$, the radical in (28) vanishes, and the boundary curve becomes imaginary. This clearly represents an absolute upper limit on ρ. (The other real branch of the boundary curve for even larger ρ can be shown to lie entirely in a region of $z'>1$.) The physical significance of this limit is quite simply seen if we write it as $\alpha=\alpha_1+(\gamma-1)$. At this limit the scattered photon has picked up *all* of the electron kinetic energy in the collision.

This limit is not usually reached in any situation that will interest us, since it only occurs when the initial photon momentum is of the order of or greater than that of the electron. Figures 2(a), 2(b), and 2(c) illustrate the three different situations that can exist. It is quite obvious that the usual situation in astrophysics

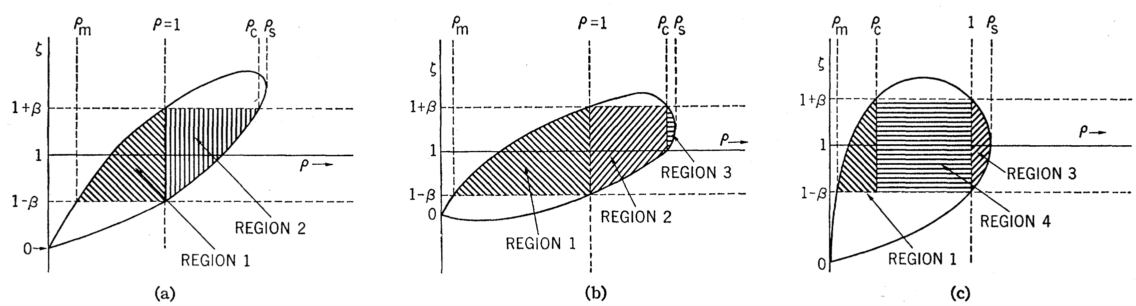

FIG. 2. Integration boundaries of the variable ζ drawn as a function of $\rho=\alpha/\alpha_1$. (For illustration only; not an accurate plot.) (a) $\epsilon<\beta/[1+\gamma(1+\beta)]$. (b) $\beta/[1+\gamma(1+\beta)]<\epsilon<\beta$. (c) $\beta<\epsilon$.

will be that depicted in Fig. 2(a), where for $\alpha_1 \leq \gamma\beta/[1+\gamma(1+\beta)]$ the maximum value of ρ is given by the point where the lower boundary curve intersects the upper boundary line; $\zeta_-(\rho)=1+\beta$, or

$$\rho_c=(1+\beta)/(1-\beta+2\alpha_1/\gamma).$$

In the relativistic limit, $1-\beta\approx 1/2\gamma^2$, and we have

$$\rho_c\approx 4\gamma^2/(1+4\alpha_1\gamma), \quad \text{or} \quad \alpha_{\max}\approx 4\alpha_1\gamma^2/(1+4\alpha_1\gamma),$$

which is just the result derived in Sec. II. The minimum value of ρ is always given by

$$\rho_m=(1-\beta)/(1+\beta 2\alpha_1/\gamma),$$

which in the relativistic limit is

$$\rho_m\approx 1/4\gamma^2 \quad \text{or} \quad \alpha_{\min}\approx \alpha_1/4\gamma^2,$$

also a result of Sec. II.

The formula given in expressions (23)–(27) are not very useful in most astrophysical applications, either for insight, since they are quite complex, or for direct computation, since they require that terms of the order of $(\gamma/\alpha)^2$ be balanced out to yield a true leading term of order of $\gamma^2\alpha_1/\alpha$. This requires computation to be carried out to an accuracy of $(\alpha\alpha_1 p)$ % to obtain an answer that is correct to an accuracy of p%. Since the quantity $\alpha\alpha_1$ can often be quite small, direct application of expressions (23)–(27) is, in general, not very satisfactory.

In Sec. IV, we shall discuss various expansions of the function $F(\zeta)$ which will be useful not only for computing the spectrum to any desired order, but also for recovering a simple approximation with a wide range of validity.

IV. EXPANSIONS OF EXACT FORMULA

The chief difficulty in computing directly with our exact formula is the fact that the quantity $\epsilon\rho=\alpha/\gamma$ is often quite small and appears as $(\epsilon\rho)^{-1}$, $(\epsilon\rho)^{-2}$ in some of the terms. This in itself suggests the way out, namely, an expansion in this or some other small quantity. The quantity that turns out to be most useful as an expansion quantity is $\epsilon=\alpha_1/\gamma$. This quantity is small in almost all physical applications and becomes smaller the more energetic the electron.

The first step in this procedure is to expand the functions $E_1^{-1/2}$, $E_2^{-1/2}$, $\cosh^{-1}$, and $\sinh^{-1}$ as power series in the quantities $2\epsilon\zeta/[(1+\epsilon)^2-1/\gamma^2]$ and $2\epsilon\rho/[\gamma^2(1-\epsilon\rho)^2-1]\zeta$.

The expansion in the first quantity can be easily shown to be convergent for all allowed values of the parameters, and the second expansion is convergent so long as we have the condition

$$\alpha_1<\frac{2(1+\beta)^{1/2}}{7+9\beta}-\frac{2+3\beta}{\gamma(1+\beta)(7+9\beta)}$$
$$\sim\tfrac{1}{8}\sqrt{2}-5/32\gamma \quad \text{for} \quad \gamma\gg 1.$$

This will be true in most cases of interest, and in those situations where it is not true, the expansion will still converge as long as

$$\epsilon\rho<2+\beta-2(1+\beta)^{1/2}\sim 0.172-0.146/\gamma^2 \quad \gamma\gg 1.$$

If this condition is violated, we see that $\epsilon\rho$ is not a small number, and there is no real need for the expansion. In the following, we shall always assume that α_1 is small enough so that this first expansion is fairly rapidly convergent.

At this point, we have an expression for the function $F(\zeta)$ of the form

$$F(\zeta)=\sum_{n=-\infty}^{\infty} K_n(\epsilon)\left(\frac{\zeta^n}{n}\right), \tag{29}$$

where the term $\zeta^0/0\equiv\ln\zeta$ arises as the leading term in the expansions of $\cosh^{-1}$ and $\sinh^{-1}$. There are no terms independent of ζ, since all we are interested in is the quantity $F(\zeta_+)-F(\zeta_-)$, and such terms would make no contribution. The coefficient $K_n(\epsilon)$ is a rather complicated function of ϵ, whose dominant term is of the order of $\epsilon^{|n-1|-2}$.

$K_n(\epsilon)$ contains the two denominators which are functions of ϵ:

$$\begin{aligned} D_1&=(1+\epsilon)^2-1/\gamma^2,\\ D_2&=(1-\epsilon\rho)^2-1/\gamma^2. \end{aligned} \tag{30}$$

These denominators appear in half powers of various orders (typically $\frac{1}{2}|n|$) and are the next items on the list to be expanded in ϵ. Before proceeding, however, we must first decide how we are to order the quantity $\rho=\alpha/\alpha_1$. Recalling the limits on ρ, $1/4\gamma^2\leq\rho\leq 4\gamma^2/(1+4\alpha_1\gamma)$, we see the question hinges on the magnitude of $\alpha_1\gamma$. If $\alpha_1\gamma\ll 1$, we have $1/4\gamma^2\leq\rho\leq 4\gamma^2$ and $\epsilon\ll 1/\gamma^2$. We may then consider ρ to be of $O(1)$ and expand in ϵ and $\epsilon\rho$ as well. On the other hand, if $\alpha_1\gamma\gtrsim 1$, then $\epsilon\gtrsim 1/\gamma^2$ and $O(\epsilon)\leq\rho\leq O(1/\epsilon)$, and the quantity $\epsilon\rho$ ranges from $O(\epsilon^2)$ to $O(1)$. In this case we must expand in quite a different manner.

We shall now consider the case where $\alpha_1\gamma\ll 1$ and treat $\epsilon\rho$ as $O(\epsilon)$. First, noting that

$$D_1=(1+\epsilon)^2+1/\gamma^2=(1+1/\gamma+\epsilon)(1-1/\gamma+\epsilon),$$

we may expand the denominator as

$$D_1^{-m}=\sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{(1-1/\gamma^2)^{n+m}}P_{n,m}(1/\gamma^2), \tag{31}$$

where

$$P_{n,m}(1/\gamma^2)=\sum_{p+q=n}\alpha_m(p)\alpha_m(q)(1-1/\gamma)^p(1+1/\gamma)^q, \tag{32}$$

and the $\alpha_m(p)$ are the expansion coefficients of

$$(1-x)^{-m}=\sum_p \alpha_m(p)x^p.$$

The $\alpha_m(p)$ are given by

$$\alpha_m(p)=m(m+1)(m+2)\cdots(m+p-1)/p!, \tag{33}$$

where $\alpha_m(0)$ and $\alpha_m(-n)\equiv 0$. We may also write $P_{n,m}(1/\gamma^2)$ as

$$P_{n,m}(1/\gamma^2)=\sum_{p=0}^{p'}\alpha_m(p)\alpha_{2m+2p}(n-2p)(1/\gamma^2)^p, \qquad (32')$$

where p' is the largest integer not greater than $\frac{1}{2}n$.

A completely equivalent expansion exists for D_2, so that our coefficient $K_n(\epsilon)$ may be in turn written as a power series in ϵ of the form

$$K_n(\epsilon)=\sum_p K_{n,p}\epsilon^{n+p}.$$

If we now regroup the terms in powers of ϵ, we may write

$$F(\zeta)=\sum_{M=-2}^{\infty}\epsilon^M F_M(\zeta). \qquad (34)$$

$F_M(\zeta)$ is a rather complex function of ζ, but for completeness we shall give its general form for arbitrary M. First, however, we note that the expansion of $F(\zeta_+)-F(\zeta_-)$ is obtained by simply replacing ζ^n by $\zeta_+{}^n-\zeta_-{}^n$ wherever it appears in $F_M(\zeta)$. In what follows, we shall use the notation

$$Z(n)\equiv(\zeta_+{}^n-\zeta_-{}^n)/n \quad \text{for} \quad n\neq 0,$$
$$Z(0)\equiv\ln(\zeta_+/\zeta_-).$$

We may now write the general expression for $F_M(\zeta_+)-F_M(\zeta_-)=F_M$:

$$\begin{aligned}F_M=(-1/\beta^2)^M\sum_{N=-2}^{M}(-2)^{N+2}\alpha_{1/2}(N+2)\Bigg\{&\frac{P_{M-N,N+5/2}(1/\gamma^2)}{\beta^5}\left[\frac{(N+1)}{\rho^2}Z(N+3)+\frac{(N+2)}{\rho}Z(N+1)\right.\\&\left.-\frac{(N+2)(1+\rho)}{\rho^2}Z(N+2)+(-\rho)^M(1/\gamma^2)\frac{Z(-N-1)}{(\gamma^2)^{N+1}}\right]+\frac{P_{M-N,N+3/2}(1/\gamma^2)}{\beta^3}\left[\frac{2(N+1)(N+2)}{(2N+3)\rho}Z(N)\right.\\&\left.-\frac{(N+1+N\rho)(N+2)}{(2N+3)\rho^2}Z(N+1)+(-\rho)^M\frac{[N+3+(N+1)\gamma^2]}{(2N+3)}\frac{Z(-N-1)}{(\gamma^2)^{N+1}}\right]\\&+\frac{P_{M-N,N+1/2}(1/\gamma^2)}{\beta}\left[\frac{(N+1)(N+2)(1+N/\rho)}{(2N+1)(2N+3)}Z(N-1)+(-\rho)^{M-1}\frac{N(N+1)(N+2)}{(2N+3)}\right.\\&\left.\times\left(\frac{Z(1-N)}{(\gamma^2)^{N-1}}+\gamma^2(1-\rho)\frac{Z(-N)}{(\gamma^2)^N}\right)+(-\rho)^M N(N+1)(N+2)\gamma^2\frac{Z(-N-1)}{(\gamma^2)^{N+1}}\right]\\&+\beta P_{M-N,N-1/2}(1/\gamma^2)\left[(-\rho)^{M-1}\frac{(N-1)(N)(N+1)(N+2)}{(2N-1)(2N+3)}(\gamma^2)\frac{Z(1-N)}{(\gamma^2)^{N-1}}\right]\Bigg\}. \qquad (35)\end{aligned}$$

In deriving the above expression, much use has been made of the recursion relations for the various $\alpha_n(p)$ coefficients, i.e.,

$$\alpha_{1/2}(N)=\frac{2N+2}{(2N+1)(2N+3)}\alpha_{3/2}(N+1), \text{ etc.}$$

Explicit calculation of the $M=-2$, -1 terms show that they are identically zero, so that the leading term in our expansion is of zero order in ϵ. Our series may then be written

$$F(\zeta_+)-F(\zeta_-)=\sum_{M=0}^{\infty}F_M\epsilon^M.$$

Note that there are terms with $(-\rho)^M$ appearing in F_M, and in the case $\rho\approx O(1/\epsilon)$, each term in the series will be $O(1)$, and our expansion breaks down completely.

Although the expression for F_M looks rather formidable, it is a straightforward matter to program a computer to evaluate our function to any order in ϵ that is desired.

It is interesting to examine the zero-order term F_0:

$$\begin{aligned}F_0=\beta^{-5}&\left(2+\frac{2}{(\gamma\beta)^2}+\frac{3}{2(\gamma\beta)^4}\right)Z(-1)\\&-\beta^{-7}\left(2+\frac{3}{(\gamma\beta)^2}\right)\left(1+\frac{1}{\rho}\right)Z(0)\\&+\beta^{-9}\left(1+\frac{6}{\rho}+\frac{1}{\rho^2}+\frac{\rho^2+1}{2\gamma^2\rho^2}\right)Z(1)\\&-\beta^{-9}\left(\frac{3(1+\rho)}{\rho^2}\right)Z(2)+\beta^{-9}\left(\frac{3}{2\rho^2}\right)Z(3). \qquad (36)\end{aligned}$$

If we also assume that $\beta\gg\epsilon$, we have $\zeta_+=1+\beta$, and to zero order in ϵ, $\zeta_-=\rho(1-\beta)$ for $\rho\geq 1$, so that

$$\begin{aligned}
Z(-1)&=(\gamma^2/\rho)[1+\beta-\rho(1-\beta)],\\
Z(0)&=-\ln\left[\frac{\rho}{\gamma^2(1+\beta)^2}\right],\\
Z(1)&=1+\beta-\rho(1-\beta),\\
Z(2)&=\tfrac{1}{2}(1+\beta)^2-\tfrac{1}{2}\rho^2(1-\beta)^2,\\
Z(3)&=\tfrac{1}{3}(1+\beta)^3-\tfrac{1}{3}\rho^3(1-\beta)^3.
\end{aligned}\tag{37}$$

Expressions (36), (37), and (23) may be combined to obtain an approximate spectrum that is valid for $\beta\gg\epsilon$ and $\alpha_1\ll 1$.

We may further simplify this approximation by assuming that the electron is relativistic, $\beta\approx 1$, $1-\beta\approx 1/2\gamma^2$. We then have, neglecting terms of order $1/\gamma^2$ as compared to 1,

$$\frac{d^2N}{dtd\alpha}=\frac{2\pi r_0{}^2c}{\alpha_1\gamma^2}[2q\ln q+(1+2q)(1-q)+O(1/\gamma^2)],\tag{38}$$

where $q=\alpha/(4\alpha_1\gamma^2)$. This is just the approximate spectrum of Ginzburg and Syrovatskii.[20] For $\rho\leq 1$, $\zeta_+=\rho(1+\beta)$ and $\zeta_-=1-\beta$. This gives

$$\begin{aligned}
Z(-1)&=(\gamma^2/\rho)[\rho(1+\beta)-(1-\beta)],\\
Z(0)&=\ln[\rho\gamma^2(1+\beta)^2],\\
Z(1)&=\rho(1+\beta)-(1-\beta),\\
Z(2)&=\tfrac{1}{2}[\rho^2(1+\beta)^2-(1-\beta)^2],\\
Z(3)&=\tfrac{1}{3}[\rho^3(1+\beta)^3-(1-\beta)^3].
\end{aligned}\tag{39}$$

Once again neglecting terms of order $1/\gamma^2$, we have

$$\frac{d^2N}{dtd\alpha}=\frac{\pi r_0{}^2c}{2\gamma^4\alpha_1}[(q'-1)(1+2/q')-2\ln q'+O(1/\gamma^2)],\tag{40}$$

where $q'=4\gamma^2\rho=4\gamma^2\alpha/\alpha_1$. Note that this is just our approximate spectrum, expression (8), with additional correction terms that become important at the bottom of the spectrum, where $\alpha/\alpha_1\approx 1/4\gamma^2$. These correction terms are expected on the basis of the discussion at the end of Sec. II.

We now consider the case where $\alpha_1\gamma$ is of order unity or greater and $\epsilon\rho$ may become of order unity. We first note that if $\alpha_1\gamma\gtrsim 1$, then $1/\gamma^2\gtrsim\epsilon$, and it would be inconsistent not to expand in $1/\gamma^2$ as well as in ϵ.

The expansion procedure is very much the same as before. The denominators D_1 and D_2 [expression (30)] may be first expanded in $1/\gamma^2$, and then the term $(1+\epsilon)^2$ that comes from D_1 is expanded in ϵ. The term $(1-\epsilon\rho)^2$ arising from D_2, however, is *not* expanded, and all factors of ρ are combined with an ϵ to make terms of $O(1)$. The resulting expressions are then grouped according to the power of ϵ and the power of $1/\gamma^2$ to give a double power series expansion of $F(\zeta)$ as

$$F(\zeta_+)-F(\zeta_-)=\sum_{N,P=0}^{\infty}F_{N,P}(1/\gamma^2)^N\epsilon^P,$$

where $F_{N,P}$ is given by

$$\begin{aligned}
F_{N,P}=\sum_{p=0}^{P}\alpha_{1/2}(P-p)\alpha_{P-p+1/2}(N)\alpha_{2(N+P-p)+1}(P)2^{P-p}(-1)^P\Bigg\{&\left[1-\frac{(P-p)(P-p-1)(P-p+N)}{2(P+N-\frac{1}{2}p+\frac{1}{2})(P+N-\frac{1}{2}p)(\epsilon\rho)^2}\right]Z(P-p-1)\\
&+\left[\frac{(P-p+2N+2)}{\epsilon\rho}-\frac{P-p}{(\epsilon\rho)^2}\right]Z(P-p)+\left[\frac{P-p-1}{(\epsilon\rho)^2}-\frac{(2N+2P-p+1)(2N+2P-p+2)}{2(N+P-p+1)\epsilon\rho}\right]Z(P-p+1)\Bigg\}\\
&+\frac{1}{(1-\epsilon\rho)^{2N}}\sum_{m=0}^{N}\frac{\alpha_{3/2}(N-m)\alpha_{N-m+1/2}(m)(-2\epsilon\rho)^{N-m}}{(1+2N-2m)}\Bigg\{\frac{\delta_{P0}}{(\epsilon\rho)^2(1-\epsilon\rho)}Z(m-N+1)\\
&+\left(\frac{\delta_{P0}[2(N-m)^2+N-m-2]}{(\epsilon\rho)(1-\epsilon\rho)}-\frac{\delta_{P1}(1+2N)}{(1-\epsilon\rho)^2}\right)Z(m-N)+\Bigg(\delta_{P0}\left[\frac{1+2N}{(1-\epsilon\rho)^2}-\frac{(1+2N-2m)(N-m)}{(1-\epsilon\rho)}\right]\\
&+\delta_{P1}\frac{(1+2N)(1+\epsilon\rho)}{(1+2N-2m)(1-\epsilon\rho)^3}\Bigg)Z(m-N-1)\Bigg\},
\end{aligned}\tag{41}$$

where

$$\begin{aligned}
\delta_{m,n}&=1 \quad\text{if}\quad m=n\\
&=0 \quad\text{if}\quad m\neq n.
\end{aligned}$$

Once again we examine the lowest-order term $F_{0,0}$:

$$\begin{aligned}
F_{0,0}&=Z(-1)+\frac{2}{\epsilon\rho}Z(0)-\frac{(1+\epsilon\rho)}{(\epsilon\rho)^2}Z(1)\\
&\quad+\frac{Z(1)}{(\epsilon\rho)^2(1-\epsilon\rho)}-\frac{2}{\epsilon\rho(1-\epsilon\rho)}Z(0)+\frac{Z(-1)}{(1-\epsilon\rho)^2}\\
&=\frac{Z(1)}{1-\epsilon\rho}-\frac{2Z(0)}{(1-\epsilon\rho)}+\left[\frac{2}{(1-\epsilon\rho)}+\frac{(\epsilon\rho)^2}{(1-\epsilon\rho)^2}\right]Z(-1).
\end{aligned}\tag{42}$$

To lowest order in ϵ and $1/\gamma^2$, the boundaries of ζ are given by $\zeta_+=1+\beta\approx 2$, $\zeta_-=\rho/2\gamma^2(1-\epsilon\rho)$, so that we have

$$\begin{aligned}
F_{0,0}=\frac{2}{(1-\epsilon\rho)}\ln\left[\frac{\rho}{4\gamma^2(1-\epsilon\rho)}\right]&+\left[\frac{4\gamma^2(1-\epsilon\rho)}{\rho}+2\right]\\
&\times\left[1-\frac{\rho}{4\gamma^2(1-\epsilon\rho)}\right](1-\epsilon\rho)^{-1}\\
&+\frac{1}{2}\frac{(\epsilon\rho)^2}{(1-\epsilon\rho)^2}\left[\frac{4\gamma^2(1-\epsilon\rho)}{\rho}-1\right].
\end{aligned}\tag{43}$$

Combining this with expression (23), we have

$$\frac{d^2N}{dtd\alpha}\approx\frac{2\pi r_0^2c}{\alpha_1\gamma^2}\left\{2q''\ln q''+(1+2q'')(1-q'')+\frac{1}{2}\frac{(4\alpha_1\gamma q'')^2}{(1+4\alpha_1\gamma q'')}(1-q'')\right\}, \quad (44)$$

where now

$$q''=\rho/4\gamma^2(1-\epsilon\rho)=\alpha/4\alpha_1\gamma^2(1-\alpha/\gamma).$$

We see that we have recovered our approximate spectrum of Sec. II, expression (9). Now, however, it appears in a complete mathematical setting as the lowest-order term in a double expansion in ϵ and $1/\gamma^2$, where $\alpha/\alpha_1=O(1/\epsilon)$.

We have seen that the assumption that $\alpha/\alpha_1\gg1$ is necessary in deriving this formula. We now may ask what happens to this approximation when we no longer have $\alpha/\alpha_1=O(1/\epsilon)$ but return to the region where $\epsilon\rho=O(\epsilon)$. We can see at once from expression (41) that because of terms containing various powers of $\epsilon\rho$ and denominators $(1-\epsilon\rho)$ to various powers, a term that was originally of a given order will now contain contributions of all higher orders in ϵ. For any approximation to a given order, this does not cause any loss of accuracy. What does hurt is the presence of terms containing $(\epsilon\rho)^{-1}$ and $(\epsilon\rho)^{-2}$. This means that any given order now has contributions from terms that were previously as much as two orders higher. To maintain our zero-order approximation, we must now include those parts of ϵF_{01} and ϵ^2F_{02} that contribute to zero order in ϵ. These terms may be found in a straightforward manner, and we find that ϵF_{01} to zero order in ϵ gives $(1/\rho)[6Z(1)-3Z(2)-2Z(0)]$, and ϵ^2F_{02} gives

$$(1/\rho^2)[Z(1)-3Z(2)+\tfrac{3}{2}Z(3)].$$

These terms are the same terms in $1/\rho$ and $1/\rho^2$ that appeared in expression (36) (negelcting $1/\gamma^2$). They did not appear in expression (38), however, since they are always at least $(1/\gamma^2)$ smaller than the leading term in $Z(-1)$, which is $2\gamma^2/\rho$. For that reason they should not be included in our present formula.

We therefore offer expression (44) as an approximation to the spectrum of inverse-Compton-scattered photons that is accurate to zero order in ϵ and $1/\gamma^2$ for values of α such that $\alpha_1\leq\alpha\leq4\alpha_1\gamma^2/(1+4\alpha_1\gamma)$, and for $\alpha_1\gamma$ as large as desired. Expression (44) is not an entirely consistent formula in that it always contains contributions from higher (and hence negligible) orders, but is *complete* in that it always includes *all* zero-order contributions.

In the situation where $\rho=O(\epsilon)$, the exact formula should be expanded once again, this time considering $\epsilon/\rho=O(1)$. However, if we are interested only in the lowest-order approximation, a simple inspection of expression (35) will suffice. Keeping in mind that for $\rho\leq1$,

$$Z(N)=[(4\gamma^2\rho)^N-1]/N(2\gamma^2)^N,$$

we see that for every value of M there are terms of order ρ^{-1} and higher but of no lower order. Therefore expression (35) to zero order in ϵ and lowest order in $1/\gamma^2$ will give us our dominant term. This exactly is what we obtained in expression (40), so we see that this formula gives the correct approximation to lowest orders in ϵ and $1/\gamma^2$, no matter what the magnitude of $\alpha_1\gamma$. This could have been expected from our discussion in Sec. II.

In Fig. 3, we compare the approximate spectrum of

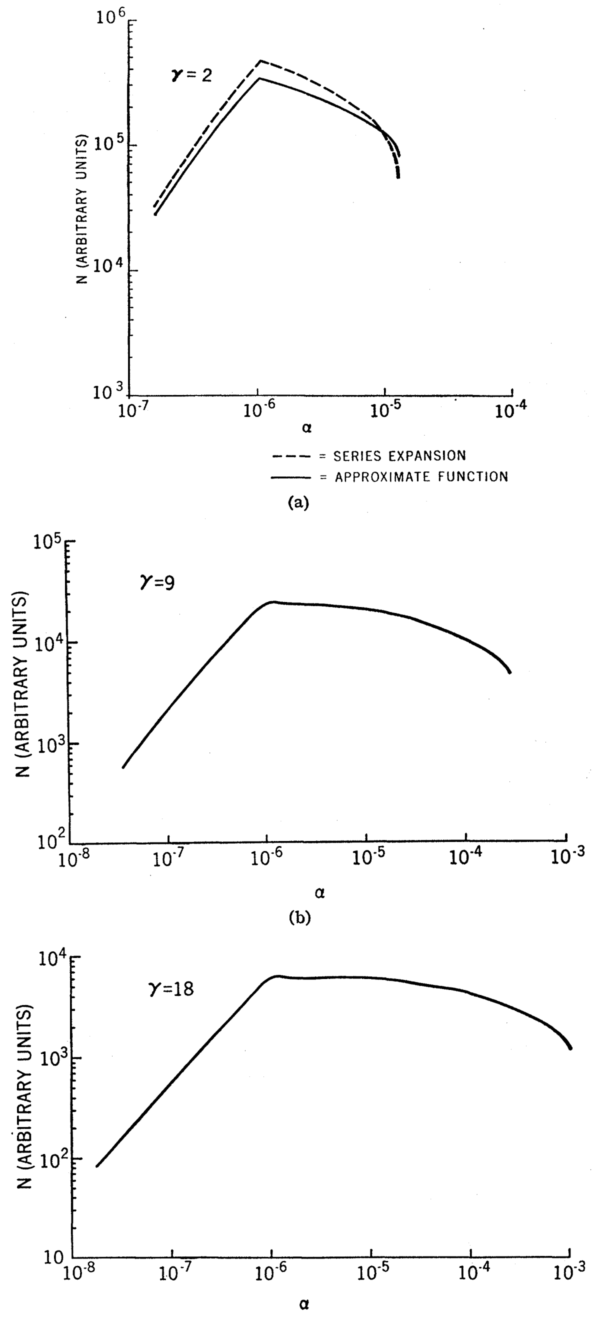

FIG. 3. Comparison of approximate and exact expressions for scattered spectrum from monoenergetic electrons. Initial photon energy $\alpha_1=10^{-6}$. Electron energy is γ, and $D=\max[\text{approx/exact}-1]$. (a) $\gamma=2$, $D=0.54$. (b) $\gamma=9$, $D=0.024$. (c) $\gamma=18$, $D=0.0055$.

expressions (40) and (44) with a computer calculation of expression (35) correct to order ϵ^5. We see that the electron does not have to be extremely relativistic for the approximate spectrum to give a good representation.

V. ASTROPHYSICAL APPLICATIONS

In astrophysics, inverse Compton scattering provides a mechanism for energy loss of high-energy cosmic-ray electrons and a source of x and γ radiation whenever energetic electrons and soft photons exist together in a region of space. The energy-loss effect has been calcu- exactly by the author in a previous publication.[9] However, it would be of interest to see how well our spectrum, expression (44), serves in giving the correct energy-loss formula. In keeping with the spirit of our approximation, we shall assume that expression (44) is valid for $0 \leq q'' \leq 1$, even though we know it is quite invalid for $q'' < 1/4\gamma^2$. When we consider effects that depend on the entire spectrum, the region $0 \leq q'' < 1/4\gamma^2$ contributes a part that is $O(1/\gamma^2)$ and is hence negligible.

The energy loss is given by

$$\left\langle -\frac{dE}{dt}\right\rangle_{\text{av}} = \int_{\alpha_1}^{\alpha_{\max}} \alpha\left(\frac{d^2N}{dtd\alpha}\right)d\alpha \approx \int_0^1 \alpha(q'')\left(\frac{d^2N}{dtdq''}\right)dq''$$
$$= 3\sigma_T c b\gamma \int_0^1 \left\{\frac{2q''^2 \ln q''}{(1+bq'')^3} + \frac{q''(1+2q'')(1-q'')}{(1+bq'')^3} + \frac{b^2q''^3(1-q'')}{2(1+bq'')^4}\right\} dq'', \quad (45)$$

where $\sigma_T = (8/3)\pi r_0^2$ and $b = 4\alpha_1\gamma$. The indicated integrals are performed in a straightforward manner to give, after some rearranging of terms,

$$\left\langle -\frac{dE}{dt}\right\rangle = \frac{3\sigma_T c\gamma}{b^2}\{(\tfrac{1}{2}b+6+6/b)\ln(1+b) - [(11/12)b^3+6b^2+9b+4](1+b)^{-2} - 2+2\,\mathrm{Li}_2(-b)\}, \quad (46)$$

where Li_2 is the dilogarithm. If we make the substitution $b = 2a$, we have

$$\left\langle -\frac{dE}{dt}\right\rangle_{\text{av}} = \tfrac{1}{2}\pi r_0^2 c\left(\frac{F(a)}{\alpha_1^2\gamma^2}\right),$$
$$F(a) = \gamma\{(a+6+3/a)\ln(1+2a) - [(22/3)a^3+24a^2+18a+4](1+2a)^{-2} - 2 - 2\,\mathrm{Li}_2(-2a)\}. \quad (47)$$

This expression may be directly compared to expressions (13) and (14) of Ref. 9. It can be seen that the present result is equal to that previously calculated for (the monoenergetic-background case) if one sets $\epsilon = 1/\gamma^2 = 0$ in the exact formula.

In considering inverse Compton scattering as a source of x and γ rays, we are interested in the radiation from electrons with a wide distribution of energies. In astrophysics the inverse power law is one of the most commonly occurring distributions, so we shall consider the spectrum

$$(2\pi r_0^2 c)R(\alpha) = \int_1^\infty \left(\gamma^{-\Gamma}\frac{d^2N}{dtd\alpha}\right)d\gamma. \quad (48)$$

If we note that

$$\gamma = \tfrac{1}{2}\alpha\left[1+\left(\frac{1+pq''}{pq''}\right)^{1/2}\right],$$
$$d\gamma = \tfrac{1}{4}\alpha^2\alpha_1\frac{dq''}{(pq'')^{3/2}(1+pq^{2'})^{1/'}},$$

where $p = \alpha\alpha_1$, we may write expression (48) as

$$R(\alpha) = \frac{2^\Gamma\alpha_1^{(\Gamma-1)/2}}{\alpha^{(\Gamma+1)/2}}F(p,\Gamma), \quad (49)$$

where

$$F(p,\Gamma) = \int_0^1 \frac{q''^{(\Gamma-1)/2}[2q''\ln q'' + (1+2q'')(1-q'')+4pq''(1-q'')]}{\{1+[pq''/(1+pq'')]^{1/2}\}^{\Gamma+2}(1+pq'')^{(\Gamma+3)/2}}dq''. \quad (50)$$

It is easy to see that for $p = \alpha\alpha_1 \ll 1$, $F(p,\Gamma)$ is essentially independent of p, and we obtain

$$R(\alpha) = 2^{\Gamma+1}\left(\frac{1}{\Gamma+3}+\frac{1}{\Gamma+1}-\frac{2}{\Gamma+5}-\frac{4}{(\Gamma+3)^2}\right) \times \alpha_1^{(\Gamma-1)/2}\alpha^{-(\Gamma+1)/2} = c'\alpha^{-(\Gamma+1)/2}. \quad (51)$$

This is just the well-known approximate spectrum of Felten and Morrison, which they obtained by approximating $d^2N/dtd\alpha$ by a δ function $\delta(\alpha - \frac{4}{3}\alpha_1\gamma^2)$. It can be shown[21] that this spectrum is a good approximation for a much more general assumption about the form of $d^2N/dtd\alpha$ than that made by Felten and Morrison.

For $p \gg 1$, on the other hand, there is no expansion in powers of p or inverse powers of p that will be valid for the entire range of the integration in q''. However, the integral may be broken up into two pieces as

$$\int_0^{1/p} + \int_{1/p}^1 ,$$

and appropriate expansions made in each range. The leading terms in p may be extracted, and after a certain amount of resumming of coefficients, we obtain the asymptotic form of $F(p,\Gamma)$ for $p\gg 1$ as

$$F(p,\Gamma)\sim\frac{\ln p - c(\Gamma)}{2^{\Gamma}p^{(\Gamma+1)/2}}, \qquad (52)$$

where, unfortunately,

$$c(\Gamma)=\frac{\Gamma}{\Gamma=1}+2\int_0^1\frac{x-[2x/(1+x)]^{\Gamma+2}}{1-x^2}dx. \qquad (53)$$

Inserting (52) in (49), we have for the case $\alpha\gg(\alpha_1)^{-1}$,

$$R(\alpha)\sim(c'/\alpha_1)[\ln(\alpha\alpha_1)+c(\Gamma)]\alpha^{-(\Gamma+1)}. \qquad (54)$$

$c(\Gamma)$ may be computed by numerical integration and is plotted in Fig. 4.

In the intermediate region $p\approx 1$, $F(p,\Gamma)$ must be computed numerically. In Fig. 5, we have plotted $F(p,\Gamma)$ as a function of p for various values of Γ, and in Fig. 6, we have the complete spectra $R(\alpha)$ for the same values of Γ.

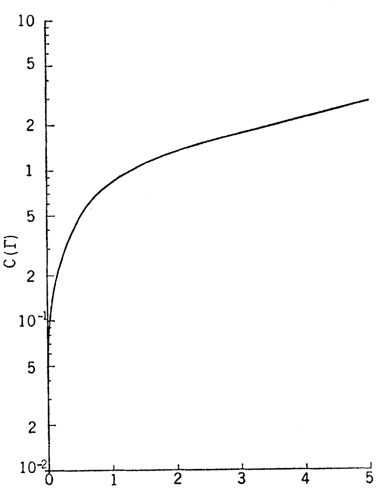

FIG. 4. Plot of $C(\Gamma)$ versus Γ.

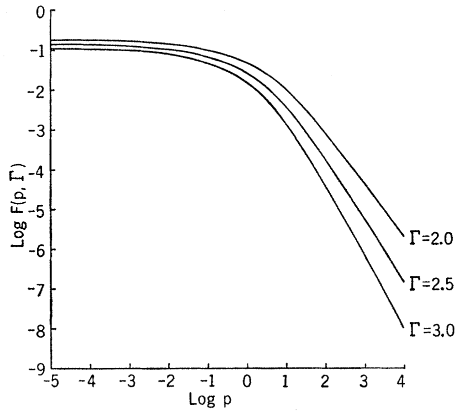

FIG. 5. Plot of $F(p,\Gamma)$ as a function of p for $\Gamma=2$, 2.5, and 3. The base of the lograithms is 10.

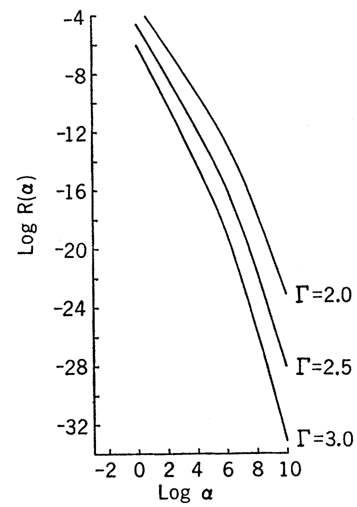

FIG. 6. Plot of $R(\alpha)$ as a function of α for $\Gamma=2$, 2.5, and 3. The base of the logarithms is 10.

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