Brillouin Scattering in Birefringent Media*

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The theory of Brillouin scattering is extended to incorporate birefringence. The necessary modifications to the Bragg laws are derived by the methods of physical optics. An integral equation method is used to calculate intensities of the first-order lines scattered by longitudinal and transverse acoustic waves in an infinite slab of birefringent crystal. This calculation also differs from those of previous authors in taking full account of internal reflection. Depletion of the incident beam is accounted for. No restriction is made to acoustic propagation parallel to the crystal faces. The intensity formulas for various cases of acoustic mode and incident optical polarization are found to differ in geometrical structure, and the differences can alter the intensities substantially.

I. INTRODUCTION

BRILLOUIN scattering of light by acoustic waves is analogous to Bragg scattering of x rays by a crystal lattice. The effect is characterized by the appearance of a small number of scattered lines, usually only those corresponding to first-order scattering. Brillouin scattering occurs only if a vector relation among the acoustic, incident optical, and scattered optical propagation vectors in the medium is satisfied.[1] This relation, which in optically isotropic media is equivalent to the usual Bragg laws, permits inference from the directions of incidence and scattering of the frequency and propagation axis of the acoustic wave involved. If the relation between the acoustic intensity and the scattered light intensity is known, Brillouin scattering may be used to determine how the acoustic power is distributed among the different frequencies. Extensive use has been made of this tool.[2]

Assumptions made in earlier theoretical work[3,4] on Brillouin scattering are not valid for some crystals of current experimental interest since they are birefringent and have high refractive indices. In this paper the theory of Brillouin scattering is extended to incorporate birefringence and internal reflection. The Bragg laws are modified to allow for birefringence in Sec. II.[5a] In Sec. III the electric field within the crystal is expanded in a series of plane waves, and the integral equation for the electric field is used to derive recursion relations among the partial amplitudes of the series. Relations equivalent to boundary conditions at the crystal faces, as well as an expression for the emerging electric field in terms of the partial amplitudes, are obtained. In Sec. IV the partial amplitudes and scattered intensities are evaluated for some important special geometries in hexagonal crystals. In each case of scattering by transverse acoustic waves, we find that there is an optical polarization rotation of 90° upon scattering. The intensity formulas we derive for various combinations of acoustic mode and optical polarization differ by simple but important geometrical factors. As in the isotropic case, no qualitative difference is found between the intensities of light scattered by longitudinal and transverse acoustic waves.

The observation of very intense acoustic disturbances in some piezoelectric semiconductors subjected to large electric fields[5b,2b] has focused interest on the Brillouin-scattering theory presented in this paper. The acoustic flux, which is due to amplification of noise or an input signal by means of the acoustoelectric effect, is accompanied in some cases by strains large enough to cause severe crystal damage. The flux is distributed over a range of acoustic frequencies and propagation directions. For example, in some of the samples of Zucker and Zemon, there is significant flux from 100 to 1500 MHz propagating at angles up to 15° from the electric-field direction. Brillouin scattering is a convenient probe with which to investigate these acoustic waves. It does not disturb the processes responsible for the acoustic flux. It is a highly selective interaction, scattering significantly only if the modified Bragg laws are satisfied, so that one combination of acoustic frequency and propagation direction may be examined at a time. The experimenter adjusts his angles of incidence and scattering to those appropriate to this combination and measures the ratio of scattered to incident intensity. If

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[1] An inequality involving the wavelengths of sound and light, and the amplitude and width of the sound beam must also be satisfied. See G. W. Willard, J. Acoust. Soc. Am. **21**, 101 (1949).

[2] (a) See, for example, M. G. Cohen and E. I. Gordon, Bell System Tech. J. **44**, 693 (1965); C. F. Quate, C. D. W. Wilkinson, and D. K. Winslow, Proc. IEEE **53**, 1604 (1965); G. B. Benedek and K. Fritsch, Phys. Rev. **149**, 647 (1966). (b) J. Zucker and S. Zemon, Appl. Phys. Letters **9**, 328 (1966).

[3] H. Mueller, Proc. Roy. Soc. (London) **A166**, 425 (1938); Benedek and Fritsch, Ref. 2.

[4] A. B. Bhatia and W. J. Noble, Proc. Roy. Soc. (London) **A220**, 356 (1953); **A220**, 369 (1953).

[5] (a) After the completion of this part of the work, it was discovered that the modified Bragg laws had been given by V. Chandrasekharan, Current. Sci. (India) **12**, 371 (1950); J. Phys. (Paris) **26**, 655 (1965). They have since been discussed in detail by R. W. Dixon, J. Quant. Electron. **QE-3**, 85 (1965). (b) See, for example, A. R. Hutson, J. H. McFee, and D. L. White, Phys. Rev. Letters **7**, 237 (1961); J. H. McFee, J. Appl. Phys. **34**, 1548 (1963); P. O. Sliva and R. Bray, Phys. Rev. Letters **14**, 372 (1965); W. H. Haydl, Phys. Letters **24**(A), 413 (1967).

the relevant photoelastic coefficients are known, he can use the formulas we derive to evaluate the strain amplitude. He can translate his sample to study the variation of the strain with distance from the cathode. We modify Brillouin scattering theory to accommodate various acoustic propagation directions. The other modifications we make to the theory are necessary because of the nature of the materials manifesting a strong acoustoelectric effect. The most important of these are birefringent, having the wurtzite crystal structure. They have large refractive indices so that internal reflections must be corrected for, since antireflection coatings efficacious for a wide range of angles of incidence and scattering are not experimentally feasible. The need to include birefringence and transverse acoustic polarization leads us to a tensor theory.

II. BRAGG LAWS

In the theory of x-ray diffraction, the Bragg laws are derived as criteria for constructive interference. This approach is adapted in this section to light scattering in birefringent media.[6] The conditions which must be satisfied if constructive interference is to occur are: (1) Light scattered from different parts of the same wavefront must be in phase; (2) light scattered from different wavefronts must be in phase. Consider condition (1) first. In Fig. 1, WW' is the wavefront of a plane acoustic wave. A plane light wave is incident with wave normals A_1A_2 and B_1B_2 at an angle θ_i, and is scattered along A_2A_3 and B_2B_3 at an angle θ_s. Condition (1) will be satisfied if the optical path lengths along $A_1A_2A_3$ and $B_1B_2B_3$ are equal, the optical path length being the integral of the refractive index along the path in space traversed by the wave-normal. Let n_i and n_s denote the indices of refraction before and after scattering, respectively. Condition (1) is equivalent to

$$n_i \cos\theta_i = n_s \cos\theta_s. \tag{1}$$

We now examine condition (2). In Fig. 2, WW' and XX' are successive acoustic wavefronts, separated by the acoustic wavelength Λ. Constructive interference will occur if the optical path lengths along $A_1A_2A_3$ and $A_1B_2B_3$ differ by $m\lambda$, where m is an integer and λ the wavelength in vacuum of the light. This is equivalent

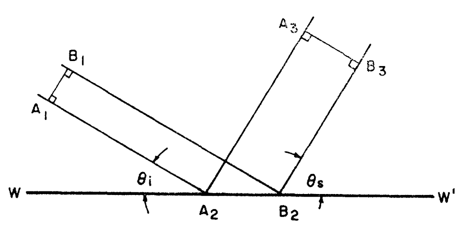

FIG. 1. Scattering at one wavefront.

[6] This approach was first used by V. Chandrasekharan. See Ref. 5a.

to the condition

$$\Lambda(n_i \sin\theta_i + n_s \sin\theta_s) = m\lambda. \tag{2}$$

In what follows we shall limit ourselves to $m = \pm 1$, since the higher orders are faint when conditions for Brillouin scattering are satisfied.

The modified Bragg laws (1) and (2) take a particularly simple form when expressed in terms of propagation vectors. Let $\mathbf{k}_i$, $\mathbf{k}_s$, and $\mathbf{Q}$ be the propagation vectors in the medium of the incident light, scattered light, and sound. In terms of these vectors, (1) and (2) are equivalent to

$$\mathbf{k}_s = \mathbf{k}_i \pm \mathbf{Q}, \tag{3}$$

which states the conservation of quasimomentum for the process. The condition that energy must be conserved in the scattering process may be written

$$\omega_s = \omega_i \pm \Omega, \tag{4}$$

where ω_i, ω_s, and Ω are the angular frequencies of the incident, scattered, and acoustic waves. Since $\Omega \ll \omega_i$, we can for most purposes neglect the difference between ω_i and ω_s.

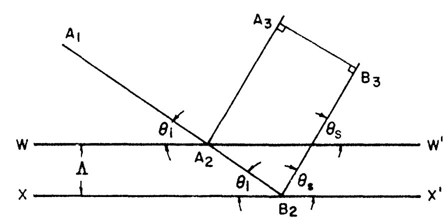

FIG. 2. Scattering at successive wavefronts.

In an optically isotropic medium, where the index of refraction is independent of polarization, n_i equals n_s, and (1) and (2) reduce to the usual Bragg laws. In an isotropic medium $|\mathbf{k}_i|$ equals $|\mathbf{k}_s|$, but in a birefringent medium they can be quite different if the polarization of the scattered light differs from that of the incident light. As illustrated in Fig. 3, the effect of birefringence can be large even though the difference in the indices for different polarizations is small.

III. INTEGRAL EQUATION AND TRIAL SOLUTION

The integral equation upon which our theory is based may be derived from Maxwell's equations.[7] We introduce the polarization density $\mathbf{P}(r,t)$, defined in terms of $\mathbf{D}(\mathbf{r},t)$, the electric displacement, and $\mathbf{E}(\mathbf{r},t)$, the average or macroscopic electric field, by

$$4\pi\mathbf{P}(\mathbf{r},t) \equiv \mathbf{D}(\mathbf{r},t) - \mathbf{E}(\mathbf{r},t) = [\boldsymbol{\varepsilon}(\mathbf{r},t) - \mathbf{I}] \cdot \mathbf{E}(\mathbf{r},t), \tag{5}$$

[7] The integral equation was introduced by C. G. Darwin, Trans. Cambridge Phil. Soc. 23, 137 (1924). It is discussed at length by M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1965), 3rd ed., Chap. 2. Our development differs slightly from that of the above authors in that we do not exclude a small volume at $R=0$ from the region of integration.

where $\boldsymbol{\varepsilon}$ is the dielectric tensor of the medium and $\mathbf{I}$ the unit tensor. Substituting (5) into Maxwell's equations for nonmagnetic media, we obtain an inhomogeneous wave equation for $\mathbf{E}$:

$$(c^{-2}\partial^2/\partial t^2-\nabla^2)\mathbf{E}(\mathbf{r},t)=4\pi(-c^{-2}\partial^2/\partial t^2+\text{grad div})\mathbf{P}(\mathbf{r},t). \quad (6)$$

With the aid of the kernel function for the wave equation, we rewrite (6) in integral form as

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_{\text{inc}}(\mathbf{r},t)+\int\int\int\int R^{-1}\delta(t-t'-R/c)\times(-c^{-2}\partial^2/\partial t'^2+\text{grad}'\,\text{div}')\mathbf{P}(\mathbf{r}',t')dV'dt', \quad (7)$$

where $\mathbf{R}=\mathbf{r}'-\mathbf{r}$, $R=|\mathbf{R}|$. The incident electric field, $\mathbf{E}_{\text{inc}}$, is a homogeneous solution of the wave equation. The integral is taken over all space-time. Using Gauss's theorem and neglecting noncontributing surface terms at infinity,[8] we may transform (7) into

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_{\text{inc}}(\mathbf{r},t)+\int\int\int\int \mathbf{P}(\mathbf{r}',t')\times(-c^{-2}\partial^2/\partial t'^2+\text{grad}'\,\text{div}')\times[R^{-1}\delta(t-t'-R/c)]dV'dt'.$$

It is readily seen that

$$\frac{\partial^2}{\partial t'^2}\delta(t-t'-R/c)=\frac{\partial^2}{\partial t^2}\delta(t-t'-R/c).$$

Similarly, if $F(R)$ is a differentiable function of R, it is easily shown that

$$\frac{\partial}{\partial r_i'}\frac{\partial}{\partial r_j'}F(R)=\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}F(R).$$

These results enable us to rewrite our integral equation as

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_{\text{inc}}(\mathbf{r},t)+\int\int\int\int \mathbf{P}(\mathbf{r}',t')(-c^{-2}\partial^2/\partial t^2+\text{grad div})\times[R^{-1}\delta(t-t'-R/c)]dV'dt'. \quad (8)$$

The differential operator $(-c^{-2}\partial^2/\partial t^2+\text{grad div})$ commutes with the integration since the limits of integration do not depend on r or t. Rewriting (8) so that integration precedes differentiation, and performing the trivial integration over t', we obtain

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_{\text{inc}}(\mathbf{r},t)+(-c^{-2}\partial^2/\partial t^2+\text{grad div})\times\int\int\int R^{-1}\mathbf{P}(\mathbf{r}',\,t-R/c)dv'. \quad (9)$$

[8] See Darwin, Ref. 7, for a discussion of surface terms at infinity.

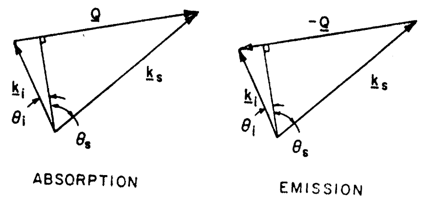

FIG. 3. Vector triangle representation of the Bragg laws.

The acoustic disturbance enters the theory through $\boldsymbol{\varepsilon}$, in which it induces fluctuations that depend on space and time. The acoustic amplitude may be written as

$$\mathbf{u}(\mathbf{r},t)=\mathbf{A}\sin(\mathbf{Q}\cdot\mathbf{r}-\Omega t). \quad (10)$$

The components of the strain tensor $\mathbf{e}$ due to the acoustic wave are

$$e_{ij}=S_{ij}\cos(\mathbf{Q}\cdot\mathbf{R}-\Omega t), \quad (11)$$

where

$$S_{ij}=\tfrac{1}{2}(A_iQ_j+A_jQ_i). \quad (12)$$

The strain induces a small periodic fluctuation $\delta\boldsymbol{\varepsilon}(\mathbf{r},t)$ in $\boldsymbol{\varepsilon}$. The usual description of this effect is in terms of Pockel's photoelastic coefficients p_{ijkl},[9] which relate the change in the inverse of the dielectric tensor to the strain by the equation

$$[\delta(\boldsymbol{\varepsilon}^{-1})]_{ij}=p_{ijkl}e_{kl}. \quad (13)$$

Since the derivative of the product of a matrix and its inverse is zero, we deduce from (12)

$$(\delta\boldsymbol{\varepsilon})_{ij}=(\boldsymbol{\varepsilon}_1)_{ij}\cos(\mathbf{Q}\cdot\mathbf{r}-\Omega t), \quad (14)$$

where

$$(\boldsymbol{\varepsilon}_1)_{ij}=-(\boldsymbol{\varepsilon}_0)_{ik}(\boldsymbol{\varepsilon}_0)_{jl}p_{klmn}S_{mn}, \quad (15)$$

$\boldsymbol{\varepsilon}_0$ being the dielectric tensor in the absence of strain.

The crystal in which the scattering occurs is taken to be a slab bounded by the planes $y=0$ and $y=d$. The acoustic disturbance is assumed to fill the sample. Without loss of generality we may take the incident light to be a linearly polarized plane wave entering the crystal at the $y=0$ face. Since the scattered electric field is linear in the amplitude of the incident electric field, solutions for more general incident fields may be obtained by superposition. The incident field is

$$\mathbf{E}_{\text{inc}}=\mathbf{B}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]. \quad (16)$$

As in the theory of Bhatia and Noble,[4] a sum of plane waves propagating in the x-y plane is adopted as the trial solution of (9). We write

$$\mathbf{E}(\mathbf{r},t)=\sum_{lms}\mathbf{V}_{lms}\exp[i(\mathbf{K}_{lms}\cdot\mathbf{r}-\omega_l t)]. \quad (17)$$

The components of $\mathbf{K}_{lms}$ are denoted by

$$\mathbf{K}_{lms}=(r_l,q_{lms},0). \quad (18)$$

[9] F. Pockels, *Lehrbuch der Kristalloptik* (Teubner, Leipzig, 1906). This work is described by J. F. Nye, *Physical Properties of Crystals* (Oxford University Press, London, 1957).

In the summations l and m take on integer values, while s takes only the values $+1$ and -1. Waves will be traveling toward both faces of the crystal because of internal reflection. The partial wave of amplitude $\mathbf{V}_{lm+}$ is traveling toward the $y=d$ face; that of amplitude $\mathbf{V}_{lm-}$ toward the $y=0$ face. In the absence of strain,

$$q_{lms}=sq_{lm},\quad q_{lm}=|q_{lms}|\,. \tag{19}$$

The polarization density associated with the trial solution is

$$\mathbf{P}=(4\pi)^{-1}\sum_{lms}[\boldsymbol{\varepsilon}_0-\mathbf{I}+\boldsymbol{\varepsilon}_1\cos(\mathbf{Q}\cdot\mathbf{r}-\Omega t)]\mathbf{V}_{lms} \times\exp[i(\mathbf{K}_{lms}\cdot\mathbf{r}-\omega_l t)]\,. \tag{20}$$

When (19) is substituted into (9), integrals appear having the form

$$J\equiv(4\pi)^{-1}\int\int\int R^{-1}\exp i(\mathbf{A}\cdot\mathbf{r}'+BR)dV', \tag{21}$$

where $\mathbf{A}$ is a vector having components $(A_1,A_2,0)$. The exact form taken by J depends on the region of space containing the observation point.[10] For $y<0$,

$$J=[2b(A_1,B)h(\mathbf{A},B)]^{-1}\{\exp[ih(\mathbf{A},B)d]-1\} \times\exp i[A_1x-b(A_1,B)y]\,. \tag{22a}$$

For $y>d$,

$$J=[2b(A_1,B)g(\mathbf{A},B)]^{-1}\{\exp[ig(\mathbf{A},B)d]-1\} \times\exp i[A_1x+b(A_1,B)y]\,. \tag{22b}$$

For $0\leq y\leq d$,

$$\begin{aligned}J=&[g(\mathbf{A},B)h(\mathbf{A},B)]^{-1}\exp i(A_1x+A_2y)\\&-[2b(A_1,B)g(\mathbf{A},B)]^{-1}\exp i[A_1x+b(A_1,B)y]\\&+[2b(A_1,B)h(\mathbf{A},B)]^{-1}\\&\times\exp i[A_1x-b(A_1,B)y+h(\mathbf{A},B)d]\,.\end{aligned} \tag{22c}$$

The quantities $b(A_1,B)$, $g(\mathbf{A},B)$, and $h(\mathbf{A},B)$ are defined by

$$b(A_1,B)=(B^2-A_1{}^2)^{1/2}, \tag{23a}$$

$$g(\mathbf{A},B)=A_2-b(A_1,B)\,,\quad h(\mathbf{A},B)=A_2+b(A_1,B)\,. \tag{23b}$$

The positive square root is used in (23a).

Evaluating the integrals for the region $0\leq y\leq d$, we find that (16) is a solution of the integral equation if

$$\begin{aligned}&\sum_{lms}\mathbf{V}_{lms}\exp i(\mathbf{K}_{lms}\cdot\mathbf{r}-\omega_l t)-\mathbf{B}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)-\sum_{lms}(\bar{\omega}_l{}^2+\text{grad div})(\boldsymbol{\varepsilon}_0-\mathbf{I})\cdot\mathbf{V}_{lms}\\&\times\{(r_l{}^2+q_{lms}{}^2-\bar{\omega}_l{}^2)^{1/2}\exp[i(\mathbf{K}_{lms}\cdot\mathbf{r}-\omega_l t)]-[2b(r_l,\bar{\omega}_l)g(\mathbf{K}_{lms},\bar{\omega}_l)]^{-1}\exp i[r_lx+b(r_l,\bar{\omega}_l)y-\omega_l t]\\&+[2b(r_l,\bar{\omega}_l)h(\mathbf{K}_{lms},\bar{\omega}_l)]^{-1}\exp i[r_lx-b(r_l,\bar{\omega}_l)y-\omega_l t+h(\mathbf{K}_{lms},\bar{\omega}_l)d]\}\\&-\tfrac{1}{2}\sum_{lms}\sum_{n}[(\bar{\omega}_l+n\Omega)^2+\text{grad div}]\boldsymbol{\varepsilon}_1\cdot\mathbf{V}_{lms}\{[(r_l+n\hat{\imath}\cdot\mathbf{Q})^2+(q_{lms}+n\hat{\jmath}\cdot\mathbf{Q})^2-(\bar{\omega}_l+n\bar{\Omega})^2]^{-1}\\&\times\exp i[(\mathbf{K}_{lms}+n\mathbf{Q})\cdot\mathbf{r}-(\omega_l+n\Omega)t]-[2b(r_l+n\hat{\imath}\cdot\mathbf{Q},\,\bar{\omega}_l+n\bar{\Omega})g(\mathbf{K}_{lms}+n\mathbf{Q},\,\bar{\omega}_l+n\bar{\Omega})]^{-1}\\&\times\exp i[(r_l+n\hat{\imath}\cdot\mathbf{Q})x+b(r_l+n\hat{\imath}\cdot\mathbf{Q}\bar{\omega}_l+n\bar{\Omega})y-(\omega_l+n\Omega)t]+[2b(r_l+n\hat{\imath}\cdot\mathbf{Q},\,\bar{\omega}_l+n\Omega)h(\mathbf{K}_{lms}+n\mathbf{Q},\,\bar{\omega}_l+n\bar{\Omega})]^{-1}\\&\times\exp i[(r_l+n\hat{\imath}\cdot\mathbf{Q})x-b(r_l+n\hat{\imath}\cdot\mathbf{Q},\,\bar{\omega}_l+n\bar{\Omega})y+h(\mathbf{k}_{lms}+n\mathbf{Q},\,\bar{\omega}_l+n\bar{\Omega})d-(\omega_l+n\Omega)t]\}=0\,,\end{aligned} \tag{24}$$

where $\hat{\imath}$, $\hat{\jmath}$ represent unit vectors in the x and y directions, respectively, and n takes the two values $+1$, -1. We have defined $\bar{\omega}_l=\omega_l/c$. It is evident from (24) that ω_l must vary in steps of Ω and $\mathbf{K}_{lms}$ must vary in steps of $\mathbf{Q}$ as l is varied if the integral equation is to be satisfied. Further, for some value L of l, ω_L must equal ω and r_L must equal $\hat{\imath}\cdot\mathbf{k}$ if $\mathbf{B}$ is nonzero. We choose $L=0$, obtaining

$$\omega_l=\omega+l\Omega,\quad r_0=\hat{\imath}\cdot\mathbf{k},\quad \mathbf{K}_{lms}=\mathbf{K}_{0ms}+l\mathbf{Q}\,. \tag{25}$$

The coefficient of any exponential differing from all of the others in argument must vanish separately. Setting the coefficients equal to zero, performing the grad div operation, and simplifying the resulting equations, we obtain

$$(\bar{\omega}_l{}^2\mathbf{I}-\mathbf{k}_{l+}\mathbf{k}_{l+})\cdot\sum_{ms}[(\boldsymbol{\varepsilon}_0-\mathbf{I})\cdot\mathbf{V}_{lms} +\tfrac{1}{2}\boldsymbol{\varepsilon}_1\cdot(\mathbf{V}_{l-1,ms}+\mathbf{V}_{l+1,ms})](2b_lg_{lms})^{-1}=\mathbf{B}\delta_{l0}\,, \tag{26a}$$

$$(\bar{\omega}_l{}^2\mathbf{I}-\mathbf{k}_{l-}\mathbf{k}_{l-})\cdot\sum_{ms}[(\boldsymbol{\varepsilon}_0-\mathbf{I})\cdot\mathbf{V}_{lms} +\tfrac{1}{2}\boldsymbol{\varepsilon}_1\cdot(\mathbf{V}_{l+1,ms})](2b_lh_{lms})e^{ik_{lms}d}=0\,, \tag{26b}$$

$$\bar{\omega}_l{}^2[\boldsymbol{\varepsilon}_0\cdot\mathbf{V}_{lms}+\tfrac{1}{2}\boldsymbol{\varepsilon}_1\cdot(\mathbf{V}_{l-1,ms}+\mathbf{V}_{l+1,ms})] =(\mathbf{K}_{lms}{}^2\mathbf{I}-\mathbf{K}_{lms}\mathbf{K}_{lms})\cdot\mathbf{V}_{lms}\,, \tag{27}$$

where

$$\mathbf{k}_{l\pm}=(r_l,\,\pm b_l,\,0)\,. \tag{28}$$

We have used the abbreviations

$$b_l=b(r_l,\bar{\omega}_l)\,,\quad g_{lms}=g(\mathbf{K}_{lms},\bar{\omega}_l)\,,\quad h_{lms}=h(\mathbf{K}_{lms},\bar{\omega}_l)\,. \tag{29}$$

Equations (26a) and (26b) are equivalent to the boundary conditions on $\mathbf{E}$ and $\mathbf{H}$ at the crystal faces. Equation (27) is the recursion relation among the $\mathbf{V}_{lms}$ and could have been derived from Maxwell's equations in the medium. It is also possible, although tedious, to derive (26a) and (26b) without resort to the integral equation.

The integral equation also determines the scattered electric field E_s in the region beyond the crystal. The

[10] This integral was evaluated by Darwin, Ref. 7, and is discussed in Born and Wolf, Ref. 7, p. 772.

polarization density vanishes in this region, and there is no distinction between average and effective fields. Using (22b) and (26a), we derive

$$\mathbf{E}_s=\sum_l (\omega_l^2\mathbf{I}-\mathbf{k}_{l+}\mathbf{k}_{l+})\cdot\sum_{ms}[(\boldsymbol{\varepsilon}_0-\mathbf{I})\cdot\mathbf{V}_{lms}+\tfrac{1}{2}\boldsymbol{\varepsilon}_1\cdot(\mathbf{V}_{l-1,ms}+\mathbf{V}_{l+1,ms})]\cdot(2b_lg_{lms})^{-1}\times\exp i(\mathbf{k}_{l+}\cdot\mathbf{r}+g_{lms}d-\omega_lt).\quad(30)$$

With this equation and the conditions (26) and (27), $\mathbf{E}_s$ is completely determined. The propagation direction of the lth contribution to $\mathbf{E}_s$ may be read off from (30) if (28) and (25) are used.

IV. EVALUATION OF THE SCATTERED LIGHT INTENSITY

We have derived a complete set of equations for the partial amplitudes $\mathbf{V}_{lms}$ and an expression for $\mathbf{E}_s$. In this section we shall calculate the $\mathbf{V}_{lms}$, the first-order term of $\mathbf{E}_s$, and the intensity due to this term for some important special cases of crystal symmetry, acoustic mode, and incident light propagation direction. Most of the crystals in which Brillouin scattering experiments have been done have had cubic or hexagonal (wurtzite) lattices, and further discussion is limited to the wurtzite structure, which is uniaxial. Intensity formulas for the cubic case may be obtained by setting the extraordinary and ordinary refractive indices equal. The difference between the frequencies of the incident and scattered light will again be neglected.

Case I

The first case we treat is one in which the crystal birefringence plays no role. The optic axis is taken in the plane of the slab faces. The acoustic wave is longitudinal and propagates in the basal plane, i.e., the plane perpendicular to the optic axis. The acoustic propagation vector is inclined to the slab faces at an angle β (see Fig. 4). Coordinates are chosen such that the z direction is along the optic axis. The incident light propagates in the basal plane, is incident at an angle θ_i, and is polarized with its electric vector along the optic axis.

Our choice of geometry reduces $\boldsymbol{\varepsilon}_0$ to a diagonal tensor of the form $n_0^2(\hat{\imath}\hat{\imath}+\hat{\jmath}\hat{\jmath})+n_e^2\hat{k}\hat{k}$, where n_0 and n_e are the ordinary and extraordinary indices of refraction. The nonvanishing components of the strain tensor due to the acoustic wave are S_{11}, S_{12}, S_{21}, and S_{22}. The photoelastic constants are such that $\boldsymbol{\varepsilon}_1$ takes the form

$$\boldsymbol{\varepsilon}_1=-n_0^4[(p_{1111}S_{11}+p_{1122}S_{22})\hat{\imath}\hat{\imath}+2p_{1212}S_{12}(\hat{\imath}\hat{\jmath}+\hat{\jmath}\hat{\imath})+(p_{2211}S_{11}+p_{2222}S_{22})\hat{\jmath}\hat{\jmath}]-n_e^4p_{3311}(S_{11}+S_{22})\hat{k}\hat{k}.$$

If any vector $\mathbf{A}$ lies along the z axis, the products $\boldsymbol{\varepsilon}_0\cdot\mathbf{A}$ and $\boldsymbol{\varepsilon}_1\cdot\mathbf{A}$ will also lie along the z axis. None of our equations relating the $\mathbf{V}_{lms}$ to each other and to the incident amplitude couples the z direction to any other,

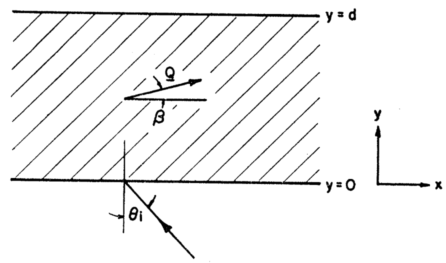

FIG. 4. The basal plane.

and the E polarization of the incident light will be unchanged by the scattering. We are left with a scalar problem, and the tensors $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_1$ may be replaced in all of our equations by the scalars n_e^2 and ϵ_1, respectively, where $\epsilon_1=-n_e^4p_{3311}(S_{11}+S_{22})$.

The recursion relation (27) takes the simple form

$$(r_l^2+q_{lms}^2-n_e^2\bar{\omega}^2)\mathbf{V}_{lms}=\tfrac{1}{2}\epsilon_1\bar{\omega}^2(\mathbf{V}_{l-1,ms}+\mathbf{V}_{l+1,ms}).\quad(31)$$

The subscript l on $\bar{\omega}$ has been omitted because we neglect the difference in frequency of incident and scattered light. With the aid of this relation, we may reduce (26a), (26b), and (30) to

$$\sum_{ms}[2(\bar{\omega}^2-r_l^2)^{1/2}g_{lms}]^{-1}(r_l^2+q_{lms}^2-\bar{\omega}^2)\mathbf{V}_{lms}=\mathbf{B}\delta_{l0},\quad(32a)$$

$$\sum_{ms}[2(\bar{\omega}^2-r_l^2)^{1/2}h_{lms}]^{-1}\times(r_l^2+q_{lms}^2-\bar{\omega}^2)\mathbf{V}_{lms}\exp(ih_{lms}d)=0,\quad(32b)$$

$$\mathbf{E}_s=\sum_{lms}[2(\bar{\omega}^2-r_l^2)^{1/2}g_{lms}]^{-1}(r_l^2+q_{lms}^2-\bar{\omega}^2)\mathbf{V}_{lms}\times\exp[i(\mathbf{k}_{l+}\cdot\mathbf{r}+g_{lms}d-\omega t)].\quad(33)$$

We now derive intensity formulas assuming that the Bragg laws are satisfied for the $l=1$ line.[11,12] Satisfaction of these laws is the criterion for $\mathbf{V}_{1ms}$ to be large. We neglect $\mathbf{V}_{lms}$ for l not equal to zero or unity. Then (31) may be written as

$$(r_0^2+q_{0ms}^2-n_e^2\bar{\omega}^2)\mathbf{V}_{0ms}=\tfrac{1}{2}\epsilon_1\bar{\omega}^2\mathbf{V}_{1ms},\quad(34a)$$

$$(r_1^2+q_{1ms}^2-n_e^2\bar{\omega}^2)\mathbf{V}_{1ms}=\tfrac{1}{2}\epsilon_1\bar{\omega}^2\mathbf{V}_{0ms}.\quad(34b)$$

There are two situations that can occur. In the first, the internally reflected light also satisfies the $l=1$ Bragg condition. This occurs when $\beta=0$. In this case the light traveling toward the $y=0$ face interacts strongly with the acoustic wave. For $\beta\neq0$, the light traveling toward the $y=0$ face is not influenced by the acoustic wave, and a separate treatment is required. We treat the $\beta=0$ case first. Combining (34a) and (34b), and using the Bragg laws to express q_{0ms} in terms of q_{1ms}, or vice versa, one obtains a quadratic equation for q_{lms}. Denoting one root by $m=0$ and the other by $m=1$, we obtain for the

[11] The geometry is essentially that of Bhatia and Noble, Ref. 3.
[12] The calculation for the $l=-1$ line is identical.

possible q_{lms} values:

$$q_{00s}=s[K_0+\tfrac{1}{4}\epsilon_1\bar{\omega}^2K_0^{-1/2}K_1^{-1/2}], \quad (35a)$$

$$q_{01s}=s[K_0-\tfrac{1}{4}\epsilon_1\bar{\omega}^2K_0^{-1/2}K_1^{-1/2}], \quad (35b)$$

$$q_{10s}=s[K_1+\tfrac{1}{4}\epsilon_1\bar{\omega}^2K_0^{-1/2}K_1^{-1/2}], \quad (35c)$$

$$q_{11s}=s[K_1-\tfrac{1}{4}\epsilon_1\bar{\omega}^2K_0^{-1/2}K_1^{-1/2}], \quad (35d)$$

where

$$K_0=(n_e^2\bar{\omega}^2-r_0^2)^{1/2}, \quad K_1=(n_e^2\bar{\omega}^2-r_1^2)^{1/2}. \quad (36)$$

We define a parameter α by

$$\alpha=\tfrac{1}{2}\bar{\omega}^2\epsilon_1K_0^{-1/2}K_1^{-1/2}, \quad (37)$$

and note that

$$q_{00s}-q_{01s}=q_{10s}-q_{11s}=s\alpha. \quad (38)$$

Equations (31) and (35) imply

$$\mathbf{V}_{10s}=K_0^{1/2}K_1^{-1/2}\mathbf{V}_{00s}, \quad \mathbf{V}_{11s}=-K_0^{1/2}K_1^{-1/2}\mathbf{V}_{01s}. \quad (39)$$

The boundary conditions (26a) and (26b) provide further relations among the amplitudes. If the refractive indices are close to unity and the angles of incidence are small, the internally reflected amplitudes $\mathbf{V}_{lm-}$ are negligible and may be dropped from the theory. In this case, (32b) provides no information and (32a) only need be considered. Some materials of current interest have large refractive indices, e.g., about 2.5 for CdS, and internal reflection is appreciable. For this reason we retain (32b) and solve for $\mathbf{V}_{lm-}$ as well as $\mathbf{V}_{lm+}$. When solving (32a), (32b), and (39) for $\mathbf{V}_{0ms}$ and $\mathbf{V}_{1ms}$, we may drop the term in q_{lms} of first order in ϵ_1 everywhere except in the phase factors $\exp(iq_{lms}d)$. Although this term is much smaller than the zero-order terms, its product with d may be of order unity.

We have defined θ_i as the angle of incidence and derived the result $r_0=\hat{\imath}\cdot\mathbf{k}$. For the incident direction chosen in Fig. 4, $r_0=-\bar{\omega}\sin\theta_i$. In order to simplify later formulas, we now introduce other angles (see Fig. 5). Define θ_s, θ_i', and θ_s' by

$$r_1=\bar{\omega}\sin\theta_s, \quad n_e\sin\theta_i'=\sin\theta_i, \quad n_e\sin\theta_s'=\sin\theta_s. \quad (40)$$

It is evident from (30) and (40) that the scattered light corresponding to $l=1$ will emerge at the angle θ_s. Substituting our solutions for q_{0ms} and q_{1ms} to zeroth order in ϵ_1 into (23), remembering the definition (29), and using (40), we obtain

$$g_{0m+}=-h_{0m-}=\bar{\omega}(n_e\cos\theta_i'-\cos\theta_i), \quad (41a)$$

$$g_{0m-}=-h_{0m-}=-\bar{\omega}(n_e\cos\theta_i'+\cos\theta_i), \quad (41b)$$

$$g_{1m+}=-h_{1m-}=\bar{\omega}(n_e\cos\theta_s'-\cos\theta_s), \quad (41c)$$

$$g_{1m-}=-h_{1m+}=-\bar{\omega}(n_e\cos\theta_s'+\cos\theta_s). \quad (41d)$$

It is convenient to introduce the usual E polarization (electric vector perpendicular to the plane containing the incident and reflected wave vectors) reflection

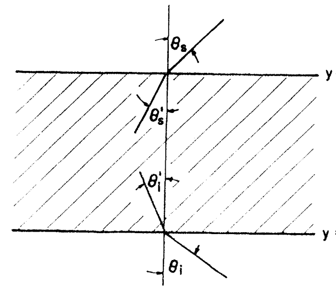

FIG. 5. Angles of incidence and scattering.

coefficients for a refractive index n.[13] These are

$$R_i^E(n)=(n\cos\theta_i'-\cos\theta_i)(n\cos\theta_i'+\cos\theta_i)^{-1}, \quad |\theta_i|\leq\tfrac{1}{2}\pi \quad (42a)$$

$$R_s^E(n)=(n\cos\theta_s'-\cos\theta_s)(n\cos\theta_s'+\cos\theta_s)^{-1}, \quad |\theta_s|\leq\tfrac{1}{2}\pi. \quad (42b)$$

For use in later examples, we also define the H polarization (electric vector in the plane containing the incident and reflected wave vectors) reflection coefficients

$$R_i^H(n)=R_i^E(n)\cos(\theta_i+\theta_i')/\cos(\theta_i-\theta_i'), \quad |\theta_i|\leq\tfrac{1}{2}\pi \quad (42c)$$

$$R_s^H(n)=R_s^E(n)\cos(\theta_s+\theta_s')/\cos(\theta_s-\theta_s'), \quad |\theta_s|\leq\tfrac{1}{2}\pi. \quad (42d)$$

With the aid of (39), (41), and (42) we may express (32a) and (32b) in the form

$$R_i^E(n_e)V_{10+}e^{iq_{00}d}-V_{10-}e^{-iq_{00}d}-R_i^E(n_e)V_{11+}e^{iq_{01}d}+V_{11-}e^{-q_{01}d}=0, \quad (43a)$$

$$R_s^E(n_e)V_{10+}e^{iq_{10}d}-V_{10-}e^{-iq_{10}d}+R_s^E(n_e)V_{11+}e^{iq_{11}d}-V_{11-}e^{-iq_{11}d}=0, \quad (43b)$$

$$V_{10+}-R_i^E(n_e)V_{10-}-V_{11+}+R_i^E(n_e)V_{11-}=b, \quad (43c)$$

$$V_{10+}-R_s^E(n_e)V_{10-}+V_{11+}-R_s^E(n_e)V_{11-}=0, \quad (43d)$$

where

$$b=2(K_0/K_1)^{1/2}(n_e^2-1)^{-1}\cos\theta_i(n_e\cos\theta_i'-\cos\theta_i)B. \quad (44)$$

The $l=1$ scattered electric field becomes

$$(\mathbf{E}_s)^1=2(n_e^2-1)\cos\theta_s(n_e\cos\theta_s'-\cos\theta_s)^{-1}\mathbf{W}, \quad (45)$$

where

$$\mathbf{W}=\mathbf{V}_{10+}e^{iq_{10}d}-R_s^E(n_e)\mathbf{V}_{10-}e^{-iq_{10}d}+\mathbf{V}_{11+}e^{iq_{11}d}-R_s^E(n_e)\mathbf{V}_{11-}e^{-iq_{11}d}. \quad (46)$$

The ratio of the scattered light intensity I_1, to the incident light intensity $I_{\rm inc}$ is

$$I_1/I_{\rm inc}=(n_e^2-1)^2[2\cos\theta_s(n_e\cos\theta_s'-\cos\theta_s)]^{-2}\times\mathbf{W}\cdot\mathbf{W}^*(\mathbf{B}\cdot\mathbf{B}^*)^{-1}. \quad (47)$$

[13] See, for example, Born and Wolf, Ref. 7.

Solving (43) and evaluating $\mathbf{W}\cdot\mathbf{W}^*$, we obtain

$$\mathbf{W}\cdot\mathbf{W}^*=\{1-[R_s^E(n_e)]^2\}^2b^2\times\sin^2\alpha d\ F[R_i^E(n_e),R_s^E(n_e)],\tag{48}$$

where

$$F[\lambda_1,\lambda_2]=[1+2\lambda_1\lambda_2\cos(2K_0d)+\lambda_1^2\lambda_2^2]\{[1-2\lambda_1^2\cos(2K_0d)+\lambda_1^4][1-2\lambda_2^2\cos(2K_1d)+\lambda_2^4]$$
$$+(\lambda_1+\lambda_2)^2\sin^4\alpha d[\lambda_1^2+2\lambda_1\lambda_2\cos(2K_0d-2K_1d)+\lambda_2^2]$$
$$-2(\lambda_1+\lambda_2)\sin^2\alpha d[(\lambda_1^3+\lambda_2^3)-\lambda_1(1+\lambda_1\lambda_2^3)\cos(2K_0d)$$
$$-\lambda_2(1+\lambda_1^3\lambda_2)\cos(2K_1d)+\lambda_1\lambda_2(\lambda_1+\lambda_2)\cos(2K_0d-2K_1d)]\}^{-1}.\tag{49}$$

Finally, for the case $\beta=0$, the ratio of the emergent intensity in the $l=1$ mode to the incident light intensity becomes

$$I_1/I_{\rm inc}=\{1-[R_s^E(n_e)]^2\}^2\sin^2\alpha d(K_0/K_1)(\cos\theta_i/\cos\theta_s)^2$$
$$\times[(n_e\cos\theta_i'-\cos\theta_i)/(n_e\cos\theta_s'-\cos\theta_s')]^2F[R_i^E(n_e),R_s^E(n_e)].\tag{50}$$

This intensity formula and the ones to be derived subsequently are used as follows: Starting with the θ_i and θ_s for which the intensity is to be calculated, obtain θ_i' and θ_s' from Snell's law. Evaluate α or α', K_0, K_1, and the appropriate reflection coefficients. Substitute these quantities into (50). Bhatia and Noble[4] calculated $I_1/I_{\rm inc}$ for the case $\beta=0$, $R_i^E(n_e)=0$ and $R_s^E(n_e)=0$ in a nonbirefringent medium. Our result, since for this case there is no effect due to birefringence, reduces to theirs if the limits $R_i^E(n_e)\to 0$, $R_s^E(n_e)\to 0$ are taken. The factor $\{1-[R_s^E(n_e)]^2\}^2$ reduces the emergent intensity to zero if the geometry is such that the scattered light is totally reflected internally. The parameters of the acoustic wave enter I_1 through the quantity α, which is of the form $\bar{\omega}ps$, where ps is some combination of photoelastic coefficients and strain amplitudes. If the sample width d, the strains, and the photoelastic coefficients are such that αd is small, the scattered intensity is proportional to $(psd)^2$. It is possible to measure αd directly as the phase retardation induced by the acoustic wave in the emergent $l=0$ light. A typical value for αd found by Zucker and Zemon[2] in such an experiment on CdS is 0.4 rad. This was measured for light passing through an acoustic domain, or region of high acoustic strain, in a sample with $d=1$ mm. Unfortunately, the relevant photoelastic coefficients are not known.

A calculation similar to that above shows that the intensity of the unscattered ($l=0$) light is modulated by a factor $\cos^2\alpha d$. The maxima of I_1 at $\alpha d=\pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$, etc. are thus accompanied by complete depletion of the incident beam, and the incident light is completely converted to scattered light.

Due to internal reflection, some light corresponding to both the $l=0$ and $l=1$ lines emerges at the $y=0$ crystal face (see Fig. 4). The intensity of this light may also be calculated by the methods of this paper.

If $\beta\neq 0$, the Bragg conditions are not satisfied for the reflected light, i.e., for $s=-1$. A somewhat lengthy extension of the above calculation shows that (50) holds if $F[\lambda_1,\lambda_2]$ is replaced by $G[\lambda_1,\lambda_2]$, where

$$G[\lambda_1,\lambda_2]=\{1-2\lambda_1^2\lambda_2^2\cos^2(K_0d+K_1d)+\lambda_1^4\lambda_2^4$$
$$-2(1-\lambda_1^2\lambda_2^2)[\lambda_1^2\cos(2K_0d)+\lambda_2^2\cos(2K_1d)]$$
$$+\cos^2(\alpha d)[\lambda_1^2+\lambda_2^2+2\lambda_1^2\lambda_2^2\cos(2K_0d-2K_1d)]\}^{-1}.\tag{51}$$

Figure 6 illustrates the variation of the scattered intensity described by Eqs. (50) and (51) with acoustic frequency and propagation direction in the small strain limit ($\alpha d\ll 1$). In this limit we may write $I_1/I_{\rm inc}=(\gamma d)^2Z$, where $\gamma=\frac{1}{2}\epsilon_1\bar{\omega}/n_e$. It will be recalled that ϵ_1 is linear in strain amplitudes and photoelastic coefficients. The variable Z, which carries the dependence on acoustic frequency, has been evaluated numerically using the parameters appropriate to the experiment of Zucker and Zemon[2] in CdS. The light wavelength in air is taken to be 6328 Å, the speeds of the transverse and longitudinal sound waves are 1.80×10^5 cm/sec and 4.22×10^5 cm/sec, respectively, n_0 is 2.453, n_e is 2.470, and d is 0.10 cm. Each graph of Z versus acoustic frequency is plotted for the range of frequencies for

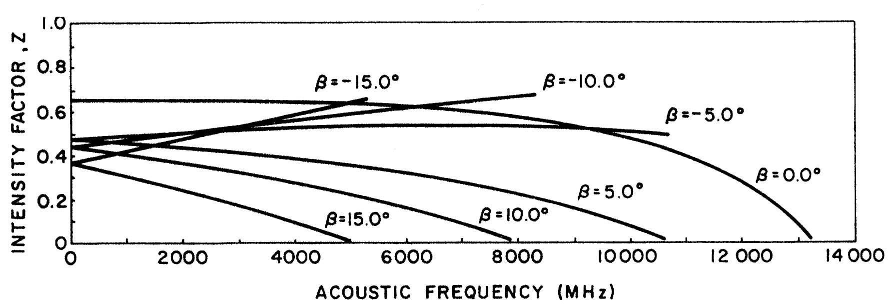

FIG. 6. Acoustic frequency dependence of scattered intensity. Small strain limit. Case I.

which both reflection coefficients are less than unity. A similar plot for the theory of Bhatia and Noble, which ignores reflection and is restricted to acoustic frequencies low enough so that $\theta_i \approx 0$ and $\theta_s \approx 0$, is a horizontal line at $Z=1.0$. The geometrical structure of (51) is such that the curves for β negative lie above the corresponding curves for β positive.

Case II

In this case, the longitudinal wave is replaced by a shear wave propagating at an angle β to the x axis in the basal plane, as shown in Fig. 4, and polarized along the optic axis. The incident light is H polarized with its electric field in the plane of incidence, i.e., the basal plane. The tensor $\boldsymbol{\varepsilon}_0$ is unchanged in this case, but according to (11b) and (14) $\boldsymbol{\varepsilon}_1$ becomes $\epsilon_1(\hat{k}\mathbf{Q}+\mathbf{Q}\hat{k})/Q$, where $Q=|\mathbf{Q}|$ and $\epsilon_1=-n_0^2 n_e^2 p_{44} QA_3$. The customary abbreviation $p_{44}=p_{2323}$ and the fact that $p_{1313}=p_{2323}$ have been used in deriving the above form for ϵ_1. We again assume that the modified Bragg conditions are met for $l=1$. Because of the birefringence, it is possible in this case to satisfy these conditions for both $l=-1$ and $l=1$ for a particular (nonzero) acoustic frequency. For the present, we consider the case in which this does not happen. The case in which it does happen will be discussed at the end of the section.

As in the theory of Bhatia and Noble, the plane of polarization is unchanged upon refraction into the crystal, so that $\hat{k}\cdot\mathbf{V}_{0ms}=0$. This may be understood as the consequence of reflection symmetry in the basal plane. It also emerges from a longer version of the calculation to follow. As in case I, we retain $l=0$ and $l=1$ terms only in (27), obtaining

$$\bar{\omega}^2(\boldsymbol{\varepsilon}_0\cdot\mathbf{V}_{0ms}+\tfrac{1}{2}\boldsymbol{\varepsilon}_1\cdot\mathbf{V}_{1ms}) = (\mathbf{K}_{0ms}{}^2\mathbf{I}-\mathbf{K}_{0ms}\mathbf{K}_{0ms})\cdot\mathbf{V}_{0ms}, \quad (52a)$$

$$\bar{\omega}^2(\boldsymbol{\varepsilon}_0\cdot\mathbf{V}_{1ms}+\tfrac{1}{2}\boldsymbol{\varepsilon}_1\cdot\mathbf{V}_{0ms}) = (\mathbf{K}_{1ms}{}^2\mathbf{I}-\mathbf{K}_{1ms}\mathbf{K}_{ims})\cdot\mathbf{V}_{1ms}. \quad (52b)$$

Forming the scalar products of $\hat{k}$ with (52a) and $\mathbf{K}_{1ms}$ with (52b), we obtain $\mathbf{Q}\cdot\mathbf{V}_{1ms}=\mathbf{K}_{1ms}\cdot\mathbf{V}_{1ms}=0$. Since $\mathbf{Q}$ is not parallel to $\mathbf{K}_{1ms}$, and since $\mathbf{Q}$ and $\mathbf{K}_{1ms}$ lie in the basal plane, the scattered amplitudes $\mathbf{V}_{1ms}$ are perpendicular to the basal plane. Thus the light polarization is rotated 90° upon scattering by the transverse wave. This is a result that was derived earlier by Mueller[3] for isotropic materials. Forming the scalar product of $\mathbf{K}_{0ms}$ with (52a), we obtain

$$n_0^2\mathbf{K}_{0ms}\cdot\mathbf{V}_{0ms}+\tfrac{1}{2}Q^{-1}\epsilon_1\mathbf{Q}\cdot\mathbf{K}_{0ms}\hat{k}\cdot\mathbf{V}_{1ms}=0. \quad (53)$$

The electric field within the crystal is not transverse because of the fluctuations in the dielectric tensor. Forming the scalar product of $\mathbf{Q}$ with (52a) and using (53), we obtain

$$(\mathbf{K}_{0ms}{}^2-n_0^2\bar{\omega}^2)\mathbf{Q}\cdot\mathbf{V}_{0ms} -\tfrac{1}{2}\epsilon_1\bar{\omega}^2[Q-(n_0^2\bar{\omega}^2 Q)^{-1}(\mathbf{Q}\cdot\mathbf{K}_{0ms})^2]\hat{k}\cdot\mathbf{V}_{1ms}=0, \quad (54a)$$

while the scalar product of $\hat{k}$ with (52b) becomes

$$\tfrac{1}{2}\epsilon_1\bar{\omega}^2 Q^{-1}\mathbf{Q}\cdot\mathbf{V}_{0ms}-(\mathbf{K}_{1ms}{}^2-n_e^2\bar{\omega}^2)\hat{k}\cdot\mathbf{V}_{1ms}=0. \quad (54b)$$

Equations (54a) and (54b) comprise a homogeneous set of linear equations for $\mathbf{Q}\cdot\mathbf{V}_{1ms}$ and $\hat{k}\cdot\mathbf{V}_{1ms}$. A nontrivial solution exists only if the determinant of the coefficients vanishes, i.e., if

$$(\mathbf{K}_{0ms}{}^2-n_0^2\bar{\omega}^2)(\mathbf{K}_{1ms}{}^2-n_e^2\bar{\omega}^2) -\tfrac{1}{4}\epsilon_1^2\bar{\omega}^4[1-(n_0\bar{\omega}Q)^{-2}(\mathbf{Q}\cdot\mathbf{K}_{0ms})^2]=0. \quad (55)$$

As in case I, we treat first the $\beta=0$ situation, where the modified Bragg conditions are satisfied for the reflected light. We have $\mathbf{Q}\cdot\mathbf{K}_{0ms}=Qr_0$, where r_0 is again equal to $\bar{\omega}\sin\theta_i$. In this case, the angles θ_s, θ_i', and θ_s' are defined by

$$r_1=\bar{\omega}\sin\theta_s, \quad n_0\sin\theta_i'=\sin\theta_i, \quad n_e\sin\theta_s'=\sin\theta_s. \quad (56)$$

Equation (55) reduces to

$$(q_{0ms}{}^2-K_0^2)(q_{1ms}{}^2-K_1^2)=\tfrac{1}{4}\epsilon_1^2\bar{\omega}^4\cos^2\theta_i', \quad (57)$$

where

$$K_0=(n_0^2\bar{\omega}^2-r_0^2)^{1/2}, \quad K_1=(n_e^2\bar{\omega}^2-r_1^2)^{1/2}. \quad (58)$$

Solving (57) as in case I, we obtain

$$q_{00s}=s(K_0+\tfrac{1}{2}\alpha), \quad (59a)$$

$$q_{01s}=s(K_0-\tfrac{1}{2}\alpha), \quad (59b)$$

$$q_{10s}=s(K_1+\tfrac{1}{2}\alpha), \quad (59c)$$

$$q_{11s}=s(K_1-\tfrac{1}{2}\alpha), \quad (59d)$$

where

$$\alpha=\tfrac{1}{2}\epsilon_1\bar{\omega}^2(K_0K_1)^{-1/2}\cos\theta_i'. \quad (60)$$

Substituting (59) and (60) into (54a), we derive

$$(-1)^m(K_0/K_1)^{1/2}\mathbf{Q}\cdot\mathbf{V}_{0ms}=Q\cos\theta_i'\hat{k}\cdot\mathbf{V}_{1ms}. \quad (61)$$

Using (23), (29), and (56) to evaluate g_{lms} and h_{lms}, we find that (41c) and (41d) are unchanged, while (41a) and (41b) are replaced by

$$g_{0m+}=-h_{0m-}=\bar{\omega}(n_0\cos\theta_i'-\cos\theta_i), \quad (62a)$$

$$g_{0m-}=-h_{0m+}=-\bar{\omega}(n_0\cos\theta_i'+\cos\theta_i). \quad (62b)$$

Equations (56), (58), (61), and (62) allow us to rewrite (26a) and (26b) as

$$R_i^H(n_0)\hat{k}\cdot\mathbf{V}_{10+}e^{iq_{00}d}-\hat{k}\cdot\mathbf{V}_{10-}e^{-iq_{00}d} -R_i^H(n_0)\hat{k}\cdot\mathbf{V}_{11+}e^{iq_{01}d}+\hat{k}\cdot\mathbf{V}_{11-}e^{-iq_{01}d}=0, \quad (63a)$$

$$R_s^E(n_e)\hat{k}\cdot\mathbf{V}_{10+}e^{iq_{10}d}-\hat{k}\cdot\mathbf{V}_{10-}e^{-iq_{10}d} +R_s^E(n_e)\hat{k}\cdot\mathbf{V}_{11+}e^{iq_{11}d}-\hat{k}\cdot\mathbf{V}_{11-}e^{-iq_{11}d}=0, \quad (63b)$$

$$\hat{k}\cdot\mathbf{V}_{10+}-R_i^H(n_0)\hat{k}\cdot\mathbf{V}_{10-}-\hat{k}\cdot\mathbf{V}_{11+} +R_i^H(n_0)\hat{k}\cdot\mathbf{V}_{11-}=b', \quad (63c)$$

$$\hat{k}\cdot\mathbf{V}_{10+}-R_s^E(n_e)\hat{k}\cdot\mathbf{V}_{10-}+\hat{k}\cdot\mathbf{V}_{11+} -R_s^E(n_e)\hat{k}\cdot\mathbf{V}_{11-}=0, \quad (63d)$$

where

$$b'=2\hat{\imath}\cdot\mathbf{B}(n_0\cos\theta_i'-\cos\theta_i)(K_0/K_1)^{1/2} \times[\cos(\theta_i-\theta_i')(n_0^2-1)]^{-1}. \quad (64)$$

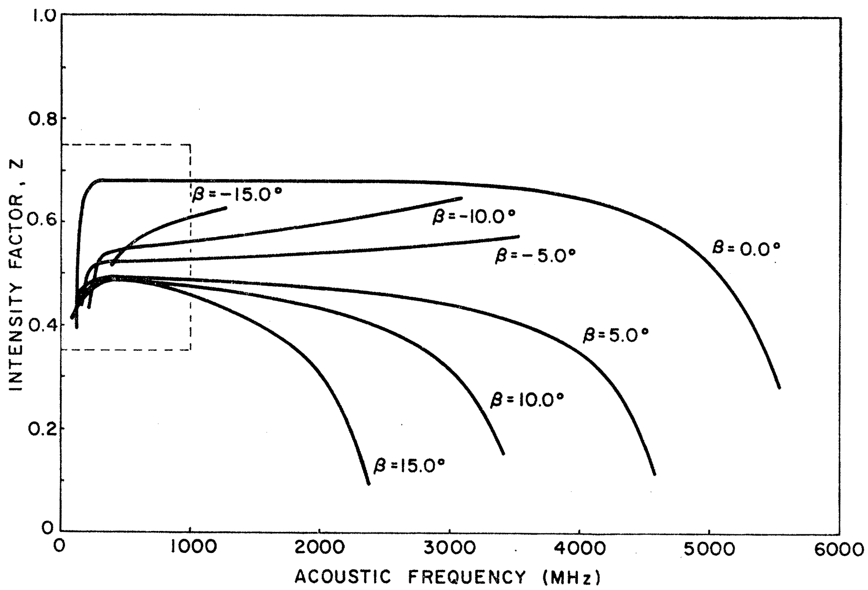

FIG. 7. Acoustic frequency dependence of scattered intensity. Small strain limit. Case II.

Solving (63) and evaluating I_1/I_{inc}, we obtain

$$I_1/I_{\text{inc}}=\{1-[R_s^E(n_e)]^2\}^2[(n_e^2-1)/(n_0^2-1)]^2\sin^2\alpha d \times(K_0/K_1)[(n_0\cos\theta_i'-\cos\theta_i)/(n_e\cos\theta_s'-\cos\theta_s)]^2 \times\cos^2\theta_i[\cos\theta_s\cos(\theta_i-\theta_i')]^{-2}F[R_i^H(n_0),R_i^E(n_e)], \tag{65}$$

where F was defined by (49).

If $\beta\neq 0$, an extension of the above calculation shows that

$$I_1/I_{\text{inc}}=\{1-[R_s^E(n_e)]^2\}^2[(n_e^2-1)/(n_0^2-1)]^2\sin^2\alpha' d \times(K_0/K_1)[(n_0\cos\theta_i'-\cos\theta_i)/(n_e\cos\theta_s'-\cos\theta_s)]^2 \times\cos^2\theta_i[\cos\theta_s\cos(\theta_i-\theta_i')]^{-2}G[R_i^H(n_0),R_i^E(n_e)], \tag{66}$$

where

$$\alpha'=\tfrac{1}{2}\epsilon_1\bar{\omega}^2(K_0K_1)^{-1/2}\cos(\theta_i'-\beta) \tag{67}$$

and G was defined by (51).

Figure 7 is a plot of the intensity factor Z for case II. Compared with those of Fig. 6, these curves are compressed horizontally. This is due to the lower speed of transverse sound, since the acoustic wave vector rather than the frequency determines the scattered intensity. The low-frequency region enclosed by the dashed lines is expanded in Fig. 8. This region exhibits more structure than the corresponding region of Fig. 7. The rapid drop in intensity as the frequency is decreased is due to birefringence. At low-acoustic frequencies in birefringent media θ_i and θ_s and the reflection coefficients grow large.

The Bragg conditions for the $l=1$ and $l=-1$ lines are met simultaneously if $\sin(\theta_i'-\beta)=0$. If this occurs, $V_{-1,ms}$ may not be neglected. When the appropriate scattering angles are used, $I_{\pm 1}/I_{\text{inc}}$ is given by (65) for $\beta=0$ if a factor of $\frac{1}{2}$ is introduced on the right-hand side and α is replaced by $\alpha/\sqrt{2}$. The $l=1$ and $l=-1$ lines have equal intensity and appear symmetrically about the unscattered beam. For $\beta\neq 0$, $I_{\pm 1}/I_{\text{inc}}$ is given by (66) if a factor of $\frac{1}{2}$ is introduced and α' is replaced by $\alpha'/\sqrt{2}$.

Since $n_0\approx n_e$ in most materials, the factor $[(n_e^2-1)/(n_0^2-1)]$ is approximately equal to unity. The birefringence influences the scattered intensity through the geometrical factors, since, as seen earlier, θ_i, θ_i', θ_s, and θ_s' are quite different in birefringent and isotropic media. The factor $\cos(\theta_i'-\beta)$ in α' arises because of the H polarization of the incident light. The scattered light is E polarized, and the refractive index for E polarization in this case is n_e, so that $R_s^E(n_e)$ is the appropriate

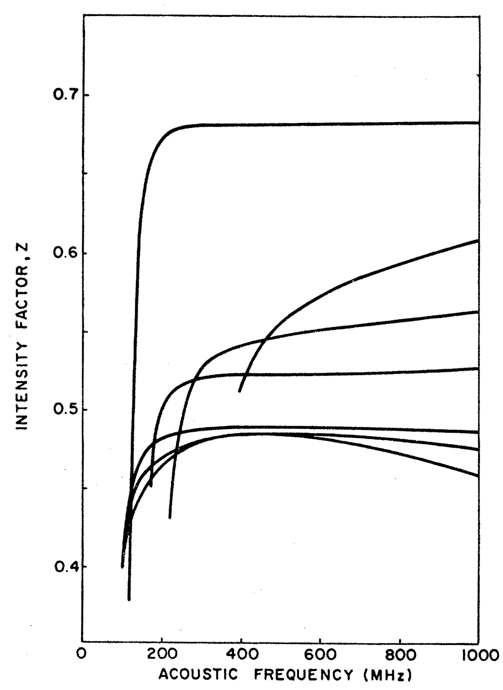

FIG. 8. Low-frequency section of Fig. 7 expanded.

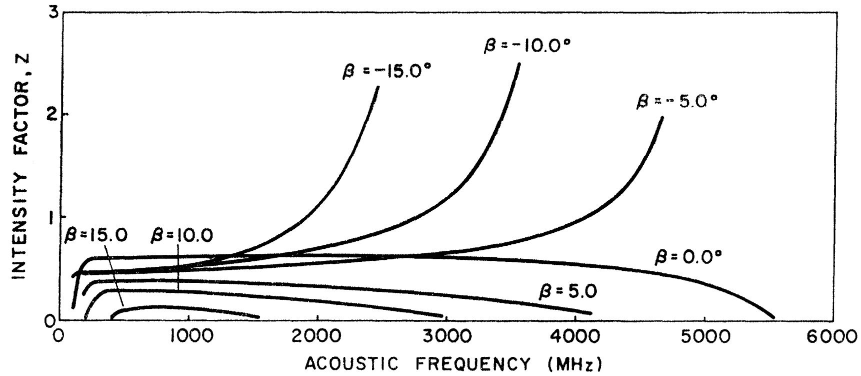

FIG. 9. Acoustic frequency dependence of scattered intensity. Small strain limit. Case III.

reflection coefficient, as in case I. The incident light, however, is H polarized and its refractive index is n_0, so that $R_i^H(n_0)$ must be used.

Case III

The H polarization of case II is changed to E polarization in case III, so that the incident electric field lies along the optic axis. An analysis similar to that of case II shows that the scattered light is again polarized at right angles to the incident light. If $\beta=0$, the intensity ratio is

$$I_1/I_{\rm inc}=\{1-[R_s^H(n_0)]^2\}^2[(n_0^2-1)/(n_e^2-1)]^2\sin^2\alpha d \times(K_0/K_1)[(n_e\cos\theta_i'-\cos\theta_i)/(n_0\cos\theta_s'-\cos\theta_s)]^2 \times[\cos\theta_i\cos(\theta_s-\theta_s')]^2[\cos\theta_s]^{-2}F[R_i^E(n_e),R_s^H(n_0)], \tag{68}$$

where

$$\alpha=\tfrac{1}{2}\epsilon_1\bar{\omega}^2(K_0K_1)^{-1/2}\cos\theta_s', \tag{69}$$

$$K_0=(n_e^2\bar{\omega}^2-r_0^2)^{1/2},\quad K_1=(n_0^2\bar{\omega}^2-r_1^2)^{1/2}. \tag{70}$$

For $\beta\neq0$,

$$I_1/I_{\rm inc}=\{1-[R_s^H(n_0)]^2\}^2[(n_0^2-1)/(n_e^2-1)]^2\sin^2\alpha d \times(K_0/K_1)[(n_e\cos\theta_i'-\cos\theta_i)/(n_0\cos\theta_s'-\cos\theta_s)]^2 \times[\cos\theta_i\cos(\theta_s-\theta_s')]^2(\cos\theta_s)^{-2}G[R_i^E(n_e),R_s^H(n_0)], \tag{71}$$

where

$$\alpha'=\tfrac{1}{2}\epsilon_1\bar{\omega}^2(K_0K_1)^{-1/2}\cos(\theta_s'-\beta).$$

In case III, the incident light is E polarized, and has refractive index n_e, while the scattered light is H polarized, and has refractive index n_0. Therefore, $R_i^H(n_0)$ and $R_s^E(n_0)$ of case II become $R_i^E(n_e)$ and $R_s^H(n_0)$ in case III. The factor $\cos(\theta_i'-\beta)$ in (67) for α' of case II, having its origin in the H polarization of the incident light in that case, becomes $\cos(\theta_s'-\beta)$ here because the scattered light is H polarized. Figure 9 is a plot of the intensity factor Z for case III. These curves are quite similar to those of case II.

Case IV

In case IV, the acoustic shear wave is polarized along the x axis and propagates along the optic axis. The incident light propagates in the yz plane and has H polarization. There is again a polarization flip of 90° upon scattering. The intensity is given by (65) if the appropriate changes in refractive index are made, i.e.,

$$n_0\rightarrow(\cos^2\theta_i'/n_e^2+\sin^2\theta_i'/n_0^2)^{-1/2},$$
$$n_e\rightarrow n_0.$$

No other change is necessary.

Case V

The incident H polarization of case IV is changed to E polarization in case V. The intensity is given by (68) if

$$n_e\rightarrow(\cos^2\theta_s'/n_0^2+\sin^2\theta_s'/n_e^2)^{-1/2},$$
$$n_0\rightarrow n_e.$$

The polarization flip also occurs in this case.

V. SOME COMMENTS

In the cases we have treated, scattering by transverse waves has been characterized by a 90° rotation of polarization, while no polarization rotation occurred upon scattering by a longitudinal wave. It appears that these results are due to the special geometries we have assumed. In the general case, rotations other than 0° or 90° are to be expected.

The intensity formulas we have derived for the different cases differ in geometrical structure. The differences may not be ignored, but they are not extensive enough to support the contention that the mechanisms of scattering by transverse and longitudinal waves differ fundamentally.[14]

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[14] H. Kuppers, Acustica 16, 365 (1966); W. T. Maloney and H. R. Carleton, Trans. Sonics Ultrasonics 14, 135 (1967).