This momentum matrix element, which is independent of $i$, is a known function of the band masses.[14]

The final result for the absorption coefficient is

$$A(\omega)=\frac{e^2(2m_c)^{3/2}}{9m^3E_g{}^2\pi c}|\langle 0c|i\nabla|0v\rangle|^2$$

$$\times\omega\theta(\omega-E_g)\sum_{i=1,2}\int_{-\omega}^{-E_g}dE\,(E+\omega)^{1/2}$$

$$\times\mathrm{Im}\left[\frac{1}{E+E_g+(E+\omega)m_c/m_{vi}+\Sigma_{vi}(E)}\right]. \quad (33)$$

The following numerical values are used in the determination of this integral:

$$E_g=0.89\text{ eV},\quad m_c=0.041m;$$
$$m_{v1}=0.045m,\quad m_{v2}=0.35m.$$

The absorption coefficient is shown in Fig. 2. The result shown in Fig. 3 is the absorption coefficient in a semiconductor with a large electron–optical-phonon interaction (10 times that in germanium). As we see from Figs. 2 and 3, the absorption coefficient is, on the whole, reduced by the presence of the interaction, and the maximum effect occurs in the neighborhood of the photon energies given by

$$\omega-E_g=(1+m_{vi}/m_c)\omega_0.$$

These values are just the minimum photon energies which are required for the physical process in which a valence electron is scattered by the emission of a phonon and is then raised into the conduction band by the absorption of a photon. The rapid variation near these values is caused by the fact that the self-energy has infinite slope at the value corresponding to one phonon emission.

The two values used for the electron-phonon coupling constant $g$ probably encompass the whole range of values in semiconductors. Thus we see that in semiconductors in the strong coupling range the absorption coefficient is drastically affected by the interaction, and even in weakly coupled germanium the effect is certainly not negligible.

[14] M. Cardona, J. Phys. Chem. Solids **24**, 1543 (1963).

PHYSICAL REVIEW VOLUME 166, NUMBER 3 15 FEBRUARY 1968

# Statistics of 1/*f* Noise

JAMES J. BROPHY*
*Illinois Institute of Technology, Chicago, Illinois*
(Received 23 August 1967)

Probability amplitude distributions of $1/f$ noise and Nyquist noise in a 8 Hz-to-10 kHz band are examined using a multichannel pulse-height analyzer. Both types of noise obey a normal distribution law in all sample lengths investigated, 100, 10, and 1.0 sec. In the case of $1/f$ noise, the variance of the distribution itself fluctuates in identical sample intervals. The distribution of variances appears to be from an exponential population. The difference between Nyquist-noise and $1/f$-noise data suggests that the latter is not stationary in the statistical sense, but rather possesses a weaker form of stationarity.

## I. INTRODUCTION

A LARGE number of experimental measurements of $1/f$ noise have shown that the spectral power density varies approximately inversely with frequency over a wide frequency interval.[1] In particular, a limit at low frequencies has not been observed[2] even down to frequencies of the order of $10^{-4}$ Hz. If it is inferred that the spectral power density remains inversely proportional to frequency down to zero frequency, then the total noise power becomes infinite.

It has been noted[3] that this inference is appropriate only for stationary random processes and that $1/f$ noise may not be stationary in the usual sense but rather may possess a weaker form of conditional stationarity.[4] That is, the process may be a "noisy noise" in which the probability amplitude distribution, say, in a given sample interval differs from that in another interval.

There appears to be only one previous attempt to examine the statistical properties of $1/f$ noise, apart from determination of the conventional power-density spectrum.[5] The probability amplitude was shown to be

* Supported by the U. S. Office of Naval Research.

[1] See, for example, A. van der Ziel, *Fluctuation Phenomena in Semiconductors* (Academic Press Inc., New York, 1959), Chap. 5.

[2] B. V. Rollin and I. M. Templeton, Proc. Phys. Soc. (London) **B66**, 259 (1953).

[3] B. Mandelbrot, in Fluctuations in Solids Symposium, University of Minnesota, June 1966 (unpublished).

[4] J. M. Berger and B. Mandelbrot, IBM J. Res. Develop. **7**, 224 (1963).

[5] D. A. Bell, Proc. Phys. Soc. (London) **B68**, 690 (1955).

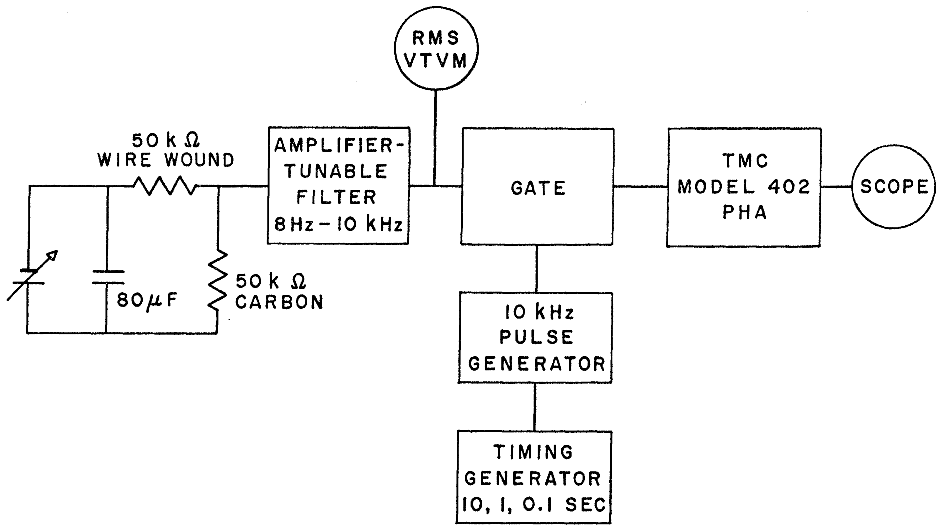


FIG. 1. Simplified block diagram of apparatus to measure probability amplitude distribution of Nyquist and $1/f$ noise.

normally distributed, as in the case of white Nyquist noise. The measurement technique employed, however, was such that statistical properties of the kind implied by a process obeying conditional stationarity would most likely not be detected. The purpose of the present study is to examine the statistical properties of $1/f$ noise in greater detail with the view of developing experimental evidence for or against a "noisy noise" process.

## II. EXPERIMENTAL TECHNIQUE

The current noise of a 50 000-Ω carbon resistor is amplified and examined with an analog-to-digital converter and a conventional multichannel pulse-height analyzer, as shown in the block diagram of Fig. 1. The resistor noise source was selected because the current-noise level is quite stable and is known to have a $1/f$ spectrum. Furthermore, the statistical properties of Nyquist noise can also be examined by simply reducing the dc current to zero. The 80-μF capacitor in the input circuit is part of a more elaborate battery filter which is not shown.

The amplifier-tunable filter in Fig. 1 is a conventional low-noise system assembled from commercial components. In conjunction with the rms vacuum-tube voltmeter (VTVM), it is possible to measure both the $1/f$-noise and Nyquist-noise spectra in standard fashion. This is useful to check on the stability of the current-noise level, to provide a convenient calibration of amplifier gain, and to yield conventional power spectra for comparison with the probability amplitude distribution data.

The analog-to-digital converter is a specially constructed transistor gate triggered by 1.2-μsec pulses from a 10-kHz multivibrator. This pulse generator is in turn activated by a timing generator with selectable interval lengths of 10, 1.0, and 0.10 sec. When the timing generator is activated manually 10-kHz pulses are fed to the gate for one sample interval of either 10, 1.0, or 0.10 sec duration. The gate thus samples the noise signal at the 10-kHz rate for this period. Output pulses from the gate are registered on a conventional TMC Model 402 multichannel pulse-height analyzer. Integral with the gate is a dc bias offset voltage so that only negative output pulses are produced even though the input noise signal has signal components of both polarities.

The pulse-amplitude distribution recorded for one sample interval by the pulse-height analyzer is displayed on an auxiliary oscilloscope and recorded photographically. For many measurements, it is sufficient to determine the width of the distribution at a given height and in this case it is more convenient to measure the width directly on the face of the oscilloscope screen. As the experiments proceeded, it became clear that sample lengths of 100, 10, and 1.0 sec are more appropriate. The 100-sec intervals are produced by activating the 10-sec timing generator ten times in succession.

An over-all voltage calibration and linearity of the system is obtained by introducing square waves of known amplitude at the amplifier input terminals. The probability amplitude distribution accuracy is checked by similarly introducing sine wave signals. Finally, operation with random-noise signals is examined by measuring the variance of the observed probability amplitude distribution in the case of Nyquist noise and comparing with the calculated value given by the Nyquist theorem. Satisfactory quantitative results are obtained, as discussed below.

## III. PROBABILITY AMPLITUDE DISTRIBUTIONS

Conventional spectra of current noise and Nyquist noise are shown in Fig. 2. The current-noise spectrum varies as $f^{-1.02}$, which is rather typical for carbon resistors.[6] The observed Nyquist-noise level is in good agreement with that predicted by the Nyquist theorem for a 25-kΩ resistor. This is expected from the input circuit of Fig. 1 since the two 50-kΩ resistors are effectively in parallel. The amplifier-noise spectrum, also shown in Fig. 2, is sufficiently below the resistor-noise level that its influence is negligible.

Typical photographs of probability amplitude distributions observed in 100-, 10-, and 1.0-sec intervals sampled are given in Fig. 3. The results for the longest interval sampled are most accurate since statistical variations are smaller when the total number of pulses is as great as $10^6$. Statistical fluctuations, in fact, make sampling intervals shorter than 1.0 sec unusable since the sampling rate is limited to 10 kHz by the characteristics of the pulse-height analyzer. Despite inaccuracies introduced by statistical fluctuations, probability amplitude distributions observed in all sample intervals appear to obey a normal distribution law.

This is best illustrated by scaling the photographs and replotting the data. Special graph paper is used such that a normal distribution law results in two straight lines, as in Fig. 4. Here, both Nyquist-noise and $1/f$-noise probability amplitude distributions for a 1.0-sec sample interval are compared with a normal distribution. The data points for both types of noise follow the normal law, within the accuracy of the measurements. To prepare the experimental data for this presentation the voltage amplitude $v$ is obtained from the over-all system calibration, and the standard deviation $\sigma$ is determined from the width of the experimental distribution at a relative probability amplitude equal to 0.368.

Similar data for 10- and 100-sec sample intervals are shown in Figs. 5 and 6, respectively. In each case the probability amplitudes are normally distributed. Some departures from the normal law are noticed at small probability amplitudes caused, apparently, by photographic distortions. Significantly, the departures in the case of $1/f$ noise are similar to those of the Nyquist-noise data. Since Nyquist noise is expected to be normally distributed, this is further evidence that $1/f$ noise also follows the normal distribution law in all three sample lengths.

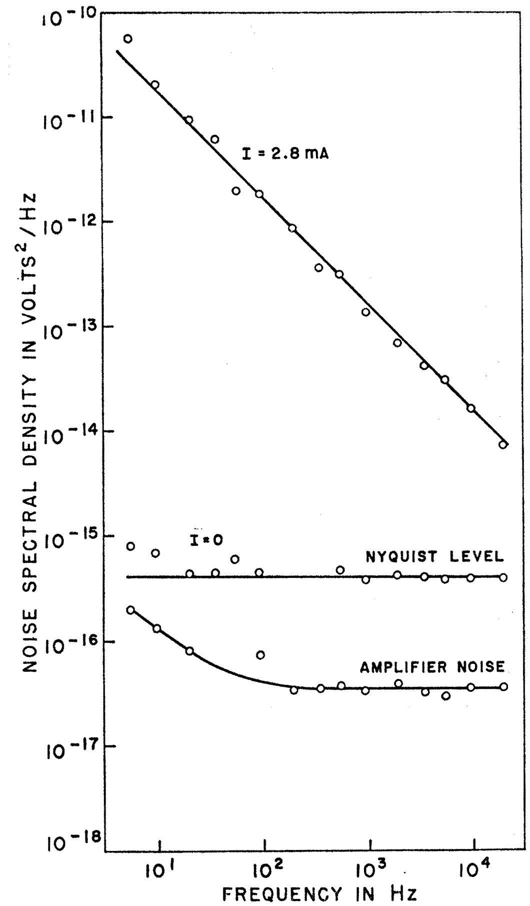


FIG. 2. Experimental spectra of Nyquist noise and $1/f$ noise of a carbon resistor.

## IV. VARIANCES

During preparation of the above data fairly significant variations in the variance, $s=\sigma^2$, were observed between supposedly similar sample intervals. Accordingly, approximately 100 variances were determined for each sample length and for both types of noise. Results in the case of Nyquist noise are shown as histograms in Fig. 7. The spread is greatest in the case of the data for a 1-sec sample interval, suggesting the presence of statistical fluctuations even with $10^4$ data points per distribution. The 10- and 100-sec data probably represent both statistical fluctuations and random experimental errors in the apparatus. In this connection, it should be noted that the data were taken over several days and in random sequence.

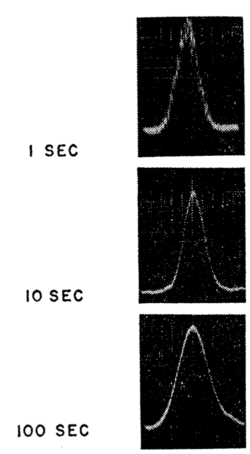


FIG. 3. Probability amplitude distributions of $1/f$ noise of a carbon resistor in a 8 Hz-to-10 kHz band for sampling intervals of 1, 10, and 100 sec.

[6] I. M. Templeton and D. K. C. MacDonald, Proc. Phys. Soc. (London) **B66**, 680 (1953).

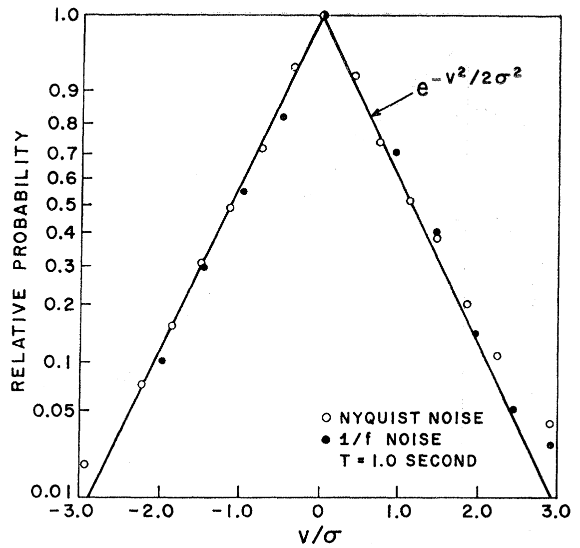


FIG. 4. Experimental probability amplitude distribution of Nyquist noise and $1/f$ noise for a 1.0-sec sample length compared to a normal distribution.

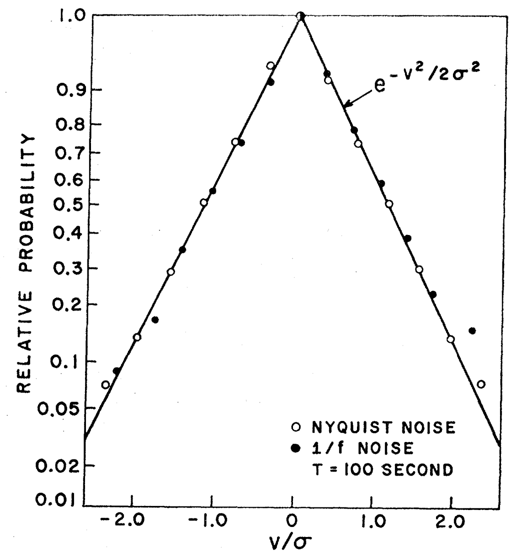


FIG. 6. Experimental probability amplitude distribution of Nyquist noise and $1/f$ noise for a 100-sec sample length compared to a normal distribution.

The magnitude of the variance is compared with that expected from the Nyquist theorem,

$$s = 4kTRB, \tag{1}$$

where $k$ is Boltzmann's constant, $T$ is the absolute temperature, $R$ is the resistance, and $B$ is the bandwidth of the amplifier. The agreement between Eq. (1) and experimental results from the probability amplitude distribution data is quite satisfactory in the case of the 10- and 100-sec sample lengths.

Although 100 samples is rather small for good statistical significance, a summary of the statistical properties of these data is given in Table I. Of particular interest is the ratio of the square of the average variance to the "variance of the variance" in column 4. This ratio increases with the length of the sample interval, indicative of improved statistical accuracy in the longer samples. The ratio is also useful in comparison with the $1/f$-noise data below. The skewness and kurtosis of the 100-sec data are relatively small, suggesting an approach to a normal distribution (in which case both the skewness and kurtosis are zero). Actually, if the two largest data points in the 10-sec data are eliminated, the skewness and kurtosis are equally small, 0.17 and

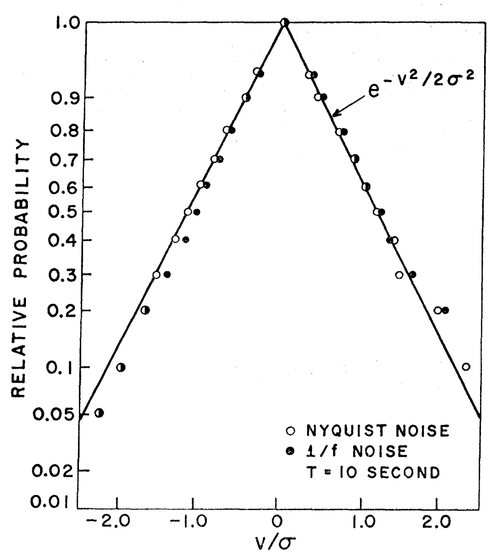


FIG. 5. Experimental probability amplitude distribution of Nyquist noise and $1/f$ noise for a 10-sec sample length compared to a normal distribution.

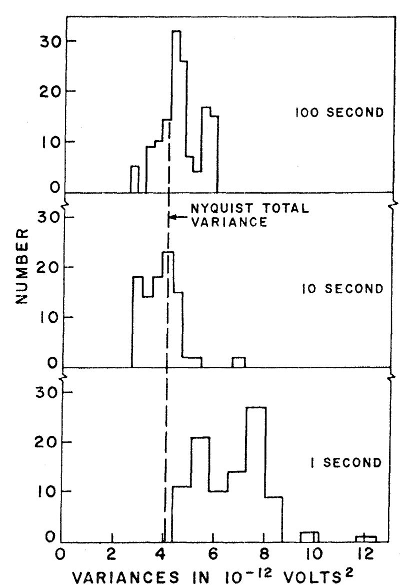


FIG. 7. Histograms of variances, $s$, observed for 1-, 10-, and 100-sec intervals of Nyquist noise in a 8 Hz-to-10 kHz band.

TABLE I. Statistical properties of Nyquist-noise measurements.

| $T$ (sec) | Mean variance $\langle s\rangle$ ($V^2$) | Variance of variance $\langle s^2\rangle$ ($V^4$) | $\langle s\rangle^2/\langle s^2\rangle$ | Skewness | Kurtosis |
|---|---|---|---|---|---|
| 1.0 | $6.7\times10^{-12}$ | $1.8\times10^{-24}$ | 25 | 0.74 | 1.8 |
| 10 | $3.9\times10^{-12}$ | $6.1\times10^{-25}$ | 25 | 1.3 | 5.4 |
| 100 | $4.5\times10^{-12}$ | $6.1\times10^{-25}$ | 33 | −0.78 | −0.34 |

0.36, respectively. This illustrates the statistical fluctuations inherent in 100-sample data.

Based on the data in Fig. 7, it is concluded that random statistical variations account for some of the spread in observed variances in the 1-sec sample data; the spread in the 10- and 100-sec data results primarily from experimental random errors; and the variances observed in all three sample lengths are normally distributed. Good agreement is observed between the average variance and that calculated from spectrum measurements using Eq. (1), at least for the 10- and 100-sec samples.

Equivalent histograms in the case of $1/f$ noise are shown in Fig. 8. Here the histograms are obviously skewed and the asymmetry appears to increase for longer sample intervals. Again reasonable agreement is observed with the total variance calculated from the spectral density

$$s=\int_{f_1}^{f_2}(K/f)df, \qquad (2)$$

where, according to Fig. 2, $K=1.6\times10^{-10}$ $V^2$ Hz, $f_1=8$ Hz, and $f_2=10$ kHz.

The statistical properties of these data are given in Table II. Note that the ratio of the mean square to the variance is an order of magnitude smaller than that corresponding to Nyquist noise. This means that $1/f$-noise variances have a much larger spread than the Nyquist variances. In addition, if the spread in the Nyquist variances is attributed to instrumental errors (for the longer samples), the smaller ratio in the case of $1/f$ noise clearly means that the observed spread is a characteristic of the noise process rather than the experimental technique. Thus, it appears that $1/f$-noise is "noisier" than Nyquist noise.

The skewness and kurtosis of the 100-sec data suggest that the variances are derived from an exponential

TABLE II. Statistical properties of $1/f$-noise measurements.

| $T$ (sec) | Mean variance $\langle s\rangle$ ($V^2$) | Variance of variance $\langle s^2\rangle$ ($V^4$) | $\langle s\rangle^2/\langle s^2\rangle$ | Skewness | Kurtosis |
|---|---|---|---|---|---|
| 1.0 | $3.0\times10^{-9}$ | $1.4\times10^{-18}$ | 6.4 | 0.91 | 1.3 |
| 10 | $2.6\times10^{-9}$ | $1.9\times10^{-18}$ | 4.5 | 0.64 | −0.26 |
| 100 | $2.0\times10^{-9}$ | $1.6\times10^{-18}$ | 2.6 | 2.4 | 6.3 |

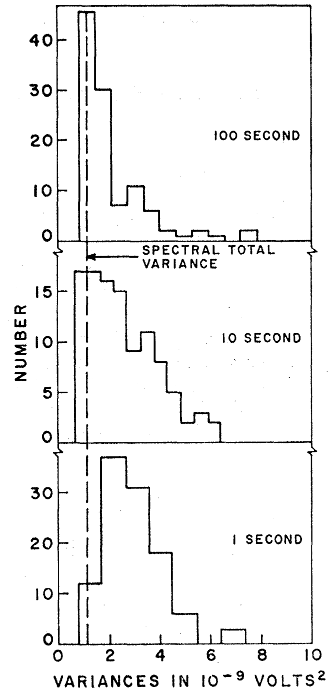


FIG. 8. Histograms of variances, $s$, observed for 1-, 10-, and 100-sec intervals of $1/f$ noise in a 8 Hz–to–10 kHz band.

population (for which the skewness is 2 and the kurtosis is 6). This is in distinct contrast to the Nyquist-noise results, which are normally distributed. The histograms for the 1- and 10-sec sample lengths in Fig. 8 suggest a trend from a normal distribution to an exponential distribution with increasing sample time.

## V. CONCLUSIONS

These results illustrate for the first time a significant statistical difference between $1/f$ noise and Nyquist noise. Although the probability amplitudes of both types of noise are normally distributed, the variances of the distributions themselves fluctuate in the case of $1/f$ noise. It appears that the variances are derived from an exponential population and that the phenomenon is best developed in the longer samples.

On the other hand, the over-all stability of the $1/f$-noise level indicates that the process is stable in some sense. That is, these results demonstrate experimentally that $1/f$ noise is "noisy," yet possesses some form of conditional stationarity. It remains to establish a more quantitative relation between the experimental data and the statistics of such processes.

## ACKNOWLEDGMENTS

The author is extremely grateful to S. L. Webb for great perseverance in acquiring the experimental data, and to E. O. Tums for his design of the analog-to-digital converter.

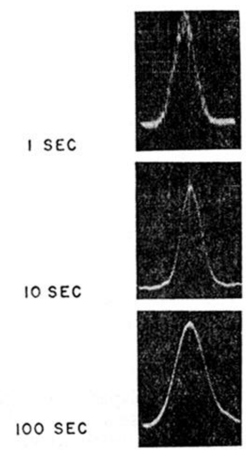


FIG. 3. Probability amplitude distributions of $1/f$ noise of a carbon resistor in a 8 Hz–to–10 kHz band for sampling intervals of 1, 10, and 100 sec.