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Flux Quantization and Dimensionality*

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A system of charged particles in a ring of radius R is considered, assuming separability of the tangential component of the total momentum. The dependence of the free energy upon the magnetic flux through the ring is shown to be strongly affected by the dimensionality. Irrespective of the specific dynamical properties of the system, one finds in the one-dimensional case that a realistic choice of R leads to the exclusion of thermodynamically stable flux trapping. The entirely different criterion for the three-dimensional case is separately discussed and is seen to be closely related to the mean-square fluctuation of the total momentum.

I. INTRODUCTION

The preceding paper by Schick deals with a one-dimensional system of interacting fermions under conditions where Tomonaga's treatment of collective modes of excitation is applicable. The evaluation of the free energy in the presence of a vector potential shows that such a system in a ring of macroscopic radius R does not exhibit thermodynamically stable flux trapping. It is the purpose of this paper to demonstrate that the basic arguments of this conclusion have a considerably more general validity than Tomonaga's method and to further clarify the essential differences between systems of one and more dimensions.

Based upon a simple relation between the energy and the total momentum of a system of particles it will be seen in particular that stable flux trapping is excluded beyond a characteristic radius R^*, which is independent of specific properties of the system. The order of magnitude of R^* is strongly affected, however, by the dimensionality. For a one-dimensional system it is found to be so small as to exclude stable flux trapping for a realistic value of R. Among other special features, the problem of long-range order in a one-dimensional system of fermions is thus not pertinent to the purposes of this paper. This conclusion does not apply to a three-dimensional system where R^* is typically found to be so large that it far exceeds any realistic radius of the ring. Stable flux trapping hinges here upon criteria which involve the different long-range characteristics of the normal and the superconductive state of a metal. It will be shown that these criteria can be reduced to refer to the mean-square fluctuation of the total momentum and that it is of decisive importance in this context whether or not the equipartition theorem is applicable.

II. EIGENVALUES OF ENERGY AND MOMENTUM

Consider a system of N particles with mass m and charge e, contained in a circular ring with inner radius R and radial width $d \ll R$. In addition to any other coordinates such as y, z, and a spin variable, a particle shall be described by its x coordinate, measured along the ring, so that all dependences on x have to be periodic with period $2\pi R$. Because of the magnetic field, caused by a current around the ring, the vector potential A points in the x direction and is independent of x. It will be assumed that the dominant contribution of the magnetic flux Φ through the ring arises from its interior so that $A = \Phi/2\pi R$ can be considered to remain constant within the ring.

The contribution to the kinetic energy of the particles due to their motion in the x direction is then given by

$$T_x = (1/2m) \sum_k (p_k - eA/c)^2, \tag{1}$$

where p_k represents the momentum conjugate to the x coordinate x_k of the kth particle ($k=1, 2, \cdots, N$). Denoting by

$$P_x = \sum_k p_k \tag{2}$$

the total momentum in the x direction, one has

$$p_k = P_x/N + p_k', \tag{3}$$

where p_k' depends only upon the relative momenta and, from Eq. (1),

$$T_x = (P_x - NeA/c)^2/(2Nm) + \sum_k p_k'^2. \tag{4}$$

The dependence upon P_x is of particular importance and it is indicated, therefore, to separate the first term on the right side of Eq. (4) from the remaining contribution of T_x to the total energy. The total Hamiltonian of the particles will thus be written in the form

$$\mathcal{H} = (P_x - NeA/c)^2/(2Nm) + \mathcal{H}', \tag{5}$$

where $\mathcal{H}'$ contains the kinetic energy of the motion in the y and z direction and of the relative motion in the x direction as well as any additional terms which arise from interactions and characterize the specific dynamical properties of the system.

As the essential feature of $\mathcal{H}'$ it will be assumed that it is independent of P_x and of the conjugate coordinate

$$X = (\sum_k x_k)/N \tag{6}$$

of the center of gravity. This assumption is evidently justified for the part of $\mathcal{H}'$ contributed by the kinetic

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energy, and it is also justified for the additional part if the underlying interactions are invariant against a common translation and a common change of velocity in the x direction, applied to all the particles. This invariance is violated inasfar as the lattice provides a fixed frame of reference and causes modifications through the presence of a periodic potential and of interactions between particles which are mediated by lattice deformations. As in the usual theory of metallic conduction, however, one can in essence account for these modifications by interpreting the mass m in Eq. (5) as an effective mass.

The relation between $\mathcal{H}$ and P_x can thus be considered to simply express the separability of the motion of the center of gravity. In order to clarify some important further implications, a set of relative coordinates, to be denoted by ξ, such as

$$\xi_l = x_1 - x_{l+1}, \qquad l = 1, 2, \cdots, N-1 \tag{7}$$

shall be introduced besides the coordinate X of the center of gravity. An eigenstate of the system with given eigenvalue P of P_x can then be represented by the wave function

$$\psi_{P,q}(X,\xi) = \exp(iPX/\hbar)\,\psi_q'(\xi), \tag{8}$$

where q represents the system of additional quantum numbers necessary in addition to P in order to fully characterize the state of the system, and where of all the arguments required by a full coordinate representation only X and ξ have been explicitly retained. The corresponding eigenvalue of $\mathcal{H}$ is then given by

$$E_{P,q} = (P - NeA/c)^2/(2mN) + E_q', \tag{9}$$

where E_q' is an eigenvalue of $\mathcal{H}'$ such that

$$\mathcal{H}'\psi_q' = E_q'\psi_q'. \tag{10}$$

It is necessary, at this point, to emphasize a difference from the familiar case of separability which is essential for the following discussion. Although the operator P_x does not occur in $\mathcal{H}'$ and its eigenvalue P has therefore been omitted in the characterization of ψ' and E', a dependence of these quantities on P has to be foreseen in view of the periodicity of the wave function around the ring. This periodicity has to be satisfied for every one of the particles of the system so that it requires the invariance of ψ under the transformation $x_k \to x_k + 2\pi R$ for each of the N possible values of k individually as well as for any number of them.

Considering first the special case where this transformation is applied to all values of k simultaneously, it is seen from Eqs. (6) and (7) that it corresponds to the transformation $X \to X + 2\pi R$, while all relative coordinates ξ remain unchanged. According to Eq. (8) the peridicity of ψ therefore merely demands in this case that the eigenvalues P have to be of the form

$$P = n\hbar/R, \tag{11}$$

with integer n.

A more interesting result is obtained if one chooses, for example, the transformation $x_2 \to x_2 + 2\pi R$, leaving all coordinates x_k for $k \neq 2$ unchanged. According to Eqs. (6) and (7), this corresponds to the transformation $X \to X + 2\pi R/N$, $\xi_1 \to \xi_1 - 2\pi R$, while all relative coordinates ξ_l for $l \neq 1$ remain unchanged. The invariance of ψ under this transformation requires in view of Eq. (8)

$$\exp[iP(X + 2\pi R/N)/\hbar]\,\psi_q'(\xi_1 - 2\pi R) = \exp(iPX/\hbar)\,\psi_q'(\xi_1)$$

or

$$\psi_q'(\xi_1 + 2\pi R) = \exp[2\pi i PR/(N\hbar)]\,\psi_q'(\xi_1). \tag{12}$$

Similarly, the individual increase of $x_3, x_4, \cdots, x_N$ by $2\pi R$ leads to the result

$$\psi_q'(\xi_l + 2\pi R) = \exp[2\pi i PR/(N\hbar)]\,\psi_q'(\xi_l), \tag{13}$$

obtained by replacing in Eq. (12) ξ_1 by $\xi_2, \xi_3, \cdots, \xi_{N-1}$ and, hence, valid for all values of l. This result represents a set of cyclical conditions to be imposed upon the solutions ψ_q' of Eq. (10) so that these solutions as well as the corresponding eigenvalues E_q' depend generally on P.

No such dependence appears in the familiar separation of the center of gravity since the conditions of Eq. (13) are here normally replaced by the boundary condition that, irrespective of the total momentum, the wave function must vanish for infinitely large values of the relative coordinates. The same conclusion would, of course, be reached if the system of particles, considered here, were confined to a region within the ring, small compared to the radius R, so that the conditions of Eq. (13) would be fulfilled as a trivial consequence of the statement that any multiple of a solution of Eq. (10) is likewise a solution. The relevance of these conditions arises from the fact that the system may coherently extend over the whole circumference of the ring and thus exhibit a long-range order which is well known to be essential for the explanation of flux quantization.[1] Another formulation can be obtained from the representation in which the momenta p_k of the individual particles are diagonal by means of the equivalent condition that the eigenvalues of each of them are integer multiples of $\hbar/R$. The coordinate representation, however, is more convenient for the purposes of this paper and the further discussion is based upon the corresponding equations (11) and (13).

Whereas the conditions expressed in Eq. (13) imply a general dependence of ψ_q' and E_q' upon P, it is to be noted that these conditions do not alter their form if P

[1] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).

is increased by an integer multiple of $N\hbar/R$, thus leaving ψ_q' and E_q' likewise unaltered. It is therefore indicated to write the integer n of Eq. (11) in the form

$$n=N\nu+\mu, \tag{14}$$

where ν is an arbitrary integer and where, assuming N to be even, μ is likewise an integer such that

$$-(N/2)<\mu\leq(N/2), \tag{15}$$

so that μ can assume N different values. One thus obtains from Eq. (11)

$$P=(N\nu+\mu)\hbar/R, \tag{16}$$

and from Eq. (13)

$$\psi_q'(\xi_l+2\pi R)=\exp(2\pi i\mu/N)\,\psi_q'(\xi_l). \tag{17}$$

The dependence of the quantities ψ_q' and E_q' upon P thus refers only to the part of P in Eq. (16) which is proportional to the integer μ. To emphasize this circumstance, the previous notation will be replaced by $\psi_{q,\mu}'$ and $E_{q,\mu}'$, where μ can assume any one of the N integer values in the range indicated by Eq. (15), and where q indicates all quantum numbers other than ν and μ. The total energy $E_{P,q}$ of the system, on the other hand, depends generally on ν as well as on q and μ, with the former dependence arising from the first part on the right side of Eq. (9). This equation shall therefore be rewritten in the form

$$E_{\nu,\mu,q}=\hbar^2N(\nu-\alpha+\mu/N)^2/(2mR^2)+E_{q,\mu}', \tag{18}$$

where Eq. (16) has been used and where

$$\alpha=2\pi RA(e/hc) \tag{19}$$

represents the flux through the ring in units of hc/e. The specific properties of the system affect only the part $E_{q,\mu}'$ of the total energy. The bearing of this circumstance upon the partition function and, hence, upon the free energy will be discussed in the following section.

III. PARTITION FUNCTION AND FREE ENERGY

Using Eq. (18), the partition function is given by

$$Z=\sum_\mu\sum_\nu\exp[-N\gamma(\nu-\alpha+\mu/N)^2]\sum_q\exp(-\beta E'_{q,\mu}), \tag{20}$$

where

$$\beta=1/kT, \tag{21}$$

$$\gamma=\hbar^2\beta/(2mR^2), \tag{22}$$

and where the summation over ν extends over all integers, while the summation over μ refers to the range of integers, limited by Eq. (15). It is convenient to introduce the notations

$$Z'=\sum_{q,\mu}\exp(-\beta E'_{q,\mu}) \tag{23}$$

and

$$z_\mu=[\sum_q\exp(-\beta E'_{q,\mu})]/Z', \tag{24}$$

noting that

$$z_\mu\geq 0 \tag{25}$$

and that

$$\sum_\mu z_\mu=1. \tag{26}$$

With the further notation

$$Z_1(\alpha)=\sum_\mu z_\mu\sum_\nu\exp[-N\gamma(\nu-\alpha+\mu/N)^2], \tag{27}$$

one has then from Eq. (20)

$$Z=Z'Z_1(\alpha), \tag{28}$$

where the factor $Z_1(\alpha)$ determines the flux dependence of the partition function. Since the specific properties of the system of particles affect only the values of $E_{q,\mu}'$, the same holds for Z' and z_μ.

The free energy $-kT\ln Z$ is thus given by the sum $F'+F_1(\alpha)$, where F' is independent of α and where

$$F_1(\alpha)=-kT\ln Z_1(\alpha). \tag{29}$$

In order to obtain the flux-dependent part of the total free energy one has to add to the contribution F_1 from the particles the energy

$$F_2=\hbar^2\alpha^2/2e^2L \tag{30}$$

stored in the magnetic field with the flux $\Phi=\alpha(hc/e)$, where L represents the self-inductance of the ring. Thermodynamically stable flux trapping is therefore determined by those values of α for which the sum

$$F(\alpha)=F_1(\alpha)+F_2(\alpha) \tag{31}$$

has a minimum. Whereas several such values exist for a superconductor with, usually, a large number of them extending over a wide range it is typical for the normal state of a metal that the only minimum of $F(\alpha)$ occurs for $\alpha=0$.

Except for the magnitude of the purely geometrical self-inductance L, this criterion is based upon the properties of $F_1(\alpha)$. In particular, stable flux trapping requires a sufficiently strong variation of $F_1(\alpha)$ or through Eq. (29) of $Z_1(\alpha)$, defined in Eq. (27), in order to prevent the dominance of the part $F_2(\alpha)$ in Eq. (31) from resulting in no other stability than that of vanishing flux.

The discussion of these properties is facilitated by applying the Poisson formula

$$\sum_{n=-\infty}^{+\infty} \exp[-\kappa(n-x)^2] = (\pi/\kappa)^{1/2} \sum_{n=-\infty}^{+\infty} \exp[-(\pi n)^2/\kappa+2\pi inx] \quad (32)$$

to the sum over ν in Eq. (27), with the result

$$Z_1(\alpha)=(\pi/N\gamma)^{1/2} \sum_{n=-\infty}^{+\infty} a_n \exp[-(\pi n)^2/(N\gamma)+2\pi in\alpha], \quad (33)$$

where

$$a_n=\sum_\mu z_\mu \exp(-2\pi in\mu/N). \quad (34)$$

One obtains from Eq. (26)

$$a_0=1, \quad (35)$$

and, in combination with Eq. (25), the more general relation

$$|a_n|^2 \leq 1. \quad (36)$$

It is evident from Eq. (33) that $Z_1(\alpha)$ is a periodic function of α with period unity and hence a periodic function of the flux with period hc/e. In addition it follows from the defining Eq. (27) that $Z_1(\alpha)$ is real and from symmetry reasons that it has to be an even function of α. The coefficients a_n must therefore be real numbers with $a_{-n}=a_n$ so that from Eqs. (33) and (35)

$$Z_1(\alpha)=(\pi/N\gamma)^{1/2}\{1+2\sum_{n=1}^{\infty} a_n \times\exp[-(\pi n)^2/N\gamma]\cos 2\pi n\alpha\}. \quad (37)$$

While the expressions for $Z_1(\alpha)$, obtained above, are quite general, specific properties of the system have to be introduced in order to obtain more detailed information. This information is contained in the quantities z_μ and hence in the coefficients a_n, defined by Eqs. (24) and (34), respectively, and some special cases of particular interest will be first considered.

IV. SPECIAL CASES

A. z_μ Independent of μ

This case arises if the eigenvalues E_q' of the internal energy, obtained from Eq. (10), do not depend upon the total momentum P so that the values $E_{q,\mu}'$ in Eq. (24) are independent of μ. As discussed in Sec. II this property, encountered in the familiar separation of the center of gravity, corresponds to the case where the system exhibits no long-range order, although it extends over the whole ring and the resulting independence of z_μ on μ may be considered, in this sense, as being characteristic of the normal state of a metal.

One has then from Eq. (26)

$$z_\mu=1/N, \quad (38)$$

and from Eq. (34)

$$a_n=(1/N)\sum_\mu \exp(-2\pi in\mu/N). \quad (39)$$

Since according to Eq. (15) the summation over μ extends over N consecutive integers, it follows that $a_n=1$ if n is an integer multiple of N of the form $n=Ng$ and that $a_n=0$ otherwise, with the result

$$Z_1(\alpha)=(\pi/N\gamma)^{1/2} \times\{1+2\sum_{g=1}^{\infty} \exp[-(\pi g)^2N/\gamma]\cos 2\pi gN\alpha\}, \quad (40)$$

obtained from Eq. (37). It is to be noted that the general periodicity in α with period unity is here accompanied by the far shorter period $1/N$ considering the large number N of particles in a macroscopic system. If any flux quantization could occur under these circumstances it would imply the existence of the practically vanishing "fluxquantum" hc/Ne.

Such a strange occurrence, however, would require totally unrealistic conditions. Indeed, one has from Eqs. (21) and (22)

$$N/\gamma=2mR^2NkT/\hbar^2, \quad (41)$$

so that at any realistic temperature of a macroscopic system this number in the exponents of Eq. (40) is very large and results through Eq. (29) in a variation of $F_1(\alpha)$, which is far too small to compete in Eq. (31) with that of $F_2(\alpha)$ and, hence, to permit the stable trapping of any finite flux. It thus follows that the case considered here leads under realistic conditions to the characteristic absence of stable flux trapping in the normal state.

B. $z_\mu=0$ for $\mu\neq 0$

This case represents the extreme opposite of the preceding case A. It would arise, for example, if the system had the strict long-range order of a completely condensed ideal Bose gas with the property that all particles are found to have the same momentum p. In fact, with the total momentum $P=Np$ and with p chosen as an integer multiple of $\hbar/R$, Eq. (16) can be satisfied only for $\mu=0$.

In view of Eq. (26) one then has

$$z_\mu=\delta_{\mu,0}, \quad (42)$$

and therefore from Eq. (34)

$$a_n=1 \quad (43)$$

for all values of n so that one obtains from Eq. (37)

$$Z_1(\alpha)=(\pi/N\gamma)^{1/2}\{1+2\sum_{g=1}^{\infty}\exp[-(\pi g)^2/N\gamma]\cos 2\pi g\alpha\} \tag{44}$$

with g instead of n used to denote the summation index.

In contrast to Eq. (40), found in the preceding case, the periodicity in α with unit period is here not accompanied by any shorter period and implies the flux quantum hc/e. Stable flux trapping requires further that $1/N\gamma$ is not too large. This requirement will be further discussed in Sec. V, but it should be remarked that, under otherwise equal conditions, one deals here with a characteristic magnitude which is N^2 times smaller than that given in Eq. (41) of the preceding case. It will be seen, in fact, that compared to Eq. (40) the order of magnitude of the exponentials for macroscopic systems is so radically changed as to greatly favor the conditions for stable flux trapping.

C. $z_\mu=0$ for $\mu\neq 0$ and $\mu\neq N/2$

In analogy to the previous example, this case arises if the system with an even number N of particles exhibits the property of $N/2$ pairs which have all the same momentum p. With the total momentum $P=Np/2$ of these pairs it is permitted, in fact, to choose in Eq. (16) either $\mu=0$ or $\mu=N/2$, noting that both choices are compatible with Eq. (15) and that p has to be an integer multiple of $\hbar/R$. This property is typical of the long-range order which can be established in a system of fermions since a common momentum of the individual particle is here ruled out by the exclusion principle.[1] Although still extreme and in some respects similar to the preceding case, the case considered here is thus far more closely related to the actual conditions of a superconductor.

Assigning the same value of z_μ to both permissible choices $\mu=0$ and $\mu=N/2$, one obtains from Eq. (26)

$$z_\mu=\tfrac{1}{2}(\delta_{\mu,0}+\delta_{\mu,N/2}), \tag{45}$$

and hence from Eq. (34)

$$a_n=\tfrac{1}{2}[1+\exp(-i\pi n)], \tag{46}$$

so that $a_n=0$ if n is odd and $a_n=1$ if n is even. Writing for the even values $n=2g$, it thus follows from Eq. (37) that

$$Z_1(\alpha)=(\pi/N\gamma)^{1/2}\times\{1+2\sum_{g=1}^{\infty}\exp[-(2\pi g)^2/N\gamma]\cos 4\pi g\alpha\}. \tag{47}$$

The properties of this expression are similar to those mentioned in the discussion of Eq. (44), with the principal difference that the periodicity in α with unit period is here accompanied by the period $\tfrac{1}{2}$ corresponding to a reduction of the flux quantum to the observed value $hc/2e$.

Instead of assuming $z_\mu\neq 0$ only for one or two values of μ as in the preceding cases B or C, respectively, the results obtained could be readily generalized by admitting a larger number of equally spaced integers μ, compatible with Eq. (15), and assigning the same value of z_μ to each of them. This would be suggested by the property of conglomerates of a correspondingly larger number of particles with the same momentum of each single conglomerate. A further reduction of the flux quantum, inversely proportional to the number of admitted integers μ, would be implied by this more general assumption and the case A can be considered as the limit which comprises all N possible values of μ.

V. CONCLUSIONS FOR $N\gamma\ll 1$ AND $N\gamma\gg 1$

The special cases of the preceding section illustrate some typical features of the partition function in its dependence upon the flux through the ring. Going back to Eq. (37) it is possible, however, to arrive at considerably more general conclusions which are primarily based upon the order of magnitude of the quantity

$$N\gamma=N\hbar^2/2mR^2kT \tag{48}$$

obtained from Eqs. (21) and (22).

An important result is directly obtained for $N\gamma\ll 1$. It is essential, for this purpose, to notice that the coefficients a_n are generally limited by Eq. (36). Irrespective of any specific properties of the system, the sum over n in Eq. (37) thus becomes negligibly small compared to the constant term due to the rapid decrease of the exponentials with decreasing $N\gamma$. In view of Eq. (29), an equally rapid decrease appears in $F_1(\alpha)$, so that the variation of $F_2(\alpha)$, given by Eq. (30), becomes easily the dominant part of $F(\alpha)$ for any conceivable magnitude of the self-inductance L. It is safe, therefore, to conclude quite generally that the condition $N\gamma\ll 1$ is sufficient to reach this point and, hence, to exclude stable flux trapping.

Turning now to the case $N\gamma\gg 1$, it is no longer sufficient to consider the general limitation of the coefficients a_n. Depending upon a more detailed information about these coefficients in accordance with the underlying properties of the system, the function $Z_1(\alpha)$ can here be found to have widely different features, ranging from a negligible to the most pronounced type of variation.

As an example of the first type, it is seen in the special case A of Sec. IV that the coefficients a_n in Eq. (37) vanish except for very large values of n so that the remaining exponentials can still be neglibly small even though $N\gamma\gg 1$. This is also shown in Eq. (40) since the much weaker simultaneous condition $\gamma/N\ll 1$ is sufficient to result in a negligible variation of $Z_1(\alpha)$. A most pronounced variation, on the other hand, is exemplified in the case B, where according to Eq. (43) all coefficients a_n have their maximum value. The result for $Z_1(\alpha)$ in Eq. (44) shows in fact that for $N\gamma \gg 1$ the exponentials do not appreciably decrease up to high terms in the sum over g. In order to discuss the actual dependence of $Z_1(\alpha)$ and, hence, of $F_1(\alpha)$, it is more convenient in this case to go back to the expression of Eq. (27) with z_μ given by Eq. (42). One thus obtains

$$Z_1(\alpha)=\sum_\nu \exp[-N\gamma(\nu-\alpha)^2]. \qquad (49)$$

For $N\gamma \gg 1$ and $|\alpha|<\frac{1}{2}$ the main contribution to the sum arises from $\nu=0$ so that one has here

$$Z_1(\alpha)\cong \exp(-N\gamma\alpha^2), \qquad (50)$$

and from Eqs. (29) and (48)

$$F_1(\alpha)\cong(N\hbar^2/2mR^2)\alpha^2. \qquad (51)$$

This value of $F_1(\alpha)$ can be interpreted as the total kinetic energy which would be obtained if all N particles moved with the same velocity $\hbar\alpha/mR$ and it can well dominate the part $F_2(\alpha)$ in Eq. (31). While such a large value of $F_1(\alpha)$ merely enhances the minimum of $F(\alpha)$ at $\alpha=0$, it has to be noted, in view of Eq. (49), that the same values of $Z_1(\alpha)$ and $F_1(\alpha)$ as those given in Eq. (50) and (51), respectively, are obtained if α is changed by an integer. Other pronounced minima of $F(\alpha)$ thus occur at integer values of α and account in this case for stable quantized flux trapping.

The preceding discussion of examples for the case $N\gamma\gg 1$ deals with two extreme opposites of a more general property. This is best explained by considering first the situation in the absence of a vector potential. Considering the sum over ν in Eq. (27) it is seen for $\alpha=0$ and with $N\gamma\gg1$ that all terms with $\nu\neq0$ are negligible since according to Eq. (15) $|\mu/N|\leq\frac{1}{2}$, so that

$$Z_1(0)=\sum_\mu z_\mu \exp[-N\gamma(\mu/N)^2]. \qquad (52)$$

The quantity

$$w_\mu=z_\mu \exp[-N\gamma(\mu/N)^2]/\{\sum_\mu z_\mu \exp[-N\gamma(\mu/N)^2]\} \qquad (53)$$

is to be interpreted as the probability that one finds a specific value of μ or, with $\nu=0$ in Eq. (16), that the momentum of the system in the x direction has the value $P=\hbar\mu/R$. In view of the sharp maximum at $\mu=0$ of the exponential in Eq. (53) one can normally expect that w_μ is negligibly small unless $|\mu|\ll N$. Writing

$$z_\mu=\exp[N\zeta(\mu/N)],$$

one obtains for $|\mu/N|\ll1$ by an expansion

$$\zeta(\mu/N)=\zeta(0)-\delta(\mu/N)^2, \qquad (54)$$

since the linear term has to vanish for symmetry reasons and hence

$$z_\mu=C\exp[-N\delta(\mu/N)^2]. \qquad (55)$$

The probability w_μ is thus seen from Eq. (53) to be proportional to $\exp[-N(\gamma+\delta)(\mu/N)^2]$ and one has

$$(\gamma+\delta)/N=1/(2\langle\mu^2\rangle_{\rm av}), \qquad (56)$$

where $\langle\mu^2\rangle_{\rm av}$ represents the mean square of μ and

$$\langle P^2\rangle_{\rm av}=\hbar^2\langle\mu^2\rangle_{\rm av}/R^2 \qquad (57)$$

the mean square of P in the absence of a vector potential. It will be shown below that the value of $\langle P^2\rangle_{\rm av}$ may be regarded as the decisive quantity required to characterize the properties of the system.

For this purpose one has to assume that $N\delta\gg1$ and that z_μ remains negligibly small for values of μ comparable to N, so that one can consider Eq. (55) to be valid within the whole permitted range of μ. In view of Eq. (26) one then has

$$1/C=\sum_\mu \exp[-N\delta(\mu/N)^2], \qquad (58)$$

and by regrouping the exponent in Eq. (27)

$$Z_1(\alpha)=C\sum_\nu \exp\{-[N\gamma\delta/(\gamma+\delta)](\nu-\alpha)^2\}$$
$$\times\sum_\mu \exp\{-N(\gamma+\delta)[(\mu/N)+\gamma(\nu-\alpha)/(\gamma+\delta)]^2\}. \qquad (59)$$

With $\gamma/N\ll1$, $\delta/N\ll1$ for large N the sums over μ in Eqs. (58) and (59) can be replaced by integrals, extending from $-\infty$ to $+\infty$. One thus obtains

$$Z_1(\alpha)=[\delta/(\delta+\gamma)]^{1/2}\sum_\nu \exp\{-[N\gamma\delta/(\gamma+\delta)](\nu-\alpha)^2\}. \qquad (60)$$

By means of the Poisson equation (32), using further Eqs. (56) and (57) to express δ in terms of $\langle P^2\rangle_{\rm av}$, and with γ given by Eqs. (21) and (22), the last equation can be rewritten in the form

$$Z_1(\alpha)=(\pi/N\gamma)^{1/2}$$
$$\times\Big(1+2\sum_{g=1}^{\infty}\exp\{-(\pi g)^2/[N\gamma(1-\rho)]\}\cos2\pi g\alpha\Big), \qquad (61)$$

where

$$\rho=\langle P^2\rangle_{\rm av}/(NmkT). \qquad (62)$$

This result includes in the limit $\rho=0$ that of Eq. (44) obtained in the special case B of Sec. IV. In fact, the corresponding assumption of vanishing $\langle P^2\rangle_{\rm av}$ is implied in Eq. (42) by admitting only the value $\mu=0$ or equivalently by going to the limit $\delta\to\infty$ in Eq. (55). Another limiting case of special significance arises for $\rho=1$. The corresponding value $\langle P^2\rangle_{\rm av}=NmkT$ is that obtained from the classical equipartition theorem by assigning the value $kT/2$ to the mean kinetic energy of the center of gravity in its motion along the x direction. Equation (61) shows, however, that $Z_1(\alpha)$ becomes independent of α as ρ increases towards unity. One is thus led to the important conclusion that stable flux trapping is excluded even for $N\gamma\gg1$ as long as the equipartition theorem is applicable.

It is true that the condition $N\delta\gg1$ required for the derivation of Eq. (61) does not strictly permit the assumption $\rho=1$ since it implies $\delta=0$. On the other hand, it is seen from Eq. (55) that a vanishing value of δ leads to the assumption of constant z_μ made in the special case A of Sec. IV with the rigorous result of Eq. (40) and with the consequence that $Z_1(\alpha)$ is practically constant if $\gamma/N\ll1$. Since this condition was likewise required in the derivation of Eq. (61) it follows that the content of this equation remains valid for $\rho=1$ so that the preceding conclusion is entirely justified.

The intermediate case $0<\rho<1$ still results in a pronounced variation of $Z_1(\alpha)$ as long as $N\gamma(1-\rho)\gg1$. This condition is satisfied even with ρ only slightly less than unity since it was assumed that $N\gamma\gg1$ and stable flux trapping can thus be expected as soon as $\langle P^2\rangle_{\text{av}}$ is found to be a small fraction below the equipartition value $NmkT$. Following the same reasoning which led to Eq. (51) one finds that the expression for $F_1(\alpha)$ is modified by the factor $(1-\rho)$. The resulting fact that $F_1(\alpha)$ vanishes in the limit $\rho=1$ is again validated by the arguments presented before and thus reaffirms the exclusion of stable flux trapping in this limit since it leads through Eq. (31) to the single minimum of $F(\alpha)$ at $\alpha=0$.

The circumstance that the result of Eq. (61) represents a generalization of that obtained in Eq. (44) for the special case B of Sec. IV rests upon the assumption, implied in Eq. (55), that z_μ is appreciable only for $|\mu|\ll N$. In particular, it is this assumption which prohibits the shorter period in α found in Eq. (47) of case C. The analogous generalization of this equation is achieved, on the other hand, if the sharp maximum of z_μ around $\mu=0$ is assumed to repeat itself around $|\mu|=N/2$. With the significance of ρ still given by Eq. (62) this assumption similarly leads to the replacement of $N\gamma$ by $N\gamma(1-\rho)$ in Eq. (47) and corresponds in the same sense to a fractional pairing in which the pairing of all particles was discussed in the special case C. In view of the periodic character of $Z_1(\alpha)$ it is to be noted that $\langle P^2\rangle_{\text{av}}$ represents not only the mean square of P in the absence of a vector potential and, hence, for $\alpha=0$, but also the mean-square fluctuation of P around the set of values $\alpha N\hbar/R$, obtained by choosing α as an integer multiple of the period.

VI. DIMENSIONALITY

The results of the preceding section call for an examination of the physical characteristics which determine the magnitude of $N\gamma$. Since the radius R of the ring is here of particular interest it is indicated to rewrite Eq. (48) in the form

$$N\gamma=R^*/R, \tag{63}$$

with

$$R^*=N\lambda^2/(2\pi R) \tag{64}$$

and

$$\lambda=(\pi\hbar^2/mkT)^{1/2}. \tag{65}$$

Given the density of the system and the cross section of the ring, N is proportional to R so that R^* represents a definite characteristic radius. Besides the macroscopic magnitude of the geometrical extensions one deals with a magnitude of atomic size by expressing the density in terms of the mean distance a between particles. In view of the different order of these magnitudes it will be seen that the dimensionality of the system is of decisive importance since it leads to widely different values of R^*. In order to distinguish between these values, an index equal to the number of dimensions shall be used below.

Considering first the case of a one-dimensional ring one has here $N=2\pi R/a$, and from Eq. (64)

$$R_1^*=\lambda^2/a. \tag{66}$$

With m and a comparable to the mass of the electron and the Bohr radius, respectively, and assuming the rather low temperature $T\cong1°\text{K}$, one obtains

$$R_1^*\cong10^{-3}\text{ cm}, \tag{67}$$

and an even smaller value for higher temperatures. Since normal radii are usually much larger, one has to expect that $R\gg R_1^*$, and hence from Eq. (63) that one deals with the condition $N\gamma\ll1$, discussed in the preceding section. Except for an abnormally small radius and irrespective of the specific properties of the system one is thus led to the exclusion of stable flux trapping in a one-dimensional ring.

While this conclusion refers to the case of a single ring it would substantially remain valid for a ring of bulk material composed of a large number n of identical one-dimensional rings in a close-packed parallel bundle. In fact, the free energy of such a system is simply n times that of a single one-dimensional ring and the same holds therefore for the part $F_1(\alpha)$ in Eq. (31). Using Eqs. (29), (37), and (63), it can be shown that the exclusion of stable flux trapping calls here for no more than the replacement of the condition $R\gg R_1^*$ by $R\gg R_1^*\ln n$. Assigning to each one-dimensional ring a cross section of atomic dimensions and to the bulk ring a cross section of linear dimensions comparable to R_1^*, n is not found to be so large as to increase the effective value of the characteristic radius by much more than a factor 10.

Much larger values appear already for a two-dimensional ring with macroscopic width $d \ll R$. Since $N = 2\pi Rd/a^2$ one has in this case

$$R_2^* = (d/a) R_1^*. \tag{68}$$

Even with d as small as 0.1 mm it is found that $R_2^* \cong 10^6 R_1^*$ so that the estimate of Eq. (67) leads to $R_2^* \cong 10$ m and to somewhat but not greatly larger values for a bulk ring with a cross section of linear dimensions comparable to d.

A further great increase is finally met in the most realistic case of a three-dimensional ring with macroscopic radial width $d_1 \ll R$ and with height d_2. In this case $N = 2\pi R d_1 d_2/a^3$ and therefore

$$R_3^* = (d_1 d_2/a^2) R_1^*. \tag{69}$$

Choosing, for example, $d_1 \cong d_2 \cong 0.1$ mm, one obtains $R_3^* \cong 10^{12} R_1^*$. The estimate of Eq. (67) yields thus $R_3^* \cong 10^4$ km and even larger values for a more sizeable cross section. In marked contrast to the one-dimensional case these large values prohibit the condition $R \gg R_3^*$ to be realistically satisfied so that the exclusion of stable flux trapping for $N\gamma \ll 1$ is here irrelevant. A realistic situation implies instead the opposite condition $R \ll R_3^*$ or, from Eq. (63), $N\gamma \gg 1$, and it was shown in the preceding section that stable flux trapping hinges here upon separate criteria.

In order to discuss the connection between these criteria and the dimensionality of the system one has to examine the value of $\langle P^2 \rangle_{\text{av}}$, since it was seen under the condition $N\gamma \gg 1$ to be the essentially determining quantity. It is indicated for this purpose to introduce the mean number

$$n_k = n(p_k) \tag{70}$$

of particles with momentum $p_k = k\hbar/R$ in the x direction so that one has quite generally

$$N = \sum_k n(p_k), \tag{71}$$

$$\langle P \rangle_{\text{av}} = \sum_k p_k n(p_k), \tag{72}$$

and

$$\langle P^2 \rangle_{\text{av}} = -mkT \sum_k p_k [\partial n(p_k)/\partial p_k], \tag{73}$$

where the last equation is in essence based upon the conservation of momentum and can be deduced from the results of Sec. III.[2]

Due to the relatively small increment $\hbar/R$ of p_k for macroscopic values of R it is normally to be expected that the difference between successive terms is sufficiently small to permit the replacement of the sums over k by integrals. Partial integration leads then from Eq. (73) together with Eq. (71) to the same value $\langle P^2 \rangle_{\text{av}} = NmkT$ as that obtained from the application of the equipartition theorem and thereby to the exclusion of stable flux trapping for $\rho = 1$ which was discussed in the preceding section. The analogous expectation of a smoothly varying velocity distribution function was shown earlier[3] to lead to a vanishing stable current. While the preceding conclusion represents a confirmation of this result it will be seen below to likewise bear upon the dimensionality.

As an adequate description of the normal state of a metal, the case of a degenerate Fermi gas shall be first considered. The most rapid variation of the summands in Eqs. (71), (72), and (73) occurs here when p_k is close to the Fermi momentum p_f and is essentially determined by the variation of $(p_k^2 - p_f^2)/(2mkT)$. With $p_k \cong p_f$ the difference between successive values of this quantity is

$$\Delta \cong p_f \hbar/(RmkT). \tag{74}$$

Using again the mean distance a between particles one has $p_f \cong \hbar/a$ and hence from Eq. (74) with the notations of Eqs. (65) and (66),

$$\Delta \cong R_1^*/R. \tag{75}$$

In order to permit the replacement of the sums by integrals it is sufficient to demand that $\Delta \ll 1$ so that the condition $R \gg R_1^*$ leads to the equipartition value of $\langle P^2 \rangle_{\text{av}}$ and hence with $\rho = 1$ to the exclusion of stable flux trapping. It is important to notice that this conclusion is based upon a generally valid estimate of the Fermi momentum p_f and that it therefore applies not only to a Fermi gas in one dimension, although the quantity R_1^* was first introduced in the consideration of the one-dimensional case.

The one-dimensional case is exceptional, however, inasfar as the condition $R \gg R_1^*$ is in this case equivalent to $N\gamma \ll 1$, so that it prohibits flux trapping without reference to any specific properties of the system. In particular, the absence of long-range order is not essential for this argument. Indeed, it is covered by the special case A of Sec. IV and can be seen from Eq. (40) to concern the preceding conclusion only through the minor observation that $\gamma/N \ll 1$ is a consequence of the condition $N\gamma \ll 1$. While the opposite condition $R \ll R_1^*$ or $N\gamma \gg 1$ is quite unrealistic one may note that, in principle, it would lead to the opposite conclusion. In fact, this condition not only establishes

[2] This can best be seen from Eq. (27) by keeping first α finite. With $p = (P/N) = (\nu + \mu/N)(\hbar/R)$ from Eq. (16), $p_\alpha = \alpha(\hbar/R)$, and with $N\gamma$ given by Eq. (48) one obtains from the first and second derivative of $\ln Z_1(\alpha)$: $\langle (p - p_\alpha)^2 \rangle_{\text{Av}} - (\langle p \rangle_{\text{Av}} - p_\alpha)^2 = (mkT/N)(\partial \langle p \rangle_{\text{Av}}/\partial p_\alpha)$, where $\langle p \rangle_{\text{Av}} = \langle P \rangle_{\text{Av}}/N = (1/N)\Sigma p_k n_k$. It also follows from Eqs. (1) and (19) that p_k appears in the Hamiltonian and hence in n_k only in the combination $p_k - p_\alpha$, so that $\partial \langle p \rangle_{\text{Av}}/\partial p_\alpha = -(1/N)\Sigma p_k(\partial n_k/\partial p_k)$. Going to the limit $\alpha \to 0$ or $p_\alpha \to 0$, one has for symmetry reasons $\langle p \rangle_{\text{Av}} = 0$ and therefore $\langle p^2 \rangle_{\text{Av}} = \langle P^2 \rangle_{\text{Av}}/N^2 = -(mkT/N^2)\Sigma p_k(\partial n_k/\partial p_k)$ in agreement with Eq. (73).

[3] F. Bloch, Phys. Rev. **137**, A787 (1965).

the value of $\langle P^2 \rangle_{av}$ as the essential criterion but it also implies $\Delta \gg 1$, so that a strict evaluation of the sum in Eq. (73) is required. It is therefore not to be expected that $\langle P^2 \rangle_{av}$ has the value which is obtained from the equipartition theorem and which would thus prohibit stable flux trapping. As an example, it can be shown for an ideal Fermi gas that the condition $\Delta \gg 1$ results in a considerable reduction from the equipartion value and thereby indicates the occurrence of stable flux trapping.[4] The same conclusion holds for an ideal one-dimensional Bose gas, and the arguments remain basically unaltered even in the presence of interactions, so that one may consider this conclusion to be likewise independent of the specific properties of one-dimensional systems.

In view of its intermediate nature the two-dimensional case shall not be further considered, and the following discussion will deal with the essentially novel features encountered in the case of three dimensions. It was seen before that a realistic radius requires here the condition $R \ll R_3^*$ or $N\gamma \gg 1$ to be satisfied so that the difference between the normal and the superconductive state has to be reflected in the evaluation of $\langle P^2 \rangle_{av}$. Since R_1^* is of very much lower order of magnitude than R_3^*, the additional condition $R \gg R_1^*$ remains equally realistic, but does not have the general relevance found in the one-dimensional case. This condition was seen, however, to be relevant to the fact that stable flux trapping cannot occur in the normal state of a metal since the nature of $n(p_k)$, characteristic of this state, permits under the equivalent condition $\Delta \ll 1$ the replacement of the sum in Eq. (73) by an integral and leads therefore to the equipartition value of $\langle P^2 \rangle_{av}$.

The opposite condition $\Delta \gg 1$ is far too unrealistic, on the other hand, to bear in any way upon the observed stable flux trapping by a three-dimensional superconductor. It was shown earlier[3] that the transition to the superconductive state has to be accompanied by characteristic sharp changes in the nature of the velocity distribution function and is the analogous change in the nature of $n(p_k)$, which no longer permits the replacement of the sum by an integral. Nevertheless, such a replacement is permitted if one considers $n(p_k)$ to consist of two parts corresponding to the two-phase model of the superconductor and assumes the momentum of the superconductive phase to vanish. The only contribution to $\langle P^2 \rangle_{av}$ in Eq. (73) arises then from the part of $n(p_k)$ which refers to the normal phase. Upon substitution of this part, to be denoted by $n''(p_k)$, the replacement by an integral is again permitted and one obtains by partial integration $(P^2)_{av} = N''mkT$, where $N'' = \sum_k n''(p_k)$ represents the number of particles in the normal phase. One has then from Eq. (62) $\rho = N''/N$ or

$$1-\rho = N'/N, \tag{76}$$

where $N' = N - N''$ represents the number of particles in the superconductive phase.

In the model of a condensed ideal Bose gas, this phase is constituted by the particles with momentum $p=0$ and one has $N' = N[1-(T/T_c)^{3/2}]$, where T_c represents the critical temperature of Einstein-Bose condensation. In view of Eq. (61), $Z_1(\alpha)$ is then given by Eq. (44) of the case B in Sec. IV with the only difference that N is to be replaced by N', so that this model leads for $T < T_c$ to stable flux trapping with the flux quantum hc/e. The entirely analogous reasoning, applied to electron pairing in a superconductor, calls for the replacement of N by N' in Eq. (47) of case C except that N' refers here to the number of paired electrons. Correspondingly, one deals here with an exponential variation of N' below the transition temperature, characteristic for the finite pairing energy, and with the observed value $hc/(2e)$ of the flux quantum.

[4] With an equal number $N/2$ of particles of opposite spin it is indicated for this purpose to assume $N/2$ to be an odd integer. The dominant contribution to the sum of Eq. (73) arises then from $|k| = N/4 \pm \frac{1}{2}$ and, using Eq. (62), yields $\rho = N\gamma \exp(^-N\gamma)$. In view of the single sharp maximum of z_μ around $|\mu| = 0$, $Z_1(\alpha)$ is given here by Eq. (61) or, since ρ is negligibly small for $N\gamma \gg 1$, by Eq. (44). The minima of $F_1(\alpha)$, obtained from Eq. (29) occur therefore at integer values of α, indicating flux quantization with the quantum hc/e. The opposite case of integer $N/2$ is somewhat more complicated since the maximum of z_μ is here centered around $|\mu| = N/2$. In view of Eq. (34) each term in the sum of Eq. (61) or (44) thus has to be multiplied with $(-1)^g$ so that the minima of $F_1(\alpha)$ do not occur at integer values of α but at values halfway between integers.