measured in kG. (The parameters for sodium were given in Sec. III.)

The transmission measurements were made at frequencies between 1.0 and 8.0 MHz at fields between 15 and 50 kG. In the standing-wave experiments the frequency ranged between 0.5 and 50 kHz while the field was varied between 4.3 and 16.7 kG. Thus Fig. 9 shows good agreement between theory and experiment over a wide range of experimental conditions.[20]

V. CONCLUSIONS

In summary, experimental results obtained with sodium and potassium closely agree with the predictions of the free-electron theory over a wide range of experimental conditions. The form of the theoretical correction to the nonlocal damping due to collision processes agree well with experiment.

[20] The nonlocal damping of helicon waves has been measured using pulse techniques. The results of the pulsed experiments are also in close agreement with the free-electron theory.

The high degree of agreement is particularly gratifying since the experiment gives a check on a rather involved transport-theory calculation in which there are no adjustable parameters. The Fermi momentum is the only parameter which cannot be measured directly by the experiments. However, the free-electron value which we have used agrees with the value measured by Shoenberg and Stiles[21] using the de Haas–van Alphen effect within 0.5%.[1]

ACKNOWLEDGMENTS

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Flux Quantization in a One-Dimensional Model*

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A one-dimensional model of interacting electrons is studied to determine whether such a system, in thermal equilibrium, can exhibit flux quantization. The free energy and current of the system are calculated and shown to be periodic functions of the flux enclosed in a ring-shaped sample with period hc/e. The Maxwell equations provide a second relation between the current and flux. It is found that at finite temperatures, the equations for the current I and the flux Φ_B have only the trivial solution $I=\Phi_B=0$ in the limit of macroscopic systems. Therefore, there is no flux quantization. The free energy is calculated by a generalization of the method of Tomonaga. This method describes the Fermi system in terms of an equivalent set of bosons which represent the collective modes of the Fermi gas. The major results of the generalization are the appearance of trilinear terms in the equivalent boson Hamiltonian and effects of a vector potential.

1. INTRODUCTION

SINCE the suggestion by Little[1] that properly synthesized long organic polymers might be superconducting at room temperatures, there has been renewed interest in the properties of one-dimensional electron systems. Recently, Hohenberg[2] has rigorously proved that a one-dimensional system cannot exhibit off-diagonal long-range order (ODLRO),[3] a property characteristic of superconductors in three dimensions. However, the absence of ODLRO does not imply the absence of other properties characteristic of superconductivity such as flux quantization and persistent currents as discussed in Sec. 4. The possibility of the existence of flux quantization and persistent currents in nonequilibrium situations has been investigated by Little,[4] and Ambegaokar and Langer.[5]

It is the purpose of this paper to examine a one-dimensional model of interacting electrons to determine whether the system can exhibit flux quantization in equilibrium.[6] The analysis is directed to the calculation of the free energy of the system (Sec. 3), which is shown

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[6] Whereas "one-dimensional" means for Refs. 4 and 5 that the transverse dimensions of the system are small compared to the temperature-dependent correlation length and penetration depth, for this paper it will be taken to mean that those states which correspond to transverse electron motion contribute negligibly to the partition function. In general, this imposes the more severe condition that the transverse extension be smaller or of the order of the Fermi wavelength.

to be a period function of the magnetic flux through a ring-shaped sample with period hc/e. From this result an expression for the current as a periodic function of the flux can be obtained. Combining this result with the Maxwell equation relating the current and flux, one can determine whether there exist stable nonvanishing solutions of these equations, indicating the existence of flux quantization and persistent currents in equilibrium. In the thermodynamic limit, it is found that no such solutions exist for finite temperatures.

The method used to obtain the free energy of the system is that of "quantized sound waves" introduced by Bloch[7] in discussing certain properties of the degenerate ideal Fermi gas and extended, under certain restrictions, by Tomonaga[8] to the case of the interacting Fermi gas. This method, which emphasizes the role of the electron density fluctuations, is particularly appropriate in light of Ferrell's remarks[9] which emphasize the importance of these fluctuations in a one-dimensional system. The basic idea of the method is to bring the low-lying states of the fermion system into a one-to-one correspondence with the states of a system of harmonic oscillators representing the density fluctuations which propagate in the Fermi gas. The original fermion operators are thus replaced by the boson operators pertaining to the system of oscillators. Instead of starting from Tomonaga's boson Hamiltonian, we formulate the procedure anew in Sec. 2 and obtain the following extensions to Tomonaga's procedure: (1) The kinetic-energy operator contains terms which are trilinear as well as terms which are bilinear in the boson operators; (2) the effects of a magnetic field are included.

2. GENERAL FORMULATION OF THE MODEL

Commutation Relations

We consider a system of N fermions of spin-$\frac{1}{2}$ constrained to a line. In order that it may sustain a closed current, the system shall be bent into a ring of radius R. The second-quantized field operators and their Hermitian conjugates are, in terms of plane-wave amplitudes,

$$\Psi_\sigma(x)=(2\pi R)^{-1/2}\sum_n \exp(inx/R)\,c_{n,\sigma},$$

$$\Psi_\sigma^*(x)=(2\pi R)^{-1/2}\sum_n \exp(-inx/R)\,c_{n,\sigma}^*,$$

where x is a length measured along the ring, n is an integer, the spin quantum number σ equals ±1, and $c_{n,\sigma}^*$, $c_{n,\sigma}$ are fermion creation and destruction operators obeying the anticommutation rules

$$\{c_{n,\sigma},\,c_{n',\sigma'}^*\}=\delta_{n,n'}\delta_{\sigma,\sigma'},\quad \{c_{n,\sigma},\,c_{n',\sigma'}\}=0. \tag{2.1}$$

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The expansion of the density operator

$$\rho(x)=\sum_\sigma \Psi_\sigma^*(x)\Psi_\sigma(x)$$

is, similarly, $\rho(x)=(2\pi R)^{-1}\sum_{n,\sigma}\rho_{n,\sigma}\exp(inx/R)$,

$$\rho_{n,\sigma}=\rho_{-n,\sigma}^*=\sum_l c_{l,\sigma}^*c_{l+n,\sigma}. \tag{2.2}$$

There is need later of more general operators $\rho_n(f_l)$, which are defined by

$$\rho_n(f_l)=\sum_\sigma \rho_{n,\sigma}(f_l)=\sum_\sigma\sum_l f_l c_{l,\sigma}^* c_{l+n,\sigma}, \tag{2.3}$$

with coefficients f_l which depend in an arbitrary manner on the summation index l. In the special case $f_l=1$ for all l, $\rho_{n,\sigma}(1)$ is simply $\rho_{n,\sigma}$ defined in Eq. (2.2), and the latter notation will be used. Following Tomonaga's procedure, the operator $\rho_{n,\sigma}(f_l)$ is decomposed into

$$\rho_{n,\sigma}(f_l)=\rho_{n,\sigma}^+(f_l)+\rho_{n,\sigma}^-(f_l), \tag{2.4}$$

$$\rho_{n,\sigma}^+(f_l)=\sum_{l\geq -n/2} f_l c_{l,\sigma}^* c_{l+n,\sigma}, \tag{2.5}$$

$$\rho_{n,\sigma}^-(f_l)=\sum_{l< -n/2} f_l c_{l,\sigma}^* c_{l+n,\sigma}. \tag{2.6}$$

Using Eq. (2.1) the commutators of the above operators can be obtained directly. These commutators simplify greatly if one considers only a restricted set of states of the Fermi system in which all single-particle levels with momentum between $-\hbar R^{-1}n^*$ and $\hbar R^{-1}n^*$ are fully occupied. The momentum $\hbar R^{-1}n^*$ must be much less than the Fermi momentum $\hbar R^{-1}n_M$. For temperatures $kT\ll\epsilon_F$ and interactions which are neither too strong nor of too short range, we expect the statistically important states to be contained within this subspace S, and we shall henceforth restrict ourselves to such states. The conditions under which the above assumption should be valid are discussed by Tomonaga and, more rigorously, by Gutfreund and Schick.[10] It is shown by the latter that the assumption is valid for sufficiently weak attractive interactions upon which the theory of superconductivity in three dimensions is normally based.

The following commutators, which are not identities but which are meant to be valid only if applied as operators upon the subset S of wave functions, are obtained directly from Eqs. (2.1) to (2.6) and the properties of the subspace S; provided that $|n|,|n'|<2n^*/3$,

$$[\rho_{n,\sigma},\,\rho_{n',\sigma'}(f_l)]=\delta_{\sigma,\sigma'}\rho_{n+n',\sigma}(f_{l+n}-f_l), \tag{2.7}$$

$$[\rho_{n,\sigma}^+,\,\rho_{n',\sigma'}^-(f_l)]=[\rho_{n,\sigma}^-,\,\rho_{n',\sigma'}^+(f_l)]=0, \tag{2.8}$$

$$[\rho_{n,\sigma}^+,\,\rho_{n',\sigma'}^+(f_l)]=\delta_{\sigma,\sigma'}[\rho_{n+n',\sigma}^+(f_{l+n}-f_l)+\delta_{n,-n'}F_n(f_l)], \tag{2.9}$$

$$[\rho_{n,\sigma}^-,\,\rho_{n',\sigma'}^-(f_l)]=\delta_{\sigma,\sigma'}[\rho_{n+n',\sigma}^-(f_{l+n}-f_l)-\delta_{n,-n'}F_n(f_l)], \tag{2.10}$$

[10] H. Gutfreund and M. Schick (to be published).

where

$$F_n(f_l)=\sum_{0>l\geq -n/2} f_{l+n}-\sum_{n/2>l\geq 0} f_l, \qquad n\geq 0$$
$$=-\sum_{-n/2>l\geq 0} f_{l+n}-\sum_{0>l\geq n/2} f_l, \qquad n<0.$$

Setting $f_l=1$ for all l in the above, one recovers the Tomonaga relations

$$[\rho_{n,\sigma}^+, \rho_{n',\sigma'}^-]=0,$$
$$[\rho_{n,\sigma}^\pm, \rho_{n',\sigma'}^\pm]=\pm n\delta_{n,-n'}\delta_{\sigma,\sigma'}. \tag{2.11}$$

It is convenient to introduce the normalized boson operators

$$a_{n,\sigma}\equiv n^{-1/2}\rho_{n,\sigma}^+, \qquad \text{for } n>0,$$
$$a_{n,\sigma}\equiv (-n)^{-1/2}\rho_{n,\sigma}^-, \qquad \text{for } n<0, \tag{2.12}$$
$$a_{n,\sigma}^*=n^{-1/2}\rho_{-n,\sigma}^+, \qquad \text{for } n>0,$$
$$a_{n,\sigma}^*=(-n)^{-1/2}\rho_{-n,\sigma}^-, \qquad \text{for } n<0. \tag{2.13}$$

From Eq. (2.11) it can be seen that these operators obey the usual boson commutation relations

$$[a_{n,\sigma}, a_{n',\sigma'}^*]=\delta_{n,n'}\delta_{\sigma,\sigma'},$$
$$[a_{n,\sigma}, a_{n',\sigma'}]=0, \tag{2.14}$$

for $|n|$, $|n'|<2n^*/3$. As previously mentioned, the definition of the subspace S requires that $n^*\ll n_M$. Except for this condition, the value of n^* and, hence, the extent of the core of occupied states does not affect the results obtained below. The corresponding restriction to values of $n\ll n_M$ will have to be noted in the following considerations.

Vacuum States

Consider the vacuum state ψ_0 of the operators $a_{n,\sigma}$, that is, the state for which $a_{n,\sigma}\psi_0=0$. It can be seen from the definition of the operators $\rho_{n,\sigma}^\pm\equiv\rho_{n,\sigma}^\pm(1)$, Eqs. (2.5) and (2.6), that the noninteracting ground state in which all single-particle levels with momentum $\hbar|n|/R\leq\hbar n_M/R$ are occupied satisfies this condition. However, if the momentum of all particles with spin σ are increased by $\hbar\nu_\sigma/R$, the resulting state will also satisfy this condition except for a few values of n of order $2(n_M-\nu_\sigma)$. Provided that the integers ν_1 and ν_{-1}, which are not necessarily positive or equal, are much less than n_M in magnitude, these values of n are larger than $2n^*/3$. For such large values of n, the commutation relations of Eqs. (2.7)–(2.10) no longer hold so that such values of n are to be excluded. With this proviso, the states representing "shifted Fermi seas" are also vacuum states. Assuming for convenience that there are an equal odd number of particles with spin $\sigma=+1$ and $\sigma=-1$, we are led to consider the set of vacuum states $\psi_0(P,\nu_\sigma)$ with all single-particle levels between $-n_M+\nu_\sigma$ and $n_M+\nu_\sigma$ occupied by particles of spin σ. These states shall be distinguished by their eigenvalues of the momentum operator P and of the operators $\rho_{0,\sigma}^\pm\equiv\rho_{0,\sigma}^\pm(1)$ which count the number of particles with spin σ and positive or negative momentum. Thus,

$$a_{n,\sigma}\psi_0(P,\nu_\sigma)=0,$$
$$P\psi_0(P,\nu_\sigma)=\hbar R^{-1}\rho_0(l)\psi_0(P,\nu_\sigma)$$
$$=P\psi_0(P,\nu_\sigma), \tag{2.15}$$

where

$$P=N\hbar R^{-1}(\nu_1+\nu_{-1})/2,$$
$$\rho_{0,\sigma}^+\psi_0(P,\nu_\sigma)=(n_M+\nu_\sigma+1)\psi_0(P,\nu_\sigma), \tag{2.16}$$
$$\rho_{0,\sigma}^-\psi_0(P,\nu_\sigma)=(n_M-\nu_\sigma)\psi_0(P,\nu_\sigma), \tag{2.17}$$

and

$$N=2(2n_M+1).$$

The same considerations which lead to Eq. (2.15) also imply the more general equation, which will prove useful,

$$\rho_{n,\sigma}^+(f_l)\psi_0(P,\nu_\sigma)=0, \qquad \text{for } n>0$$
$$\rho_{n,\sigma}^-(f_l)\psi_0(P,\nu_\sigma)=0, \qquad \text{for } n<0. \tag{2.18}$$

Boson Representation of the Kinetic Energy

By means of a more general procedure than that used by Tomonaga, it will be shown here that, within the subspace S defined previously, it is possible to express the kinetic-energy operator

$$H_{\mathrm{KE}}=\hbar^2(2mR^2)^{-1}\rho_0(l^2) \tag{2.19}$$

in terms of the operators $\rho_{n,\sigma}^\pm$. For this purpose, it is shown in the Appendix that the manifold of all many-particle states in the subspace S can be represented by a complete set of wave functions:

$$\psi_m(P',\nu_\sigma)=c\prod_\sigma\prod_n{}'\,(a_{n,\sigma}^*)^{N_{n,\sigma}}\psi_0(P,\nu_\sigma), \tag{2.20}$$

where c is a normalization constant, the prime on the product means $n\neq 0$, and the subscript m stands for a particular set of integers $N_{n,\sigma}$. Each state is an eigenstate of the operators $\hat{P}$ and $\rho_{0,\sigma}^\pm$. Indeed, from the fact that $\rho_{0,\sigma}^\pm$ commutes with all $a_{n,\sigma}^*$, as seen from Eqs. (2.11) and (2.13), and from the properties of the vacuum states, formulated in Eqs. (2.16) and (2.17), one obtains

$$\rho_{0,\sigma}^+\psi_m(P',\nu_\sigma)=(n_M+\nu_\sigma+1)\psi_m(P',\nu_\sigma),$$
$$\rho_{0,\sigma}^-\psi_m(P',\nu_\sigma)=(n_M-\nu_\sigma)\psi_m(P',\nu_\sigma). \tag{2.21}$$

Based upon the fact that the wave functions ψ form a complete set in S, it can be shown, using Eq. (2.13), that any two operators, O and O', which satisfy the conditions

$$[\rho_{n,\sigma}^\pm, O]=[\rho_{n,\sigma}^\pm, O'], \qquad \text{all } n,\sigma \tag{2.22}$$
$$O\psi_0(P,\nu_\sigma)=O'\psi_0(P,\nu_\sigma), \qquad \text{all } P,\nu_\sigma \tag{2.23}$$

are identical within the subspace S in the sense that all matrix elements of O and O' taken between states in this subspace are equal. In seeking a boson representation of the operator $O'=\rho_0(l^2)$, therefore, one has to find an operator function O of the quantities $\rho_{n,\sigma}^{\pm}$ such that

$$[\rho_{n,\sigma}^{\pm}, O]=2n\rho_{n,\sigma}^{\pm}(l+\tfrac{1}{2}n), \tag{2.24}$$

where Eqs. (2.3)–(2.10) have been used to evaluate the commutator $[\rho_{n,\sigma}^{\pm}, \rho_0(l^2)]$, and

$$O\psi_0(P, \nu_\sigma)=[\tfrac{1}{3}Nn_M^2+8n_M^2N^{-1}(\nu_1^2+\nu_{-1}^2)]\psi_0(P, \nu_\sigma), \tag{2.25}$$

where the expression in the bracket has been obtained by evaluating $\rho_0(l^2)\psi_0(P, \nu_\sigma)$. It will be convenient to obtain first a boson representation $O_{n,\sigma}^{\pm}$ of the operators $O'=\rho_{n,\sigma}^{\pm}(l+\frac{1}{2}n)$ which appear in Eq. (2.24). Proceeding as above, the operators $O_{n,\sigma}^{\pm}$ must satisfy

$$[\rho_{n,\sigma}^{\pm}, O_{n',\sigma'}^{\pm}]=n\delta_{\sigma,\sigma'}(\rho_{n+n',\sigma}^{\pm}\mp\tfrac{1}{2}\delta_{n,-n'}),$$

$$[\rho_{n,\sigma}^{\mp}, O_{n',\sigma'}^{\pm}]=0,$$

where Eqs. (2.8)–(2.10) have been used to evaluate $[\rho_{n,\sigma}^{\pm}, \rho_{n',\sigma'}^{\pm}(l+\frac{1}{2}n')]$, and

$$O_{n',\sigma'}^{\pm}\psi_0(P, \nu_\sigma)=\rho_{n',\sigma'}^{\pm}(l+\tfrac{1}{2}n')\psi_0(P, \nu_\sigma)=0, \qquad n'\gtrless 0$$

from Eq. (2.18). By means of the commutation relations Eq. (2.11) and Eq. (2.18), it can be directly verified that the operators

$$O_{n',\sigma'}^{\pm}=\pm\tfrac{1}{2}(\sum_r \rho_{-r,\sigma'}^{\pm}\rho_{r+n',\sigma'}^{\pm}\mp\rho_{n',\sigma'}^{\pm}) \tag{2.26}$$

satisfy all these conditions and are therefore the boson representation of the operators $\rho_{n',\sigma'}^{\pm}(l+\frac{1}{2}n')$. It should be noted that the momentum operator $\hat{P}$ is

$$\hat{P}=\hbar R^{-1}\sum_\sigma [\rho_{0,\sigma}^{+}(l)+\rho_{0,\sigma}^{-}(l)],$$

so that a boson representation of this operator is simply

$$\hat{P}=\hbar R^{-1}\sum_\sigma (O_{0,\sigma}^{+}+O_{0,\sigma}^{-}).$$

Setting $n'=0$ in Eq. (2.26), replacing the operators $\rho_{0,\sigma}^{\pm}$ wherever they occur by their eigenvalues as given in Eq. (2.21), and introducing the normalized boson operators of Eqs. (2.12) and (2.13), one obtains

$$\hat{P}=\hbar R^{-1}[\tfrac{1}{2}N(\nu_1+\nu_{-1})+\sum_\sigma \sum_l{}' \, l a_{l,\sigma}^* a_{l,\sigma}], \tag{2.27}$$

where the prime on the sum means $l\neq 0$. This expression for the momentum operator agrees with that of Tomonaga for the particular states he considered. These states have equal numbers of particles with positive and negative momentum for both values of σ. According to Eqs. (2.16) and (2.17) they are characterized by $\nu_1=\nu_{-1}=0$.

Returning to the problem of obtaining a boson representation for $\rho_0(l^2)$, we substitute the boson representation of $\rho_{n,\sigma}^{\pm}(l+\frac{1}{2}n)$, given by Eq. (2.26), into Eq. (2.24) and obtain

$$[\rho_{n,\sigma}^{\pm}, O]=\pm n(\sum_r \rho_{-r,\sigma}^{\pm}\rho_{r+n,\sigma}^{\pm}\mp\rho_{n,\sigma}^{\pm}). \tag{2.28}$$

To complete the task of expressing $\rho_0(l^2)$ in terms of boson operators, one has to find an operator which satisfies both Eqs. (2.28) and (2.25). Using again the relations Eq. (2.11), it can be directly verified that Eq. (2.28) is satisfied by choosing

$$O=\sum_\sigma \sum_{l,q} \tfrac{1}{3}(\rho_{-l,\sigma}^{+}\rho_{l+q,\sigma}^{+}\rho_{-q,\sigma}^{+}+\rho_{-l,\sigma}^{-}\rho_{l+q,\sigma}^{-}\rho_{-q,\sigma}^{-})$$
$$-\tfrac{1}{2}\sum_\sigma \sum_l (\rho_{-l,\sigma}^{+}\rho_{l,\sigma}^{+}-\rho_{l,\sigma}^{-}\rho_{-l,\sigma}^{-})+O_1,$$

where O_1 is a constant plus any operator which commutes with all $\rho_{n,\sigma}^{\pm}$. This expression can be simplified by replacing the operators $\rho_{0,\sigma}^{\pm}$ wherever they appear by their eigenvalues, as given in Eq. (2.21), and introducing the normalized boson operators of Eqs. (2.12) and (2.13). The quantity O_1 is then determined by requiring that the operator O satisfy Eq. (2.25). Introducing the quantities

$$\tau=8\epsilon_F/N, \tag{2.29}$$

where the Fermi energy ϵ_F is given by

$$\epsilon_F=\hbar^2 n_M^2/2mR^2\approx\hbar^2N^2/32mR^2 \tag{2.30}$$

and

$$\eta_l\equiv 1, \qquad l>0$$
$$\equiv -1, \qquad l<0$$

one obtains for $H_{\mathrm{KE}}=\rho_0(l^2)\hbar^2/2mR^2$ the boson representation valid in the subspace S:

$$H_{\mathrm{KE}}=\tfrac{1}{3}N\epsilon_F+\tau(\nu_1^2+\nu_{-1}^2)+\sum_\sigma \sum_l{}' \tau[1+4N^{-1}\nu_\sigma\eta_l]$$
$$\times|l|\, a_{l,\sigma}^* a_{l,\sigma}+H_T, \tag{2.31}$$

$$H_T=2\tau N^{-1}\sum_\sigma \sum_{l>0} \sum_{q>0} [lq(l+q)]^{1/2}(a_{l,\sigma}^* a_{q,\sigma}^* a_{l+q,\sigma}$$
$$+a_{-l,\sigma}^* a_{-q,\sigma}^* a_{-l-q,\sigma}+\mathrm{H.c.}), \tag{2.32}$$

where H.c. means Hermitian conjugate.

Ignoring all states characterized by $\nu_\sigma\neq 0$ and neglecting H_T, one recovers Tomonaga's bilinear Hamiltonian. It will be shown in Sec. 3 that it is permissible to ignore the states with $\nu_\sigma\neq 0$ provided that there is no magnetic field. The contribution of such states to the free energy is only of order $\log N$, which is negligible compared to other terms of order N in the limit of large systems. However, as shown in Sec. 2, it is necessary to include in the statistical ensemble states characterized by all values of ν_σ in order to obtain the correct magnetic flux dependence of thermodynamic quantities.

The neglect of H_T is equivalent to Tomonaga's approximation of the single-particle energy spectrum by the linear relation $\epsilon_n \propto \tau \mid n \mid$. This approximation is valid provided that the excited electrons of the system have energies sufficiently close to the Fermi energy ϵ_F. The states in the subspace S are characterized by such excitations so that the trilinear terms can be viewed as a small correction to the bilinear part of the Hamiltonian. This approximation can be checked directly in the case of the ideal Fermi gas for which the free energy is

$$F_0(T)=\tfrac{1}{3}N\epsilon_F\left[1-\tfrac{1}{4}\pi^2\left(\frac{kT}{\epsilon_F}\right)^2-\tfrac{1}{60}\pi^4\left(\frac{kT}{\epsilon_F}\right)^4+O\left(\frac{kT}{\epsilon_F}\right)^6\right].$$

The free energy as calculated from Tomonaga's bilinear Hamiltonian agrees with the first two terms of the above expression as noted by Wenzel,[11] while a straightforward application of perturbation theory shows that the lowest-order contribution of H_T to the free energy is precisely the third term of the above expression.[12] Thus the relative order of magnitude of the contributions of the trilinear and bilinear terms in this case is $(kT/\epsilon_F)^2$. As the Tomonaga model is only applicable if $(kT/\epsilon_F)\ll 1$, the correction is small as expected. In addition to corrections to all thermodynamic quantities, the trilinear terms provide a coupling between the collective modes and, hence, a damping of individual boson excitations which is reflected, for example, in the imaginary part of the dielectric response function.

Two-Particle Interactions

Having obtained a boson representation of the free-particle Hamiltonian, we proceed to obtain a similar representation of the interaction Hamiltonian which describes both electron-phonon and electron-electron interactions. Although the electron-phonon interaction is readily described in terms of bosons, as shown by several authors,[13] it will be assumed for convenience that this interaction can be replaced by an effective electron-electron interaction in the manner of Frohlich's canonical transformation.[14] As is well known, the effective interaction is nonlocal and is attractive for some values of the momenta of the two electrons. Since it is the attractive nature of the interaction which is of paramount importance for superconductivity, and not the nonlocality, no essential features of the total interparticle interaction are lost if the fermions are assumed to interact through a two-particle spin-independent local interaction. In a homogeneous space, the potential is a function only of the relative distance between particles so that the interaction Hamiltonian has the form

$$H_I=\frac{1}{2}\iint \rho(x)\rho(x')J(|x-x'|)\,dx\,dx'-\frac{1}{2}\int \rho(x)J(0)\,dx.$$

Substituting the expansion of $\rho(x)$ in plane-wave amplitudes into the above and using Eqs. (2.4), (2.12), and (2.13) to express the result in terms of the normalized boson operators, one obtains

$$H_I=\tfrac{1}{2}N(N-1)J_0+\sum_l{}' J_l\,|l|-\tfrac{1}{2}N\sum_l{}' J_l+\sum_\sigma\sum_l{}' J_l\,|l|\,a_{l,\sigma}^*a_{l,\sigma}+H_I', \tag{2.33}$$

$$H_I'=\tfrac{1}{2}\sum_\sigma\sum_l{}' J_l\,|l|\,(a_{l,\sigma}a_{-l,\sigma}+a_{l,\sigma}a_{-l,-\sigma}+a_{l,\sigma}^*a_{l,-\sigma}+\text{H.c.}), \tag{2.34}$$

where

$$J_l=J_{-l}=(2\pi R)^{-1}\int_{-\pi R}^{\pi R}\exp(ilx/R)J(x)\,dx.$$

It is to be noted that NJ_l is an intensive quantity. Combining the above with Eq. (2.31) one obtains for the total Hamiltonian

$$H=H_{\text{KE}}+H_I=H_B+H_T,$$

where the bilinear part

$$H_B=C(J)+\tau(\nu_1^2+\nu_{-1}^2)+\sum_l{}' J_l\,|l|+\sum_\sigma\sum_l{}' \tau(1+4N^{-1}\nu_\sigma\eta_l+J_l\tau^{-1})\,|l|\,a_{l,\sigma}^*a_{l,\sigma}+H_I', \tag{2.35}$$

$$C(J)=\tfrac{1}{3}N\epsilon_F+\tfrac{1}{2}N(N-1)J_0-\tfrac{1}{2}N\sum_l{}' J_l, \tag{2.36}$$

and the trilinear part H_T is given in Eq. (2.32). It is in keeping with the discussion of the trilinear part of the Hamiltonian H_T to diagonalize the bilinear part H_B and to treat H_T as a perturbation. The bilinear Hamiltonian H_B is brought into the diagonal form

$$H_B(S,D)=C(J)+2\tau(S^2+D^2)+\sum_l{}' \tau\{[\Omega_A(D,J_l)+4N^{-1}S\eta_l]\,|l|\,A_l^*A_l+[\Omega_B(D,J_l)+4N^{-1}S\eta_l]\,|l|\,B_l^*B_l$$
$$+\tfrac{1}{2}[\Omega_A(D,J_l)+\Omega_B(D,J_l)-2]\,|l|\} \tag{2.37}$$

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[14] H. Fröhlich, Proc. Roy. Soc. (London) **A215**, 291 (1952).

by the canonical transformation

$$\begin{bmatrix} A_l \\ B_l \\ A_{-l}^* \\ -B_{-l}^* \end{bmatrix} = \begin{bmatrix} R(\Omega_A, D) & R(\Omega_A, -D) & -R(-\Omega_A, -D) & -R(-\Omega_A, D) \\ R(\Omega_B, D) & R(\Omega_B, -D) & -R(-\Omega_B, -D) & -R(-\Omega_B, D) \\ -R(-\Omega_A, D) & -R(-\Omega_A, -D) & R(\Omega_A, -D) & R(\Omega_A, D) \\ -R(-\Omega_B, D) & -R(-\Omega_B, -D) & R(\Omega_B, -D) & R(\Omega_B, D) \end{bmatrix} \begin{bmatrix} a_{l,1} \\ a_{l,-1} \\ a_{-l,1}^* \\ a_{-l,-1}^* \end{bmatrix}, \tag{2.38}$$

where $l>0$ and the dependence of Ω_A on l has been suppressed. Concerning the notations used here, the quantities S, D are defined by

$$S\equiv\tfrac{1}{2}(\nu_1+\nu_{-1}), \qquad D\equiv\tfrac{1}{2}(\nu_1-\nu_{-1}). \tag{2.39}$$

From Eq. (2.21) it can be seen that a state characterized by $S\neq0$ has unequal numbers of particles with positive and negative momenta. In a state characterized by $D\neq0$, the number of particles with positive momentum and spin $\sigma=1$ is not equal to the number of particles with positive momentum and spin $\sigma=-1$.

As ν_1 and ν_{-1} are independent integers, S and D are either both integers or both half-odd integers. The dimensionless quantities $\Omega_A(D, J_l)$, $\Omega_B(D, J_l)$ are defined by

$$\Omega_{A,B}(D, J_l)\equiv\{(4N^{-1}D)^2+(1+2J_l\tau^{-1})\pm2[(4N^{-1}D)^2(1+2J_l\tau^{-1})+(J_l\tau^{-1})^2]^{1/2}\}^{1/2}. \tag{2.40}$$

A limiting form of the above which will prove particularly useful is

$$\Omega_A(0, J_l)=(1+4J_l\tau^{-1})^{1/2}, \qquad \Omega_B(0, J_l)=1. \tag{2.41}$$

Finally, the coefficients in the transformation of Eq. (2.37) are given by

$$R(\Omega, D)=(\Omega+4N^{-1}D-1)(\Omega-4N^{-1}D+1)(\Omega+4N^{-1}D+1)/M(\Omega, D^2), \tag{2.42}$$

$$M(\Omega, D^2)=2^{3/2}\{\Omega[(\Omega^2-1+16N^{-2}D^2)^2+64N^{-2}D^2(1-16N^{-2}D^2)]\}^{1/2}, \tag{2.43}$$

and are real. The operators A_l and B_l satisfy the commutation relations

$$[A_l, A_{l'}^*]=[B_l, B_{l'}^*]=\delta_{l,l'},$$

and all other commutators are zero.

Expressed in terms of these operators, the momentum operator of Eq. (2.27) becomes

$$P=\hbar R^{-1}[NS+\sum_l{}'(A_l^*A_l+B_l^*B_l)l].$$

As a special case of the above equations, it will be assumed that $D=0$, $J_l\neq0$ since it will be shown in Sec. 3 that only the contributions of states characterized by $D=0$ to the partition function need be considered in the thermodynamic limit. Setting $D=0$, one obtains from Eqs. (2.38)–(2.43)

$$A_l=2^{-1/2}[\cosh\theta_l(a_{l,1}+a_{l,-1})+\sinh\theta_l(a_{-l,1}^*+a_{-l,-1}^*)],$$

$$B_l=2^{-1/2}(a_{l,1}-a_{l,-1}), \tag{2.44}$$

for all l, where

$$\theta_l=\tfrac{1}{4}\ln(1+4J_l\tau^{-1}). \tag{2.45}$$

This transformation may be written in the explicitly unitary form

$$A_l=Ua_{l,1}U^\dagger,$$
$$A_{-l}^*=Ua_{-l,-1}^*U^\dagger,$$
$$B_l=-Ua_{l,-1}U^\dagger,$$
$$B_{-l}^*=Ua_{-l,1}^*U^\dagger, \qquad l>0$$

where $U=\exp(S_2)\exp(S_1)$, and

$$S_1=\tfrac{1}{4}\pi\sum_{l>0}(a_{l,1}a_{l,-1}^*+a_{-l,1}^*a_{-l,-1}-\text{H.c.}),$$

$$S_2=\tfrac{1}{2}\sum_{l>0}\theta_l[(a_{l,1}+a_{l,-1})(a_{-l,1}+a_{-l,-1})-\text{H.c.}].$$

From the ground-state wave function for the noninteracting system $\psi_0(0, 0)$, one obtains the corresponding wave function for the interacting system $\psi_0'(0, 0)=U\psi_0(0, 0)$. By symmetry arguments, these states are characterized by P, $\nu_\sigma=0$, or $D=S=0$.

Introduction of Vector Potential

We now consider a cylindrically symmetric magnetic field to be present with axis of symmetry perpendicular to the plane of the ring. Because of the infinitesimal width of the ring, the vector potential may be considered to be constant across it. Therefore the free-particle Hamiltonian is

$$H_{\text{KE}}(\alpha)=\hbar^2(2mR^2)^{-1}\sum_{l,\sigma}(l-\alpha)^2c_{l,\sigma}^*c_{l,\sigma}$$
$$=H_{\text{KE}}(0)-\hbar\alpha\hat{P}/mR+N\hbar^2\alpha^2/2mR^2,$$

where α is the magnetic flux Φ_B in units of hc/e. Substituting the boson representations of $H_{\text{KE}}(0)$ and P from Eqs. (2.31) and (2.27), one finds that $H_{\text{KE}}(\alpha)$ can be obtained from $H_{\text{KE}}(0)$ by the replacement $S\to S-\alpha$. In fact, this replacement is sufficient for the modification of the total Hamiltonian since H_I is unaffected by the presence of a vector potential. Thus the diagonalized bilinear Hamiltonian in the presence of a magnetic field is simply $H_B(S-\alpha, D)$ with H_B given by Eq. (2.37).

Partition Function

In view of the discussion of the trilinear part of the Hamiltonian H_T, it is permissible to retain only the bilinear part and hence to use for the partition function

$$Z(\alpha)=\exp-\beta F(\alpha)=\mathrm{Tr}\,\exp[-\beta H_B(S-\alpha, D)],$$

where $F(\alpha)$ is the free energy of the fermions, Tr signifies trace, and $\beta=(kT)^{-1}$. This may be written, with the aid of Eq. (2.37), as

$$Z(\alpha)=\sum_{i;S,D}\exp-\beta\{2\tau[(S-\alpha)^2+D^2]+\mathfrak{F}[(S-\alpha)^2, D^2]\}$$
$$+\sum_{h;S,D}\exp-\beta\{2\tau[(S-\alpha)^2+D^2]+\mathfrak{F}[(S-\alpha)^2, D^2]\}, \tag{2.46}$$

where $\sum_i$; denotes a summation over all integers $\sum_h$; denotes a summation over all half-odd integers, and

$$\mathfrak{F}[(S-\alpha)^2, D^2]=C(J)+\tfrac{1}{2}\tau\sum_l{}'[\Omega_A(D,J_l)+\Omega_B(D,J_l)-2]\,|\,l\,|$$
$$+\beta^{-1}\sum_l{}'(\ln\{1-\exp-\beta\tau[\Omega_A(D,J_l)+4N^{-1}(S-\alpha)\eta_l]\,|\,l\,|\}$$
$$+\ln\{1-\exp-\beta\tau[\Omega_B(D,J_l)+4N^{-1}(S-\alpha)\eta_l]\,|\,l\,|\}). \tag{2.47}$$

It is to be noted that the summands of Eq. (2.46) contain S and α only in the combination $(S-\alpha)$ and are even functions of this combination. Due to the fact that states characterized by all values of the parameter $S=\frac{1}{2}(\nu_1+\nu_{-1})$ have been included in the ensemble, it follows that $Z(\alpha)$ is an even function of α for the simultaneous replacement $\alpha\rightarrow-\alpha$, and $S\rightarrow-S$ leaves $Z(\alpha)$ unchanged; $Z(\alpha)$ is a periodic function of α with unit period, for S increases in both sums of Eq. (2.46) by integer steps so that the replacement $\alpha\rightarrow\alpha+1$, $S\rightarrow S+1$ leaves $Z(\alpha)$ unchanged. It is important to note that these two properties of the partition function which are equivalent to Theorem 3 of Byers and Yang[15] are not, in this case, related to the existence of a Meissner effect as in the situation discussed by Byers and Yang. Rather, these properties follow immediately from the assumption of cylindrical symmetry and negligible ring thickness; they are manifested irrespective of whether the system is superconducting or normal. What distinguishes the former from the latter is the magnitude of the periodic variations with flux. In the normal state these variations are negligibly small, while in the superconducting state they are appreciable. A criterion for distinguishing between these two cases is presented in Sec. 3.

3. RESULTS OF THE MODEL

Ground-State Energy and Current

The behavior pertaining to the absolute zero of temperature is reached if the temperature is so low that thermal excitations of all oscillators can be ignored. From the expression for the partition function, Eqs. (2.46) and (2.47), one sees that this requires $\beta\tau\gg1$, or, from Eq. (2.29), $kT\ll\epsilon_F/N$, which is unrealistic for macroscopic systems. Nevertheless it is of interest to consider this state. The factor $\exp\{-2\beta\tau[(S-\alpha)^2+D^2]\}$ in the partition function eliminates all contributions to the sums over D and S except the contribution of the ground state which is characterized by $D=0$, $S=S(\alpha)$. The quantity $S(\alpha)$ is defined as the integer nearest α. Using Eqs. (2.29), (2.36), and (2.41) one obtains for the ground-state energy

$$E_g(\alpha)=N\epsilon_F\{\tfrac{1}{3}+\tfrac{1}{2}(N-1)(J_0/\epsilon_F)$$
$$-\tfrac{1}{2}\sum_n{}'(J_n/\epsilon_F)+W(J)+16N^{-2}[S(\alpha)-\alpha]^2\}, \tag{3.1}$$

where

$$W(J)\equiv4N^{-2}\sum_n{}'[(1+NJ_n/2\epsilon_F)^{1/2}-1]\,|\,n\,| \tag{3.2}$$

is the contribution of the zero-point energy of the A mode oscillators in units of $N\epsilon_F$. It should be noted that the quantities $NJ_n/2\epsilon_F$ must, for attractive interactions, be less than unity in magnitude in order that the ground-state energy be real. This implies that the square root in Eq. (3.2) can be expanded in powers of the interaction.[16] The result of Eq. (3.1), obtained by ignoring the trilinear terms in the boson Hamiltonian, must be expected to agree with that obtained by carrying out to all orders a perturbation method which consistently approximates the electron spectrum by Tomonaga's linear relation $\epsilon_n\propto\tau\,|\,n\,|$. If the applicability criteria of Tomonaga or Gutfreund and Schick are satisfied, the fact that the ground-state energy is an analytic function of the interaction strength is a confirmation, in this model, of Hohenberg's result that there is no ODLRO in the one-dimensional system. The existence of ODLRO leads to an expression for E_g which

[15] N. Byers and C. N. Yang, Phys. Rev. Letters **7**, 46 (1961).

[16] This is also the case for electron-phonon interactions. See Engelsberg and Varga (Ref. 13).

is not analytic in the interaction strength as in the Bardeen-Cooper-Schrieffer[17] theory of superconductivity.

For repulsive interactions, the ground-state energy is always real and the strength of the Fourier components are not necessarily restricted to $NJ_n/2\epsilon_F<1$ in order that the formalism be valid. Thus the square root in Eq. (3.2) can no longer, in general, be expanded in a power series and the usual Rayleigh-Schrödinger perturbation series will fail to converge. A thorough comparison of the Tomonaga model and perturbation procedures is given by Engelsberg and Varga.[13]

The current in the ground state is obtained from

$$I_g(\alpha)=-(e/2\pi\hbar)[dE_g(\alpha)/d\alpha],$$

with the result

$$I_g(\alpha)=-2I_g^{\max}[\alpha-S(\alpha)],\qquad kT\ll\epsilon_F N^{-1},\quad(3.3)$$

where

$$I_g^{\max}=e\hbar N/4\pi mR^2.\quad(3.4)$$

Equation (2.30) has been used to obtain this result.

Free Energy and Current

The free energy and current of the system will now be obtained for more realistic temperatures in the range $1\gg kT/\epsilon_F\gg N^{-1}$. An approximation for the partition function which retains the essential periodic dependence on α can be obtained by expanding the exponent of the summand of Eq. (2.46) about its extremum value. This occurs for $D=S-\alpha=0$, and one needs to retain only quadratic terms in a power series in D and $S-\alpha$. The summand is, then, approximately

$$\exp-\beta\{\mathfrak{F}(0,0)+2\tau[C_1(J)D^2+C_2(J)(S-\alpha)^2]\},\quad(3.5)$$

where

$$C_1(J)\equiv 1+(2\tau)^{-1}\left.\frac{\partial\mathfrak{F}[(S-\alpha)^2,D^2]}{\partial D^2}\right|_{D=S-\alpha=0},$$

$$C_2(J)\equiv 1+(2\tau)^{-1}\left.\frac{\partial\mathfrak{F}[(S-\alpha)^2,D^2]}{\partial (S-\alpha)^2}\right|_{D=S-\alpha=0}.$$

The value of $\mathfrak{F}(0,0)$ is obtained from Eqs. (2.36), (2.41), and (2.47),

$$\mathfrak{F}(0,0)=N\epsilon_F\{\tfrac{1}{3}+\tfrac{1}{2}(N-1)(J_0/\epsilon_F)-\tfrac{1}{2}\sum_n{}'(J_n/\epsilon_F)$$
$$+W(J)-(\pi^2/24)(kT/\epsilon_F)^2[1+Y(J)]\},\quad(3.6)$$

where $W(J)$ is given by Eq. (3.2) and where

$$Y(J)=-(48\beta\epsilon_F/N\pi^2)\times\sum_{n>0}\ln\{1-\exp[-\beta\tau\Omega_A(0,J_n)n]\}.$$

For the special case $J_n=0$ for all n, one obtains $Y(0)=1$ since the sum may be replaced by an integral in the temperature range being considered. The neglect of higher powers of D and $(S-\alpha)$ in the exponent can be justified by carrying out the expansion to fourth order and verifying that the corrections are negligible. The functionals $C_1(J)$ and $C_2(J)$ are of order unity and must be positive. It can be shown[12] that when the formalism of Sec. 2 is valid according to Tomonaga's applicability criterion, these functionals are indeed positive.

With the approximation of Eq. (3.5) for the summand of Eq. (2.46), the partition function becomes a sum over Gaussian functions. On replacing the half-odd integral values of D and S, which appear in the second term of Eq. (2.45), by $S=l-\frac{1}{2}$, $D=m-\frac{1}{2}$, where l and m are integers, all four sums in the partition function are of the form

$$\sum_l \exp[-K(l-a)^2],$$

with K equal to $2\beta\tau C_1(J)$ or $2\beta\tau C_2(J)$ and a equal to 0, α, $\frac{1}{2}$, or $\alpha+\frac{1}{2}$. Since $C_1(J)$, $C_2(J)$ are of order unity and since it is assumed that $\beta\tau\ll 1$, one has $K\ll 1$. It is convenient therefore to make use of the Poisson sum formula[18]

$$\sum_{l=-\infty}^{\infty}\exp[-K(l-a)^2]=(\pi/K)^{1/2}\times[1+2\sum_{n=1}^{\infty}\{\exp(-\pi^2n^2/K)\}\cos 2\pi na].$$

Keeping only the term with $n=1$ on the right, which is sufficient for our purposes, and using this expression to approximate the sums in the partition function, one obtains

$$Z(\alpha)\approx\{\exp[-\beta\mathfrak{F}(0,0)]\}2[\pi/2\beta\tau C_1(J)]^{1/2}\times[\pi/2\beta\tau C_2(J)]^{1/2}$$
$$\times(1+4\{\exp[-\pi^2/2\beta\tau C_3(J)]\}\cos 2\pi\alpha),\quad(3.7)$$

where $C_3^{-1}(J)=C_1^{-1}(J)+C_2^{-1}(J)$, and is of order unity. Since $\tau=8\epsilon_F N^{-1}$, from Eq. (2.29), it can be seen that the term $[\pi/2\beta\tau C_1(J)]$, originating from the sum over D, contributes to the free energy a term of order ln which is negligible compared to other terms of order N in the thermodynamic limit. The only extensive contribution to the free energy from the sum over D is contained in the term $\mathfrak{F}(0,0)$ which is simply the contribution of the states in the ensemble characterized by $D=0$. Thus the statement made in Sec. 2 that the contribution of all states with $D\neq 0$ to the partition function can be ignored is verified. Further, if $\alpha=0$, a similar argument shows that only the contribution of states with $S=0$ need be considered. The contribution of states with $S\neq 0$ is only important when a magnetic field is present and the response of the system to that field is of interest.

[17] J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[18] H. T. Davis, *The Summation of Series* (Principia Press, Inc., San Antonio, Tex., 1962), p. 234.

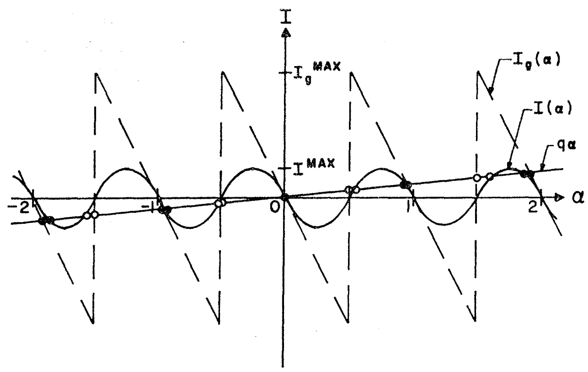

FIG. 1. Schematic plot of current I versus α, the flux measured in units of $2\pi\hbar c/e$. The intercepts of the two curves $I(\alpha)$ and $q\alpha$ represent solutions of Eqs. (3.10) and (3.12), while the intercepts of $I_g(\alpha)$ and $q\alpha$ represent solutions of Eqs. (3.3) and (3.12). In both cases, the filled circles correspond to stable solutions while the open circles correspond to unstable ones.

The free energy obtained from Eq. (3.7) is

$$F(\alpha)=-\beta^{-1}\ln Z(\alpha)\approx\mathfrak{F}(0,0)-4kT \times\{\exp[-\pi^2NkT/16\epsilon_F C_3(J)]\}\cos 2\pi\alpha, \quad (3.8)$$

where flux-independent contributions of order $\ln N$ have been ignored and where the approximation $\ln(1+x)\approx x$ for small x has been used. The current is

$$I(\alpha)=-(e/2\pi\hbar)[dF(\alpha)/d\alpha] \quad (3.9)$$

or

$$I(\alpha)=-I^{\max}\sin 2\pi\alpha, \qquad 1\gg kT/\epsilon_F\gg N^{-1}, \quad (3.10)$$

where

$$I^{\max}=I_g^{\max}(\pi NkT/2\epsilon_F)\exp[-\pi^2NkT/16\epsilon_F C_3(J)], \quad (3.11)$$

and $I_g^{\max}$ is given in Eq. (3.4).

Current and Free-Energy Minima

In addition to the relations of Eqs. (3.3) and (3.10) between the current I and the flux $\Phi_B=hc\alpha/e$, there is also the electromagnetic relation, valid in the absence of an external field,

$$\Phi_B=c\mathfrak{L}I$$

or

$$I=q\alpha, \quad (3.12)$$

where

$$q=2\pi\hbar/e\mathfrak{L}, \quad (3.13)$$

and where $\mathfrak{L}$ is the self-inductance of the system. It is easy to verify with the aid of Eq. (3.9) that the total free energy of the system $F_T(\alpha)$, given by

$$F_T(\alpha)=F(\alpha)+\tfrac{1}{2}\mathfrak{L}I^2(\alpha),$$

has an extremum with respect to variations in α when Eq. (3.12) is satisfied. The relations of Eqs. (3.3) and (3.12) or Eqs. (3.10) and (3.12) represent two equations to determine I and the corresponding flux $hc\alpha/e$. If these equations have stable nonzero solutions for the current, the total free energy of the system has relative minima at these values of the current. The periodic functions $I_g(\alpha)$ of Eq. (3.3) and $I(\alpha)$ of Eq. (3.10) are plotted schematically along with the straight line $q\alpha$ of Eq. (3.12) in Fig. 1. The stable solutions of the simultaneous equations, which correspond to a minimum of $F_T(\alpha)$, are indicated by dots. The corresponding values of α and I indicate flux quantization and persistent currents. The solutions denoted by circles are unstable and can be ignored.

It is clear from Fig. 1 that there are no solutions of the equations for values of the flux such that $I^{\max}/q<|\alpha|$. In particular, there are no nonzero solutions at all if $I^{\max}/q<\frac{1}{2}$. Since the self-inductance of a single loop of radius R and thickness $2r_0$ is approximately

$$\mathfrak{L}\approx(4\pi R/c^2)\ln(R/2r_0), \qquad R\gg r_0,$$

one obtains for the ratio $I_g^{\max}/q$, appropriate for $kT/\epsilon_F\ll N^{-1}$, from Eqs. (3.4) and (3.13),

$$I_g^{\max}/q=(e^2/mc^2)(N/2\pi R)\ln(R/2r_0).$$

Since the coefficient of the logarithm is much less than unity for reasonable densities, the ratio $I_g^{\max}/q$ is small compared to the value $\frac{1}{2}$ so that the total free energy exhibits no minimum except for vanishing current. Because $I^{\max}$, Eq. (3.11) is smaller than $I_g^{\max}$; this result is also obtained for higher temperatures.

The results are qualitatively different, however, if, instead of a single ring, a cylinder, consisting of many such rings stacked closely one upon the other, is considered. In this case one finds

$$I_g^{\max}/q=(e^2/2mc^2)(N/R)(R/r_0).$$

This ratio can exceed the value $\frac{1}{2}$ even for relatively small systems, and the total free energy will, in general, exhibit a considerable number of relative minima at nonzero values of the current and flux. The values of the latter, obtained from Eqs. (3.3) and (3.12), are

$$\alpha=S(\alpha)/[1+(q/2I_g^{\max})]. \quad (3.14)$$

In the thermodynamic limit, N, $R\to\infty$, N/R, r_0 constant, the ratio $I_g^{\max}/q$ increases without limit. The total free energy exhibits an infinite number of relative minima, which occur, according to Eq. (3.14), at integer values of α. Thus this system exhibits thermodynamically stable persistent currents and flux quantization in units of hc/e for temperatures $kT/\epsilon_F\ll N^{-1}$ which are, however, unrealistic for macroscopic systems.

For temperatures $kT/\epsilon_F\gg N^{-1}$ the ratio $I^{\max}/q$, obtained from Eq. (3.11), is

$$I^{\max}/q=(I_g^{\max}/q)(\pi NkT/2\epsilon_F) \times\exp[-\pi^2NkT/16\epsilon_F C_3(J)].$$

As $NkT/\epsilon_F\gg1$ by assumption and $C_3(J)$ is of order unity, one will normally find $I^{\max}/q$ much less than $\frac{1}{2}$ so that the total free energy will only exhibit a minimum for vanishing current. In particular this is the case in the thermodynamic limit for any geometry since $I^{\max}/q$ tends exponentially towards zero for any finite temperature with increasing N. In this limit then, the one-dimensional system does not exhibit thermodynamically stable persistent currents or flux quantization.

4. RELATION BETWEEN SUPERCONDUCTING PROPERTIES

In light of the above result, it is of interest to examine the relations between the following three properties generally associated with superconductivity:

(1) existence of ODLRO;
(2) existence of flux quantization in equilibrium;
(3) existence of persistent currents.

The relationships between these properties is such that (2) follows from (1) and (3) from (2). The fact that the existence of ODLRO implies the existence of flux quantization has been established by Yang[3] and by Bloch.[19] However, the example of the one-dimensional Fermi gas shows that the converse is not true. This system cannot exhibit ODLRO as shown by Hohenberg[2] but can, at extremely low temperatures and in particular geometries, exhibit flux quantization as shown in Sec. 3. While it is obvious that the existence of flux quantization in equilibrium implies the existence of persistent currents, the converse is again not true, since it is possible that currents can persist for long times for reasons other than thermodynamic stability. This possibility has been examined by Little,[4] and by Ambegoakar and Langer.[5]

ACKNOWLEDGMENT

The author is deeply indebted to Dr. Felix Bloch for his help and constant guidance.

APPENDIX: COMPLETENESS OF BOSON STATES

Consider the wave functions

$$\psi_m(P',\nu_\sigma)=c\prod_\sigma\prod_n{}'\,(a_{n,\sigma}^*)^{N_{n,\sigma}}\psi_0(P,\nu_\sigma)\qquad\text{(A1)}$$

defined in Eq. (2.20). Provided that the boson occupation numbers $N_{n,\sigma}$ are not too large, this state is within the subspace S. In the spirit of the Tomonaga method, we will assume that all statistically important states are in the subspace S and will, with that assumption, investigate the completeness of the states ψ_m. This will be done by comparing the number of wave functions $\psi_m(P,\nu_\sigma)$, characterized by a given momentum P and quantum numbers ν_σ, with the number of the usual wave functions $\varphi_m(P,\nu_\sigma)$, specified by fermion particle and hole occupation numbers, of the same momentum P and quantum numbers ν_σ. As the set of wave functions $\varphi_m(P,\nu_\sigma)$ are orthogonal and complete, the orthogonal set of functions of Eq. (A1) will be complete if these numbers are equal.

From Eq. (2.27) the momentum P of the state $\psi_m(P,\nu_\sigma)$ is

$$P=\hbar R^{-1}[\tfrac{1}{2}N(\nu_1+\nu_{-1})+P_{+,1}+P_{+,-1}-P_{-,1}-P_{-,-1}],$$

where

$$P_{\pm,\sigma}\equiv\sum_{l>0}lN_{\pm l,\sigma},$$

and where the integers $N_{\pm l,\sigma}$ for all l comprise the particular set of integers which characterize the state ψ_m. Suppose that the four momenta, $P_{+,\sigma}$, $P_{-,\sigma}$ have the particular values r, s, t, u, so that

$$P=\hbar R^{-1}[\tfrac{1}{2}N(\nu_1+\nu_{-1})+r+s-t-u].\qquad\text{(A2)}$$

Then, if $C(r)$ is defined as the number of configurations of the integers $N_{\pm l,\sigma}$ such that

$$\sum_{l>0}lN_{\pm l,\sigma}=r,\qquad r\geq 0$$

and

$$C(r)=0,\qquad r<0$$

the number of different states for which $P_{+,\sigma}$, $P_{-,\sigma}$ have these same values is $C(r)C(s)C(t)C(u)$. But the positive values of r, s, t, u are subject only to Eq. (A2), so that the total number of states characterized by the quantum numbers P, ν_σ is given by

$$\sum_{r,s,t}C(r)C(s)C(t)C(u).\qquad\text{(A3)}$$

The generating function $G_B(x)$ defined by

$$G_B(x)\equiv\sum_{r\geq 0}C(r)x^r,\qquad |x|<1\qquad\text{(A4)}$$

is well known from number theory where $C(r)$ is designated as the number of unrestricted partitions of the number r. The generating function is

$$G_B(x)=\prod_{r=1}^{\infty}(1-x^r)^{-1}.\qquad\text{(A5)}$$

The number of states $\psi_m(P,\nu_\sigma)$ is now known in principle. We will now count the fermion states in a similar manner.

We define the fermion particle and hole occupation numbers $n_{l,\sigma}'$, $n_{l,\sigma}''$ appropriate to a state $\varphi_m(P,\nu_\sigma)$ by

$$\begin{aligned}n_{l,\sigma}&=n_{l-f,\sigma}', & l&>f=n_M+\nu_\sigma\\ &=1-n_{l-f,\sigma}'', & f&\geqslant l\geqslant 0\\ n_{l,\sigma}&=1-n_{l+g,\sigma}'', & 0&>l\geq -g=-n_M+\nu_\sigma\\ &=n_{l+g,\sigma}', & -g&>l.\end{aligned}\qquad\text{(A6)}$$

The value $n_{l,\sigma}$ which is zero or 1, is the eigenvalue of $c_{l,\sigma}^*c_{l,\sigma}$ operating on the state $\varphi_m(P,\nu_\sigma)$. The index m stands for a particular set of values of $n_{l,\sigma}$ for all l.

[19] F. Bloch, Phys. Rev. **137**, A787 (1965).

Equal numbers of particles and holes imply the relations

$$\sum_{r>0} n_{r,\sigma}' = \sum_{r\geq 0} n_{-r,\sigma}'', \tag{A7}$$

$$\sum_{r>0} n_{-r,\sigma}' = \sum_{r\geq 0} n_{r,\sigma}''. \tag{A8}$$

It should be noted that the momenta of "holes" cannot exceed values of the order n_M as seen from Eq. (A6), so that a limit on the sums on the right side of Eqs. (A7) and (A8) should appear. However, again taking the view that those states which are most probable statistically will not have holes of such large momenta, one may set these upper limits to infinity. The momentum P of a state $\varphi_m(P, \nu_\sigma)$ is given by

$$P = \hbar R^{-1}[\tfrac{1}{2}N(\nu_1+\nu_{-1}) + P_{+,1}' + P_{+,-1}' - P_{-,1}' - P_{-,-1}'],$$

where

$$P_{+,\sigma}' = \sum_{l>0} l n_{l,\sigma}' + \sum_{l\geqq 0} l n_{-l,\sigma}''$$

and

$$P_{-,\sigma}' = \sum_{l>0} l n_{-l,\sigma}' + \sum_{l\geqq 0} l n_{l,\sigma}''.$$

The counting proceeds as before. We define $C'(r)$ as the number of sets of numbers $n_{l,\sigma}'$, $n_{l,\sigma}''$, which are zero or 1, such that Eqs. (A7) and (A8) are satisfied, and such that

$$\sum_{l>0} l n_{l,\sigma}' + \sum_{l\geqq 0} l n_{-l,\sigma}'' = r, \qquad r \geq 0$$

and

$$C'(r) = 0, \qquad r<0. \tag{A9}$$

Then the number of wave functions $\varphi_m(P, \nu_\sigma)$ is given by Eq. (A3) with all numbers C replaced by C'. Therefore, if $C(r) = C'(r)$, the number of boson and fermion states are equal and the boson states are complete. We will show that this is the case by constructing the generating function

$$G_F(x) \equiv \sum_{r\geqq 0} C'(r) x^r, \qquad |x|<1 \tag{A10}$$

and showing that it is identical to $G_B(x)$. We construct $G_F(x)$ as follows: The number of sets of integers $n_{l,\sigma}'$, for which the equation $\sum_{l>0} l n_{l,\sigma}' = t$ is satisfied, is readily seen to be equal to the coefficient of x^t in the expansion of the function

$$g_F'(x) = \prod_{l>0} (1+x^l).$$

Similarly the number of sets of numbers $n_{-l,\sigma}''$ for which the equation $\sum_{l\geq 0} l n_{-l,\sigma}'' = q$ is satisfied is the coefficient of x_q in the expansion of

$$g_F''(x) = \prod_{s\geqq 0} (1+x^s).$$

Therefore the total number of sets of numbers $n_{l,\sigma}'$, $n_{-l,\sigma}''$—such that Eq. (A9) is satisfied but not necessarily Eqs. (A7) and (A8)—is the coefficient of x^r in the expansion of $g_F'(x) g_F''(x)$. The generating function $G_F(x)$ is now seen to be given by

$$G_F(x) = (2\pi)^{-1} \int_0^{2\pi} d\varphi \prod_{l=1}^{\infty} [1+\exp(i\varphi) x^l] \times \prod_{s=0}^{\infty} [1+\exp(-i\varphi) x^s].$$

The integration over φ selects only those terms in the product with equal numbers of particle and hole contributions so that Eqs. (A7) and (A8), as well as (A9), are satisfied. By multiplying the products together and using the fact that $G_F(x)$ is real, the above expression can be written

$$G_F(x) = \pi^{-1} \int_0^{2\pi} d\varphi \cos\varphi [\cos\varphi \prod_{l=1}^{\infty} (1+2x^l \cos 2\varphi + x^{2l})].$$

The integration is immediately accomplished on using the identity

$$\cos\varphi \prod_{l=1}^{\infty} (1+2x^l \cos 2\varphi + x^{2l}) = G_B(x) x^{-1/8} \sum_{s=0}^{\infty} x^{p(s)} \cos(2s+1)\varphi,$$

where

$$p(s) = \tfrac{1}{2}(s+\tfrac{1}{2})^2,$$

and where $G_B(x)$ is given in Eq. (A5). This identity is derived by equating the definition of the second theta function of Jacobi[20] $\theta_2(\varphi, q)$ to its infinite product representation, setting $q = x^{1/2}$ and rearranging. The integration yields

$$G_F(x) = G_B(x).$$

This completes the proof that the set of functions $\psi_m(P, \nu_\sigma)$ are complete in the subspace S.

[20] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1963), 4th ed., pp. 464, 470, and 472.