Regge-Pole Model with L-S Coupling*†

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A Regge-pole model for $PN \to P'N$ and $PN \to VN$ is proposed. The model is based on the L-S coupling scheme and describes the coupling of the particles involved in the reaction to the trajectory having definite quantum numbers J^{PG}. The scheme determines the structure of the amplitude, which in turn characterizes the decay density matrix of the produced resonance. Kinematical factors and the threshold behavior are discussed on the basis of the analyticity assumption.

I. INTRODUCTION

RECENTLY, Regge-pole models[1] have been successfully applied to the analysis[2] of many elastic and inelastic two-body processes. The aim of the present paper is to develop a Regge-pole model for the reactions $PN \to P'N$ [3] and $PN \to VN$.[4] The problem of general spin effects has also been considered by other authors.[5] We have a different approach—in which the effect of the quantum numbers J^{PG} of the exchange trajectory are taken into account by imposing the restrictions of the L-S coupling scheme on the amplitude. The present paper is a continuation of our previous investigations.[6]

The notion of helicity amplitude[7] and the use of their crossing[8] and transformation properties have improved the formulation of the model. Such a formulation has been proposed by Muzinich,[9] and Gottfried and Jackson.[10] In particular, the latter authors have pointed out the relation between the decay density matrix of the produced resonance and the exchanged quantum numbers J^P. They have also given the method for calculating the density matrix, assuming that the amplitude in the crossed-channel c.m. system is known. This calculation is briefly summarized in the next section, together with the kinematics of the reaction and the conventions adopted in this paper.

In Sec. III, we derive the Regge-pole contribution which has been used in our previous work.[6] In this expression for the Regge-pole contribution, there remains an undetermined factor which exhibits the structure of the helicity amplitude. By this structure, we mean the relations between the components of the helicity amplitude. This ambiguity is due to the fact that no information about the quantum numbers J^{PG} of the exchanged trajectory has been put into the helicity amplitude. In addition, there are momentum-dependent kinematical factors. In our previous work, we introduced an *ad hoc* assumption: namely, that the structure and the kinematical factors are taken into account by the requirement that the Regge-pole contribution reduces to the corresponding Born term when it is continued to the mass shell of the crossed channel.

In Sec. IV, we discuss the role of the L-S coupling scheme in the Born term by studying some specific examples. It is found that the quantum numbers J^{PG} of the exchanged particle determine the possible orbital angular momenta L and total spins S in each vertex, and that the L-S scheme determines the structure of the Born amplitude. This scheme can be naturally incorporated in a Regge-pole model via the Sommerfeld-Watson transform.

In Sec. V, the L-S coupling scheme is introduced into our model, replacing our previous *ad hoc* assumption. The coupling coefficients in the angular momentum plane give rise to kinematic cuts, which can be eliminated consistently. If the results of Sec. V are supplemented by the kinematical factors obtained from the Born term, we have a complete rule for writing down the Regge-pole contribution.

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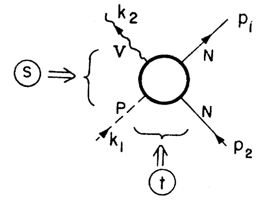

FIG. 1. Diagram for the reaction $PN \rightarrow VN$.

In Sec. VI, we discuss the kinematical factors and the threshold behavior in connection with analyticity of the amplitude. Effects of the unequal masses of the mesons are discussed briefly within the frame of the Freedman-Wang method.[11] Finally, the results are summarized in Sec. VII.

II. KINEMATICS AND HELICITY AMPLITUDES

We consider the production of a vector meson V,

$$P+N \rightarrow V+N\,, \tag{2.1}$$

where P represents a pseudoscalar particle and N the nucleon. The channel, where the Regge pole appears as an intermediate state, will be called the s channel; and the reaction channel (2.1) will be called the t channel. Since we are working with the helicity amplitudes,[7] only the momenta in the c.m. system are relevant. The notation used is shown in Fig. 1 and Table I.

The quantities in the s channel are used extensively, and their relations to the invariants s and t are given by[12]

$$z=\cos\theta_s=\frac{2t+s-\sum_i m_i^2}{4p_sk_s}\,,$$

$$p_s^2=\tfrac{1}{4}(s-4M^2)\,,$$

$$k_s^2=[s-(m_V-m_P)^2][s-(m_V+m_P)^2]/4_s\,, \tag{2.2}$$

where M = nucleon mass, m_V = vector-meson mass, and m_P = mass of the pseudoscalar particle.

The invariant t is related to the incident lab momentum p_L by

$$t=m_P^2+M^2+2M(p_L^2+m_P^2)^{1/2}\,, \tag{2.3}$$

and the invariant s is related to the production angle θ_t in the c.m. system of the channel (2.1) through

$$s=\tfrac{1}{2}[\sum_i m_i^2-t-(M^2-m_P^2)(M^2-m_V^2)/t \times 4p_tk_t\cos\theta_t]\,,$$

$$p_t^2=[t-(M-m_P)^2][t-(M+m_P)^2]/4t\,,$$

$$\mathrm{k}_t^2=[t-(M-m_V)^2][t-(M+m_V)^2]4t\,. \tag{2.4}$$

The helicity amplitude F is defined by

$$\langle f|S-1|i\rangle=-(2\pi)^4i\delta^4(P_f-P_i) \times[m/(2k_{10}2k_{20})^{1/2}p_0]\langle f|F|i\rangle\,, \tag{2.5}$$

TABLE I. Kinematics for $PN \rightarrow VN$.

Channel	Reaction	(Energy)^2	(Momentum transfer)^2	Helicity amplitude
s	$P+\bar{V}\rightarrow N+\bar{N}$	s	$-t$	$\langle\lambda_1\bar{\lambda}_2\|F^s\|\lambda_V\rangle$
t	$P+N\rightarrow V+N$	t	$-s$	$\langle\lambda_1\|F^t\|\lambda_2\lambda_V\rangle$

where $|i\rangle$ and $|f\rangle$ are specified by the helicity of the initial and final states, respectively. This convention corresponds to the normalization for the wave functions, $\bar{u}u=1$; $\bar{v}v=-1$; $\epsilon_\mu{}^\lambda\epsilon_\mu{}^{\lambda'}=\delta_{\lambda\lambda'}$. The helicity amplitude has the advantage that it transforms elegantly under crossing[8,9] and Lorentz transformation.[13] This leads to simple relations of the observables to the helicity amplitude of the s channel. The observables we are concerned with are the momentum-transfer distribution and the decay density matrix. The momentum-transfer distribution[9,10] is given by

$$\frac{d\sigma}{ds}=\frac{M^2}{16\pi p_t^2t}\frac{1}{(2s_i+1)}\sum_{\lambda_1,\bar{\lambda}_2,\lambda_V}|\langle\lambda_1,\bar{\lambda}_2|F^s|\lambda_V\rangle|^2\,, \tag{2.6}$$

and the decay density matrix of V in its rest frame is given by[10]

$$\rho_{mm'}=\frac{1}{N}\sum_{\lambda_1\bar{\lambda}_2}\langle m|F^s|\lambda_1\bar{\lambda}_2\rangle\langle\lambda_1\bar{\lambda}_2|F^s|m'\rangle\,, \tag{2.7}$$

where

$$N=\sum_{\lambda_1\bar{\lambda}_2\lambda_V}|\langle\lambda_V|F^s|\lambda_1\bar{\lambda}_2\rangle|^2.$$

For (2.6) and (2.7), it is understood that (s,t) has been continued to the physical region of (2.1).

The helicity amplitude of the reaction (2.1), in general, has 12 components. However, among these components there are only 6 which are independent, due to the P-conservation condition[7]

$$\langle-\lambda_1,-\bar{\lambda}_2|F^s|-\lambda_V\rangle=(-1)^{\lambda_i-\lambda_f}(\eta_N\eta_{\bar{N}}/\eta_P\eta_V) \times\langle\lambda_1,\bar{\lambda}_2|F^s|\lambda_V\rangle\,, \tag{2.8}$$

where η_x is the intrinsic parity of x, $\lambda_i=\lambda_1-\bar{\lambda}_2$, and $\lambda_f=\lambda_V$. Note that λ_i and λ_f are the third components of the total angular momentum of the $P\bar{V}$ and $N\bar{N}$ systems, respectively.

The structure of the helicity amplitude is shown in Table II. The observable elements of the decay density matrix[9] of V are ρ_{00}, $\rho_{1,-1}$, and Reρ_{10}. Their relation to

TABLE II. Structure of helicity amplitude of Eq. (2.1).

λ_1 / λ_2 / $\lambda_\pm$	$+\frac{1}{2}$, $+\frac{1}{2}$	$+\frac{1}{2}$, $-\frac{1}{2}$	$-\frac{1}{2}$, $+\frac{1}{2}$	$-\frac{1}{2}$, $-\frac{1}{2}$
$+1$	χ_1	χ_2	$-\chi_4$	$+\chi_3$
0	χ_5	χ_6	χ_6	$-\chi_5$
-1	χ_3	χ_4	$-\chi_2$	$+\chi_1$

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the helicity amplitude is explicitly given by

$$N\rho_{11}=|\chi_1|^2+|\chi_2|^2+|\chi_3|^2+|\chi_4|^2,$$
$$N\rho_{1,-1}=2\,\mathrm{Re}(\chi_1\chi_3^*+\chi_2\chi_4^*),$$
$$N\,\mathrm{Re}\rho_{10}=\mathrm{Re}[(\chi_1-\chi_3)\chi_5^*\times(\chi_2-\chi_4)\chi_6^*]. \quad (2.9)$$

If one uses the simple pole term then for π exchange, only $\chi_5\neq 0$ which implies $\rho_{00}=1$, $\rho_{1,-1}=\mathrm{Re}\rho_{10}=0$; and for 1^- exchange $x_5=x_6=0$ leading to $\rho_{00}=0$, $\rho_{1,-1}\neq 0$, $\rho_{10}=0$. These examples show the correlation between the density matrix and the structure of the helicity amplitude. This structure has been shown to be related to the quantum numbers J^{P10} and C-parity[14] of the intermediate system in the s channel for physical values of J.

In this paper, we wish to study the structure of the helicity amplitude in the Regge-pole model, where J may take unphysical values.

III. REGGE-POLE CONTRIBUTION

We shall discuss the Regge-pole contribution from the point of view of the Sommerfeld-Watson transform, following the approach of Ref. 15 with certain modifications. Particular attention is paid to the continuation from the s to the t channel.[16] The required analytic properties of the rotation matrix $d_{\lambda_1\lambda_2}{}^J(z)$ are summarized in Appendix A.

In the spin-zero case one can apply the Sommerfeld-Watson transform to the expansion of the amplitude in terms of the representations of the rotation group $P_l(z)$. When spin is taken into account $P_l(z)$ is replaced by the appropriate representation: $d_{\lambda_i\lambda_f}{}^J(z)$.

The partial-wave expansion of the helicity amplitude is given by

$$\langle\lambda_1\bar{\lambda}_2|F^s|\lambda_V\rangle=\sum_{J=\lambda_m}^{\infty}(2J+1)$$
$$\times\langle\lambda_1\bar{\lambda}_2JM|F^s|\lambda_VJM\rangle d_{\lambda_i\lambda_f}{}^J(z), \quad (3.1)$$

where

$$\lambda_m=\max(|\lambda_1|,|\lambda_2|).$$

In the following, we restrict ourselves to the s-channel helicity amplitude; therefore the superscript will be omitted. The explicit expression for $d_{\lambda_i\lambda_f}{}^J(z)$ is given by (A1) and (A2). The s-channel helicity amplitude contains the following kinematic singularities: (a) a common factor $[\frac{1}{2}(1-z)]^{a/2}[\frac{1}{2}(1+z)]^{b/2}$, where $a=|\lambda_1-\lambda_2|$ and $b=|\lambda_1+\lambda_2|$; (b) kinematical factors denoted by $c_{\lambda_f\lambda_i}$. They appear from the products of the helicity wave functions.[17,18] Later we shall discuss these factors by the old-fashioned method of expanding the invariant amplitude.[19] To remove these kinematical factors we introduce $f_{\lambda_f\lambda_i}(s,t)$:

$$\langle\lambda_1\bar{\lambda}_2|F|\lambda_V\rangle=c_{\lambda_i\lambda_f}[\tfrac{1}{2}(1-z)]^{a/2}$$
$$\times[\tfrac{1}{2}(1+z)]^{b/2}f_{\lambda_f\lambda_i}(s,t). \quad (3.2)$$

Then (3.1) becomes

$$f_{\lambda_f\lambda_i}(s,t)=\sum_{J=\lambda_i}(2J+1)P_{J-\lambda_i}{}^{(a,b)}(z)f'_{\lambda_f\lambda_i}{}^J(s), \quad (3.3)$$

where

$$f_{\lambda_f\lambda_i}{}'=N_{\lambda_i\lambda_f}f_{\lambda_f\lambda_i}{}^J(s);$$
$$f_{\lambda_f\lambda_i}{}^J\equiv\langle\lambda_1\bar{\lambda}_2JM|F|\lambda_VJM\rangle c_{\lambda_f}f_i^{-1}.$$

In order to perform a Somerfeld-Watson transform on (3.3), $f_{\lambda_f\lambda_i}{}'^J(s,t)$ must be regular in the J plane. Otherwise we would obtain contributions other than Regge poles and the background integral, which would arise from the kinematical branch cut in the J plane. This kinematical branch cut comes from the normalization of $d_{\lambda_i\lambda_f}{}^J(z)$, and is fixed by the $n_{\lambda_i\lambda_f}{}^J$ given in Appendix A. This condition is known to be satisfied for particles (with spin) scattering in a potential that belongs to a certain class.[20] In this section we shall assume that the condition is satisfied in general.

Using the symmetry property $P_m{}^{(a,b)}(-z)=(-1)^n\times P_n{}^{(b,a)}(-z)$ (n=integer), we define:

$$P_n{}^{(a,b)\pm}(z)=\tfrac{1}{2}[P_n{}^{(a,b)}(z)\pm P_n{}^{(b,a)}(-z)]$$
$$=P_n{}^{(a,b)}(z)\quad\text{for } n \text{ even (odd)}$$
$$=0\quad\text{for } n \text{ odd (even)}.$$

Then, as in the spin case,[21] we split $f_{\lambda_f\lambda_i}$ into positive and negative signature parts:

$$f_{\lambda_f\lambda_i}(s,t)=f_{\lambda_f\lambda_i}{}^+(s,t)+f_{\lambda_f\lambda_i}{}^-(s,t),$$

and

$$f_{\lambda_f\lambda_i}{}^\pm(s,t)=\sum_J(2J+1)P_n{}^{(a,b)\pm}(z)f_{\lambda_f\lambda_i}{}'^J(z). \quad (3.4)$$

Applying the Sommerfeld-Watson transform to (3.4), we obtain

$$f_{\lambda_f\lambda_i}{}^\pm(s,t)=-\frac{1}{2\pi i}\int_{C'}dJ(2J+1)f_{\lambda_f\lambda_i}{}'^J(s)$$
$$\times P_{J-\lambda_i}{}^{(b,a)\pm}(-z)/\sin\pi(J-\lambda_m)-\sum_k(2\alpha^\pm+1)\pi\alpha'$$
$$\times[P_{\beta(s)}{}^{(b,a)\pm}(-z)/\sin\pi\beta(s)]R_{\lambda_f\lambda_i}{}'^\alpha(s), \quad (3.5)$$

where $\beta(s)=\alpha(s)-\lambda_m$. The contour C' is chosen as usual.[15] In (3.5) we have used $(-1)^nP_n{}^{(a,b)\pm}(z)=P_n{}^{(b,a)\pm}(-z)$ to compensate the oscillating sign of $\sin\pi n$ ($n=J-\lambda_m$). Equation (3.5) is valid for the physical region of the s channel: $|z|\leq 1$. To obtain the Regge-

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[16] C.A.M., p. 170.

[17] M. S. Marinov and V. I. Roginskii, Zh. Eksperim. i Teor. Fiz. **46**, 673 (1965) [English transl.: Soviet Phys.—JETP **21**, 444 (1965)].

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[21] C. A. M., pp. 142, 168.

pole contribution to the t channel we have to continue (s,t) into the physical region of this channel and $s\to s-i0$, $t\to t+i0$. In the physical region of the t channel $z\to -|z|-i0$, with $|z|\geq 1$. Note that $P_\beta{}^{(b,a)}(z)$ has a cut on the real axis for $\text{Re}z\leq -1$. The behavior of $P_\beta{}^{(b,a)}(z)$ near the cut[22] is given by (A8). This gives

$$P_\beta{}^{(b,a)\pm}(-z)=P_\beta{}^{(b,a)}(-z)\tfrac{1}{2}(1\pm e^{-i\pi\beta}) -(1/\pi)(-1)^{\lambda_i-\lambda_f}\sin\pi\beta\; Q_\beta{}^{(b,a)}(-z). \quad (3.6)$$

When Eq. (3.6) is substituted into Eq. (3.5), both the Regge-pole term and the background integral contain a Q term. These two cancel each other, as can be seen by applying the Cauchy theorem to $f_{\lambda_f\lambda_i}{}^J(s)Q_\beta{}^{(b,a)}(-z)$ along the closed contour formed by the infinite semicircle and C'. From (3.2) and (3.5), we obtain the Regge-pole contribution to the *t channel*:

$$\langle\lambda_1\bar\lambda_2|F_R|\lambda_V\rangle=-(2\alpha+1)\pi\alpha'[(1\pm e^{-i\pi\beta})/2\sin\pi\beta] \times P_\beta{}^{(b,a)}(-z)[\tfrac{1}{2}(1-z)]^{a/2}[\tfrac{1}{2}(1+z)]^{b/2} \times c_{\lambda_f\lambda_i}R_{\lambda_f\lambda_i}{}'^{\alpha}(s). \quad (3.7)$$

From this equation we note that the signature factor is helicity-independent and asymptotically $P_\beta{}^{(b,a)}(-z)\propto(-z)^\beta\approx(-z)^{\alpha-\lambda_m}$. In (3.7) the normalization factor $N_{\lambda_i\lambda_f}$ is absorbed in $R_{\lambda_f\lambda_i}{}'^{\alpha}(s)$. In the physical region of the s channel, the Regge-pole contribution can be written as

$$\langle\lambda_1\bar\lambda_2|F_R|\lambda_V\rangle=(2\alpha+1)\pi\alpha'[(e^{i\pi\beta}\pm1)/2\sin\pi\beta] \times d_{\lambda_i\lambda_f}{}^{\alpha}(z)R_{\lambda_f\lambda_i}(s)c_{\lambda_f\lambda_i}, \quad (3.8)$$

where $R_{\lambda_f\lambda_i}{}'=N_{\lambda_i\lambda_f}R_{\lambda_f\lambda_i}(s)$. In the spin-zero case, at $s=m_{\text{ex}}{}^2$ (m_{ex}= mass of the exchanged particle), the Regge-pole expression coincides with that of the Born term. This is the basis for the speculation that on the mass shell the Regge poles appear as a particle or resonance depending on whether the trajectory is real or complex there.[23] In order to verify the analogous situation in the nonzero-spin case, we consider the expression for the Born term in the s channel. The Born term is given by

$$\langle\lambda_1\bar\lambda_2|F_B|\lambda_V\rangle=\frac{1}{s-m_{\text{ex}}{}^2}\langle\theta,\lambda_1\bar\lambda_2|B|0,\lambda_V\rangle$$

$$=\frac{1}{s-m_{\text{ex}}{}^2}\sum_{JMJ'M'}\langle\theta,\lambda_1\bar\lambda_2|JM\lambda_1\bar\lambda_2\rangle \times\langle JM\lambda_1\bar\lambda_2|B|J'M'\lambda_V\rangle\langle J'M'\lambda_V|\lambda_V,0\rangle, \quad (3.9)$$

where $\langle\theta,\lambda_1\bar\lambda_2|$ and $\langle 0,\lambda_V|$ are the helicity wave functions of the initial and final two-particle system in the s-channel c.m. frame. The matrix B depends on the spin and the quantum members of the exchanged particle, and it contains the projection operator for the intermediate state. By the property of the helicity representation, $\langle\theta\lambda_1\lambda_2|JM\lambda_1'\lambda_2'\rangle=\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}d_{m\lambda}{}^J(z)N'$, where $\lambda=\lambda_1-\lambda_2$, and by rotational invariance,

$$\langle JM\lambda_1\bar\lambda_2|B|J'M'\lambda_V\rangle=b_{\lambda_i\lambda_f}{}^J\delta_{JJ'}\delta_{J\sigma}, \quad (3.10)$$

where σ the spin of the exchanged particles; we obtain

$$\langle\lambda_1\bar\lambda_2|F_B|\lambda_V\rangle=1/(s-m_\alpha{}^2)b_{\lambda_i\lambda_f}{}^\sigma d_{\lambda_i\lambda_f}{}^\sigma(z). \quad (3.11)$$

We note that Eq. (3.10) is correct only if we choose a certain form for the propagator. For example, if we have a vector-meson intermediate state, the projection operator for the vector meson should be taken as

$$\Theta^{\mu\nu}=\delta_{\mu\nu}-q_\mu q_\nu/q^2,$$

where q=4-momentum of the intermediate state. If, instead of q^2, we take $-m^2$ in the expression for $\Theta^{\mu\nu}$, then off the mass shell $\Theta^{\mu\nu}$ is not a projection operator and Eq. (3.10) is not valid in general. The similarity in form of (3.8) and (3.11) allows us to extend the above speculation to the general case. In (3.8), the $c_{\lambda_f\lambda_i}$ remain undetermined, because no information about the quantum numbers of the intermediate state has been put into the partial-wave expansion. We have proposed[6] the identification

$$c_{\lambda_f\lambda_i}R_{\lambda_f\lambda_i}{}^\alpha|_{\alpha=\sigma}=b_{\lambda_f\lambda_i}{}^\sigma \quad (3.12)$$

in order to determine $c_{\lambda_f\lambda_i}$, since $b_{\lambda_f\lambda_i}{}^\sigma$ can be obtained from the Born term and $R_{\lambda_f\lambda_i}{}^\alpha$ can be fixed by the threshold behavior. It has been indicated before[6] that the identification is meant to approximate the residue near the pole $\alpha=\sigma$, and to take into account the restrictions imposed by the selection rule on the helicity dependence of the residue. These restrictions are of a kinematical nature. In the next section we shall discuss another covariant description which can take these restrictions into account.

IV. STRUCTURE OF THE BORN TERM

It is well known that there are two terms in the coupling between ρ and $N\bar N$. These are usually attributed to the "^3S-wave" and "^3D-wave" couplings of the ρ meson to the $N\bar N$ system. We ask ourselves whether such an interpretation is relativistically covariant and to what degree it is reasonable.

In relativistic kinematics there are two useful representations: the canonical representation[24] and the helicity representation.[7,13] The single-particle canonical is obtained from the rest state by applying the Lorentz transformation.[24] On the other hand, in the helicity representation, the single-particle helicity state is obtained from the rest state by aligning the quantization axis along the direction of the momentum, and then boosting this state in this direction.[25] Both representations are relativistically covariant.

[22] C. A. M., p. 170.

[23] C. A. M., pp. 152, 171; G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962); R. Oehme, Phys. Rev. **130**, 434 (1963).

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[25] E. P. Wigner, Rev. Mod. Phys. **29**, 255 (1957).

In the canonical representation the orbital angular momentum and the total spin of the *two-particle* system can be defined *in the c.m. frame* of these particles.[24,26] In the s channel, this frame is the rest frame of the intermediate state and has a similar role to that of the rest frame in the single-particle-state representations. We denote the canonical state of incident and outgoing particles by $|J\lambda_i L_i S_i\rangle$ and $|J\lambda_f L_f S_f\rangle$, respectively, where $L_{i(f)}$ is the orbital angular momentum and $S_{i(f)}$ is the total spin of the incoming (outgoing) particles in the c.m. frame. The helicity states of the incoming particles are denoted by $|0,\lambda_V\rangle$ and $|\theta,\lambda_1\bar{\lambda}_2\rangle$, respectively.

To separate the kinematical factors of the Born term, we introduce $B_{\lambda_f\lambda_i}{}^{\sigma}$ related to $b_{\lambda_f\lambda_i}{}^{\sigma}$ of Eq. (3.11) by $b_{\lambda_f\lambda_i}{}^{\sigma}=c_{\lambda_f\lambda_i}B_{\lambda_f\lambda_i}{}^{\sigma}$. Now, we decouple $B_{\lambda_f\lambda_i}{}^{\sigma}$ into the canonical states, which gives

$$B_{\lambda_f\lambda_i}{}^{\sigma}=\sum_{if}\langle\sigma\lambda_f\lambda_1\bar{\lambda}_2|\sigma\lambda_f,L_fS_f\rangle\langle\sigma\lambda_fL_fS_f|B|\sigma\lambda_i,L_iS_i\rangle \times\langle\sigma\lambda_iL_iS_i|\sigma\lambda_i,\lambda_V\rangle. \quad (4.1)$$

It has been shown[27] that the relativistic coupling coefficient $\langle\sigma\lambda_f\lambda_1\bar{\lambda}_2|\sigma\lambda_fL_fS_f\rangle$ is the same as the nonrelativistic one,[7] apart from an s-dependent factor, which appears because of a difference in normalization between the nonrelativistic and relativistic wave functions. Omitting this factor, the general coupling coefficient is given by

$$\langle JMLS|JM\lambda_1\lambda_2\rangle=[2L+1)/(2J+1)]^{1/2} \times C(L,S,J;\lambda_1-\lambda_2)C(s_1,s_2,S;\lambda_1,-\lambda_2), \quad (4.2)$$

where $C(j_1j_2, j_1+j_2; m_1m_2)$ is the usual Clebsch-Gordan coefficient. The matrix element $\langle\sigma\lambda_fL_f|B|\sigma\lambda_iL_iS_i\rangle$ is now helicity-independent; it is the reduced matrix element, depending only on the orbital angular momenta of the initial- and final-state systems. Eqs. (3.11) and (4.1) give a covariant description based on the LS coupling scheme. We shall study how far these equations are satisfied by the Born term constructed from the conventional effective Lagrangian.

We consider first the restriction imposed by G-parity conservation. In the meson vertex, G-parity conservation tells us only whether the coupling is allowed or forbidden. The situation in the nucleon vertex is different: Mesons with various quantum numbers J^{PG} can couple to $N\bar{N}$ with different L and S. The allowed L and S are obtained from $P=(-1)^{L+1}$ and $G=(-1)^{L+S+T}$.[27] Using the known G parity, we select a Dirac matrix Γ in $\bar{u}\Gamma v$ such that it has the required transformation property under G conjugation. The results are summarized in Table III. *A priori*, is it not necessary that the two descriptions agree. To illustrate the situation we consider two examples, the π-N charge exchange scattering with ρ exchange and V production with 2^{-+} exchange.

In π-N charge exchange scattering, $\pi\pi$ is coupled to ρ with $L=1$. From Table III, we see that $N\bar{N}$ is coupled to ρ in ^3S and ^3D states. The effective Lagrangian can be written as

TABLE III. Meson-nucleon coupling with $T=1$.

Meson J^{PG}	S	L	Γ
0^{--}	0	0	γ_5
1^{-+}	1	0,2	γ_μ, p_μ
1^{++}	0	1	$\gamma_5\sigma_{\mu\nu}q_\nu$
1^{+-}	1	1	$\gamma_5\gamma_\mu$
2^{+-}	1	1,3	$\gamma_\mu p_\nu, p_\mu p_\nu$
2^{-+}	1	2	$\gamma_5\gamma_\mu p_\nu$
2^{--}	0	2	$\gamma_5 p_\mu p_\nu$

$$\mathfrak{L}=j_\mu{}^N\phi_\mu+j_\mu{}^P\phi_\mu,$$
$$j_\mu{}^N=\bar{u}[(g_V+g_T)\gamma_\mu+g_T(p_\mu/M^2)]v,$$
$$j_\mu{}^\pi=2fk_\mu, \quad (4.3)$$

where

$$k_1=\tfrac{1}{2}(k_1-k_2)_\mu, \quad p_\mu=\tfrac{1}{2}(p-p_2)_\mu. \quad (4.4)$$

The helicity amplitude of the Born term is then

$$\langle\lambda_1\bar{\lambda}_2|B|0\rangle=2fk_\mu\bar{u}_{\lambda_1}[(g_V+g_T)\gamma_\mu +g_T(p_\mu/M^2)]v_{f_2}, \quad (4.5)$$

and the helicity wave functions are

$$u_{\lambda_1}=\left(\frac{p_0+M}{2M}\right)^{1/2}\begin{bmatrix}\xi_{\lambda_1}\\ \dfrac{2\lambda_1 p}{p_0+M}\quad\xi_{\lambda_1}\end{bmatrix};$$

$$v_{\lambda_2}=(-1)^{s_2-\lambda_2}\left(\frac{p_0+M}{2M}\right)^{1/2}\begin{bmatrix}-\dfrac{2\lambda_2 p}{p_0+M}\quad\xi_{\lambda_2}\\ \xi_{\lambda_2}\end{bmatrix};$$

$$\xi_{+1/2}=\begin{pmatrix}\cos\frac{1}{2}\theta\\ \sin\frac{1}{2}\theta\end{pmatrix}, \quad\text{and}\quad \xi_{-1/2}=\begin{pmatrix}-\sin\frac{1}{2}\theta\\ \cos\frac{1}{2}\theta\end{pmatrix}.$$

A straightforward calculation gives the following relations for $B_{\lambda_f\lambda_i}$ of Eq. (4.1):

$$c_{00}B_{00}=2f[(g_V+g_T)k+(g_T/M^2)p^2k],$$
$$c_{10}B_{10}=2f[(g_V+g_T)k\sqrt{2}](p_0/M). \quad (4.6)$$

From (4.1) and Appendix B, we obtain

$$B_{00}{}^{\sigma}=[1/2(2\sigma+1)]^{1/2}[\sigma^{1/2}b_S-(\sigma+1)^{1/2}b_D],$$
$$B_{10}{}^{\sigma}=[1/2(2\sigma+1)]^{1/2}[(\sigma+1)^{1/2}b_S+\sigma^{1/2}b_D], \quad (4.7)$$

where b_S and b_D are the reduced matrix elements corresponding to the ^3S and the ^3D states, respectively. Comparing (4.6) and (4.7) with $\sigma=1$, we see immediately that the γ_μ term in (4.6) represents the S-wave coupling of $N\bar{N}$ to ρ. However, the term proportional to p_μ in (4.6) cannot be identified as the ^3D wave coupling, since this term has no component with $\lambda_f=1$. The presence of such a term is necessary due to the fact that $S=1$ and $\lambda_f=S_3$. We saw that the LS coupling scheme has a remarkable basis, so that the only way to restore

[26] A. McKerrel, Nuovo Cimento 34, 1287 (1964).
[27] J. J. Sakurai, *Invariance Principles and Elementary Particles* (Princeton University Press, Princeton, New Jersey, 1964), p. 223.

the agreement is to change the effective Lagrangian. In order to modify the ^3D-coupling term in (4.3) we use the representations $S=1\leftrightarrow\gamma_\mu$ and $L=2\leftrightarrow(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2)$. Then $j_\mu{}^N$ corresponding to ^3D coupling is represented by the inner product of these two representations;

$$j_\mu{}^N(D)=(g_D/M^2)\gamma_\nu(p_\mu p_\nu-\tfrac{1}{3}\delta_{\mu\nu}p^2)\,.$$

Inserting this into Eq. (4.5), we obtain instead of Eq. (4.6),

$$c_{00}B_{00}=2fk(g_S+\tfrac{1}{3}g_Dp^2/M^2)\,,$$
$$c_{10}B_{10}=2fk(g_S-\tfrac{1}{3}g_Dp^2/M^2(p_0/M)\sqrt{2}\,. \quad (4.8)$$

Comparison of (4.7) and (4.8) shows that the relative magnitude and phase of the ^3S and ^3D to both helicity components is consistent. We also note in (4.6) and (4.7) that $c_{01}B_{01}$ contains an additional factor p_0/M, which arises from the product of the wave functions. We shall take $c_{00}=1$ and $c_{10}=p_0/M$ as kinematical factors. The products of momenta which are multiplied by g_S and g_D, respectively, represent the threshold behavior. We shall see that the same factors appear when we write the invariant amplitude as $T=-A+i\gamma_\mu k_\mu B$.

In the above example the possibility of restoring the consistency of (4.6) and (4.7) depends on the equality of the masses of the incoming and outgoing particles, so that $p_1-p_2=p$ can be used to represent the relative angular momenta of the $N\bar{N}$ system. Next we shall examine what happens if the two incoming particles have different masses as in vector-meson production. We take a nontrivial example of a 2^{-+} meson exchange, which is allowed as far as selection rules are concerned.

The most general effective Lagrangian for this process is

$$\mathfrak{L}=j_{\mu\nu}{}^M\phi_{\mu\nu}+j_{\mu\nu}{}^N\phi_{\mu\nu}\,,$$
$$j_{\mu\nu}{}^M=f_1k_\mu\omega_\nu+(f_2/m_0{}^2)k_\mu k_\nu k_\sigma\omega_\sigma\,,$$
$$j_{\mu\nu}{}^N=\gamma_5\gamma_\mu p_\nu\,, \quad (4.9)$$

where ω_μ is the polarization vector of V, and k_μ is arbitrarily taken to be the P momentum.[28] The appearance of f_1 and f_2 in $j_{\mu\nu}{}^M$ is due to the fact that there are two possible couplings between the mesons: the P- and the F-wave coupling. The $N\bar{N}$ can only be coupled to the 2^{-+} meson in the ^3D state. Then the Born term is given by

$$\langle\lambda_1\bar{\lambda}_2|B|\lambda_V\rangle=[f_1k_\mu\omega_\nu{}^{\lambda_V}+(f_2/m_0{}^2)k_\mu k_\nu k_\sigma\omega_\sigma{}^{\lambda_V}]\times P_2{}^{\mu\nu\mu'\nu'}\bar{u}_{\lambda_1}\gamma_5\gamma_{\mu'}p_{\nu'}v_{\lambda_2}\,,$$

where

$$P_2{}^{\mu\nu\mu'\nu'}=\tfrac{1}{2}(\Theta^{\mu\mu'}\Theta^{\nu\nu'}+\Theta^{\mu\nu'}\Theta^{\nu\mu'})-\tfrac{1}{3}\Theta^{\mu\nu}\Theta^{\mu'\nu'} \quad (4.10)$$

is the spin-2 projection operator with $\Theta^{\mu\nu}=[\delta_{\mu\nu}-(p_1+p_2)_\mu(p_1+p_2)_\nu/(p_1+p_2)^2]$. Using the u_{λ_1} and the v_{λ_2} given in (4.5), the vector-meson helicity wave functions

$$\omega_\nu{}^{+1}=\tfrac{1}{2}\sqrt{2}(-1,\,-i,\,0,\,0),\quad \omega^{-1}=\tfrac{1}{2}\sqrt{2}(+1,\,i,\,0,\,0)\,,$$

and

$$\omega^0=(1/m)(0,0,k_0,ik)\,,$$

we obtain after some calculation,

$$c_{11}B_{11}=f_1\sqrt{2}(p/M)pk\,,$$
$$c_{10}B_{10}=2f_1(\tfrac{2}{3})^{1/2}(p/M)pk[k_0(V)/m_V]+2f_2(p/M)pk^3[k_0(V)-k_0(p)]/m_V\,,$$
$$c_{1,-1}B_{1,-1}=f_1\sqrt{2}(p/M)pk\,,$$
$$c_{0,\lambda_V}B_{0,\lambda_V}=0\,. \quad (4.11)$$

The two terms in the expression for $c_{10}B_{10}$ do not have a common kinematic factor. The second term, proportional to $k_0(P)/m_V$, is due to the fourth component of k_μ in Eq. (4.9). If instead of the four-momentum of P, for k_μ we take only the three-momentum, this term will drop out. The three-momentum of P can be written in a covariant form as

$$\Delta_\mu=\tfrac{1}{2}\{(k_V-k_P)-[(m_V{}^2-m_P{}^2)/s](k_V+k_P)\}\,.$$

From (4.1) and Appendix II, we obtain

$$B_{11}=[1/2(2\sigma+1)]^{1/2}(\sigma+1)^{1/2}b_P+\sigma^{1/2}b_F]\,,$$
$$B_{10}=[1/2(2\sigma+1)]^{1/2}[(\sigma)^{1/2}b_P(\sigma+1)^{1/2}b_F]\,,$$
$$B_{1,-1}=[1/2(2\sigma+1)]^{1/2}[(\sigma+1)^{1/2}b_P+(\sigma)^{1/2}b_F]\,,$$
$$B_{0,\lambda_V}=0\,. \quad (4.12)$$

Again there is a discrepancy between (4.11) and (4.12). The F-wave coupling contribution is absent in the $\lambda_V=\pm1$ components. We may repeat the same trick. In order to obtain the proper F-wave representation for the second term in the expression for $j_{\mu\nu}{}^M$ we have to use

$$(f_2/m_0{}^2)\omega_\sigma[\Delta_\mu\Delta_\nu\Delta_\sigma-\tfrac{1}{5}k^2(\Delta_\mu\delta_{\nu\sigma}+\Delta_\nu\delta_{\sigma\mu}+\Delta_\sigma\delta_{\mu\nu})]\,.$$

This represents $L=3$, because $\Delta_\mu=(\mathbf{k},0)$ where $\mathbf{k}$ is the three-momentum of the incoming meson in the c.m. frame. When this expression is introduced in Eq. (4.9), we obtain

$$c_{1,1}B_{1,1}=\sqrt{2}f_1(p/M)pk-\tfrac{1}{5}\sqrt{2}f_2(p/M)p(k^3/m_0{}^2)\,,$$
$$c_{1,0}B_{1,0}=[2(\tfrac{2}{3})^{1/2}f_1(p/M)pk+(6/5)(\tfrac{2}{3})^{1/2}f_2(p/M)\times p(k^3/m_0{}^2)][k_0(V)/m_V]\,,$$
$$c_{1,-1}B_{1,-1}=\sqrt{2}f_1(p/M)pk-\tfrac{1}{5}\sqrt{2}f_2(p/M)p(k^3/m_0{}^2)\,,$$
$$c_{0,\lambda_V}B_{0,\lambda_V}=0\,, \quad (4.13)$$

and (4.13) is now consistent with (4.12). This result requires our choice of Δ_μ for two reasons: to obtain a consistent kinematic factor and to have the correct representation for $L=3$. This is not very surprising, since the same vector Δ_μ plays an important role in the canonical representations. The threshold factor $p^{L_f}k^{L_i}$ as in Eq. (4.8) also appears in Eq. (4.13).

We have also done similar calculations for 0^-, $1^\pm$, and 2^+ exchange in V production. Our calculations indi-

[28] K. J. Barnes, J. Math. Phys. **6**, 788 (1965); the expression for the vertices $2^-N\bar{N}$ used in this paper are $P_2{}^{\mu\nu\mu'\nu'}\Gamma_{\mu'\nu'}$ instead of $\Gamma_{\mu\nu}$.

cate consistency in all these cases, provided analogous modification is introduced in the effective Lagrangian. The threshold factor is also found to be $p^{L_f}k^{L_i}$.

The examples discussed in this section show that the selection rules are taken into account by imposing the L-S scheme on the helicity amplitude of the Born term.

V. STRUCTURE OF REGGE RESIDUE

In the previous section we have shown the role of the L-S coupling scheme in the Born term. We note that the helicity dependence of the Born term comes only from the kinematical factors and the coupling coefficient. The kinematical factors are helicity-dependent because they arise from the products of the helicity wave functions. The L-S coupling coefficients determine the relations between the components of the helicity amplitudes. Apart from these helicity-dependent factors, we have a helicity-independent factor; the reduced matrix element. In the Born residue, the reduced matrix element is the product of the coupling constant and the threshold factor $p^{L_f}k^{L_i}$, and the dynamics enters into the amplitude only via the reduced matrix elements. We wish to introduce a similar separation of the dynamical from the kinematical part of the Regge residue. To do this, we start from the partial-wave expansion, where the L-S coupling is exhibited explicitly, then apply the S.W. transform. We keep the helicity representation for the simplicity of crossing and Lorentz transformations and wish to use the L-S coupling to take into account the selection rules.

The partial-wave expansion, after separation of the kinematic factors, is given by (3.3). The L-S coupling scheme amounts to using the identity

$$\langle JM\lambda_1\bar{\lambda}_2|F|JM\lambda_V\rangle=\sum_{if}\langle JM\lambda_1\bar{\lambda}_2|JML_fS_f\rangle \times\langle JML_fS_f|F|JML_iS_i\rangle\langle JML_iS_i|JM\lambda_V\rangle \quad (5.1)$$

for the partial-wave amplitude. The residue of $\langle JM\lambda_1\bar{\lambda}_2|F|JM\lambda_V\rangle$ will be denoted by $\langle JM\lambda_1\bar{\lambda}_2|R\times|JM\lambda_V\rangle$, and will satisfy Eq. (5.1).

The coupling coefficients in (5.1) introduce more kinematic cuts in the J plane, leading to unwanted complications. In the potential scattering of particles with spin, the partial-wave amplitude contains trajectory-dependent factors which remove these kinematical singularities.[20] On the basis of this result, we introduce a factor $\varphi_{L_iL_f}(J)$in the reduced matrix elements:

$$\langle JML_fS_f|R|JML_iS_i\rangle r_{L_iL_f}(s)\varphi_{L_iL_f}(J)\,. \quad (5.2)$$

Notice that the helicity independence of $\varphi_{L_iL_f}(J)$ follows from the L-S coupling. The factors $\varphi_{L_iL_f}(J)$ are to be determined by the requirement that all the components of the helicity amplitudes $F_{\lambda_f\lambda_i}$ are free from kinematic singularities in the J plane. In the reaction $PN\to P'N$, we have two helicity amplitudes, F_{00} and F_{10}, and there is only one function $\varphi_{L_iL_f}(J)$ available. *A priori*, it is possible that $\varphi_{L_iL_f}(J)$, which removes the kinematic singularities of F_{00}, does not remove those of F_{10}. If this is the case, the requirement for the absence of kinematic singularities in the J plane contradicts the L-S coupling. In the following we shall see that this does not happen.

To show the α dependence of the Regge-pole contribution, (3.1) is rewritten in the following form:

$$\langle\lambda_1\bar{\lambda}_2|F_R|\lambda_V\rangle=-(2\alpha+1)\pi\alpha'(0)[(1\pm e^{-i\pi\beta})/2\sin\pi\beta] \times[\tfrac{1}{2}(1-z)]^{a/2}[\tfrac{1}{2}(1-z)]^{b/2}c_{\lambda_f\lambda_i}n_{\lambda_i\lambda_f}{}^{\alpha} \times R_{\lambda_f\lambda_i}{}^{\alpha}p_\beta(b,a;-z)\,, \quad (5.3)$$

where we have used Eq. (3.5) and Eq. (A4). In this equation $p_\beta(b,a;-z)$ is an entire function in the α plane and the cuts can only come from $n_{\lambda_i\lambda_f}{}^{\alpha}R_{\lambda_f\lambda_i}{}^{\alpha}$. According to (5.1) and (5.2),

$$R_{\lambda_f\lambda_i}{}^{\alpha}=\sum_{if}\langle\alpha M\lambda_1\bar{\lambda}_2|\alpha ML_fS_f\rangle \times\langle\alpha ML_iS_i|\alpha M\lambda_V\rangle r_{L_iC_f}\varphi_{L_iL_f}\,, \quad (5.4)$$

and

$$n_{\lambda_i\lambda_f}{}^{\alpha}R_{\lambda_i\lambda_f}{}^{\alpha}=\textstyle\sum_L\Gamma_{\lambda_f\lambda_i}{}^{\alpha}(L)\,, \quad (5.5)$$

with

$$\Gamma_{\lambda_f\lambda_i}{}^{\alpha}(L)=\langle\alpha M\lambda_1\bar{\lambda}_2|\alpha ML_fS_f\rangle \times\langle\alpha ML_iS_i|\alpha M\lambda_V\rangle r_{L_iL_f}\varphi_{L_iL_f}\,. \quad (5.6)$$

The condition to be satisfied by $\varphi_{L_iL_f}$ is that $\Gamma_{\lambda_f\lambda_i}{}^{\alpha}(L)$ is regular in the α plane. For π-N charge exchange scattering, using Appendix B, we find

$$\begin{aligned}
\Gamma_{00}(L=\alpha-1)&=\tfrac{1}{2}\sqrt{2}[\alpha/(2\alpha+1)]^{1/2}R_-\varphi_{\alpha-1}{}^{(\alpha)}\,,\\
\Gamma_{10}(L=\alpha-1)&=\tfrac{1}{2}\sqrt{2}[\alpha/(2\alpha+1)]^{1/2}(\alpha+1)R_-\varphi_{\alpha-1}{}^{(\alpha)}\,,\\
\Gamma_{00}(L=\alpha+1)&=\tfrac{1}{2}\sqrt{2}[(\alpha+1)/(2\alpha+1)]R_+\varphi_{\alpha+1}{}^{(\alpha)}\,,\\
\Gamma_{10}(L=\alpha+1)&=\tfrac{1}{2}\sqrt{2}[(\alpha+1)/(2\alpha+1)]R_+\varphi_{\alpha+1}{}^{(\alpha)}\,,
\end{aligned} \quad (5.7)$$

where $R_\pm=R(L_i=\alpha;\,L_f=\alpha\pm1)$. We note that the Table in Appendix B is valid for any J, since if we use Wigner's or Racah's closed expression for the Clebsch-Gordan coefficient, the results depend only on whether $|J-L|=0$ or 1, and that it is independent of the particular value of J. From Eq. (5.7) it follows that $\varphi_{\alpha-1}=[\alpha(2\alpha+1)]^{1/2}$ and $\varphi_{\alpha+1}=[(\alpha+1)(2\alpha+1)]^{1/2}$. This is the simplest choice that satisfies the additional property as $\alpha\to0$ the residue of the $L\approx\alpha-1$ coupling vanishes. Therefore, we obtain

$$\begin{aligned}
n_{00}{}^{\alpha}R_{00}{}^{\alpha}&=\tfrac{1}{2}\sqrt{2}[\alpha R_--(\alpha+1)R_+]\,,\\
n_{10}{}^{\alpha}R_{10}{}^{\alpha}&=\tfrac{1}{2}\sqrt{2}\alpha(\alpha+1)(R_-+R_+)\,.
\end{aligned}$$

Similar calculation for all other isovector-meson exchanges in V production can be easily performed using Appendix B. We shall write down the results for $R_{\lambda_f\lambda_i}{}^{\alpha}$ which will be needed later. For the exchange of a meson with natural parity $J^{PG}=1^{-+}$, 2^{+-}, the trajectory dependence of the residue is given by

$$\begin{aligned}
R_{0,+1}{}^{\alpha}&=\tfrac{1}{2}[\alpha(\alpha+1)]^{1/2}(R_-+R_+)\,,\\
R_{0,-1}{}^{\alpha}&=-\tfrac{1}{2}[\alpha(\alpha-1)]^{1/2}(R_-+R_+)\,,\\
R_{1,+1}{}^{\alpha}&=\tfrac{1}{2}[(\alpha+1)R_- -\alpha R_+]\,,\\
R_{1,-1}{}^{\alpha}&=-\tfrac{1}{2}[(\alpha+1)R_- -\alpha R_+]\,,\\
R_{\lambda_f,0}{}^{\alpha}&=0\,.
\end{aligned}\tag{5.8}$$

The orbital angular momenta of $N\bar{N}$ system are $L=J\pm1$, and that of P,V system is $L=J$. By G-parity conservation $1^{-+}(=\rho)$ can be exchanged in ω production and $2^{+-}(=A_2)$ can be exchanged in ρ production. As to the mesons with unnatural parity, there are two classes differing in their coupling to $N\bar{N}$ in the singlet or triplet state. Those coupled to $N\bar{N}$ in the singlet state consist of $J^{PG}=1^{++}$, 2^{--}. Both have the same residue structure given by

$$\begin{aligned}
R_{0,+1}{}^{\alpha}&=\tfrac{1}{2}\sqrt{2}[(\alpha(\alpha+1)]^{1/2}(R_-+R_+)\,,\\
R_{0,0}{}^{\alpha}&=[\alpha R_- -(\alpha+1)R_+]\,,\\
R_{0,-1}{}^{\alpha}&=\tfrac{1}{2}\sqrt{2}[\alpha(\alpha+1)]^{1/2}(R_-+R_+)\,,\\
R_{1,\lambda_i}{}^{\alpha}&=0\,.
\end{aligned}\tag{5.9}$$

In this case the orbital angular momentum of the $N\bar{N}$ system is J, and that of the meson system is $J\pm1$.

The mesons with unnatural parity of the second class, coupled to $N\bar{N}$ in the triplet state, consist of mesons with $J^{PG}=1^{+-}$, 2^{-+}. The residue structure of this class is given by

$$\begin{aligned}
R_{1,1}{}^{\alpha}&=\tfrac{1}{2}[(\alpha+1)R_-+\alpha R_+]\,,\\
R_{1,0}{}^{\alpha}&=\tfrac{1}{2}\sqrt{2}[\alpha(\alpha+1)]^{1/2}(R_- -R_+)\,,\\
R_{1,-1}{}^{\alpha}&=\tfrac{1}{2}[(\alpha+1)R_-+\alpha R_+]\,,\\
R_{0,\lambda}{}^{\alpha}&=0\,.
\end{aligned}\tag{5.10}$$

The orbital angular momentum for the $N\bar{N}$ system is J, and for the meson system $J\pm1$, as in the first class. These exhaust all possible isovector-meson exchanges in vector-meson production. Among these, those realized in nature are the ρ mesons with $J^{PG}=1^{-+}$ and the A_2 with $J^{PG}=2^{+-}$. The status of the mesons with unnatural parity is not yet established. Possibly the A_1 is associated with $J^{PG}=1^{+-}$, and if the B meson really exists it can have the quantum numbers $J^{PG}=1^{++}$ or 2^{-+}. None can be associated with 2^{--}; however, its recurrence 0^{--} is the well-established π meson.

If 0^{--} is Reggeized, then its residue is described by (5.9). On the mass shell, that is $\alpha=0$, it follows from Eq. (5.9) that only $R_{00}{}^{\alpha}\neq0$ and, among its terms, only the one with $J=L+1$ will survive. Such a structure would lead to $\rho_{11}=0$ or $\rho_{00}=1$. However, off the mass shell, $\alpha\neq0$ and $R_{0,\pm1}{}^{\alpha}\neq0$ because angular momentum is not conserved. This means that, even if π alone is exchanged, ρ_{11} does not vanish if it is Reggeized.[6] Apart from this, the A_2 can also contribute to the nonvanishing of ρ_{11} as follows from Eqs. (5.8) and (2.9).

Eqs. (5.8), (5.9), and (5.10) give the α dependence of the residues and the structure of the amplitude. In Eq. (5.3) we observe that the momentum-dependent factors $c_{\lambda_f\lambda_i}$ are left to be determined. For $c_{\lambda_f\lambda_i}$ we can use the factors obtained in the previous section and we have to add, to the reduced residue with orbital angular momenta L_i, L_f, the threshold factor $p^{L_f}k^{L_i}$. This constitutes a complete rule for writing down the Regge-pole contribution.

VI. THRESHOLD BEHAVIOR AND UNEQUAL-MASS EFFECTS IN V PRODUCTION

In this section we wish to study the threshold behavior of the reduced residues $R_\pm$ and the kinematical factors, assuming analyticity for the amplitude. We consider some specific examples.

The simplest case is that of $\pi\pi\to N\bar{N}$. By Lorentz covariance the general form of the invariant amplitude is

$$T=-A+iB\gamma\cdot k\,,\tag{6.1}$$

where $k_\mu=\frac{1}{2}(k_1-k_2)_\mu$ ($k_i=$ momentum of the π mesons in the s channel c.m. system). The functions A and B are assumed to be analytic functions and to satisfy the appropriate dispersion relations. The s-channel helicity amplitude is then given by

$$\langle\lambda_1\bar{\lambda}_2|F|0\rangle=\bar{u}_{\lambda_1}[-A+iB\gamma\cdot k]v_{\lambda_2}\,,\tag{6.2}$$

where u_{λ_1} and v_{λ_2} is given by Eq. (4.5). This gives

$$\begin{aligned}
F_{00}&=(p/M)A+Bkd_{00}{}^1(z)\,,\\
F_{10}&=\sqrt{2}Bkd_{01}{}^1(z)(p_0/M)\,.
\end{aligned}\tag{6.3}$$

The helicity amplitude $F_{\lambda_f\lambda_i}$ satisfies the partial-wave expansion (3.1). Following the treatment in Sec. III, we may remove the kinematical factors $c_{00}=1$ and $c_{10}=(p_0/M)$. These kinematical factors are the same as those obtained in Sec. III.

In order to study the threshold behavior of the reduced residue $R_\pm$, we express $R_\pm$ in terms of A and B, whose analytic properties are well known. From the results of Frazer and Fulco[29] we find the following relations:

$$\begin{aligned}
R_+&=-[1/(2J+1)]^{1/2}(p/M)A_J-kB_J\,,\\
R_-&=[\tfrac{1}{2}/(2J+1)]^{1/2}(p/M)A_J+kB_{J+1}\,,
\end{aligned}\tag{6.4}$$

where

$$\begin{aligned}
A_J&=\frac{1}{(4\pi)^{1/2}}\int_{-1}^{+1}dz\,AP_J(z)\,,\\
B_{J\pm1}&=\frac{1}{(4\pi)^{1/2}}\int_{-1}^{+1}dz\,BP_{J\pm1}(z)\,.
\end{aligned}\tag{6.5}$$

By substituting the dispersion relation for A and B in

[29] W. Frazer and J. Fulco, Phys. Rev. 117, 1603 (1960).

Eq. (6.5), we obtain the Froissart-Gribov[30] representation for A_J and B_J. Then it follows from Gribov's argument,[31] that near the threshold,

$$A_J=a(2pk/s_0)^J, \quad \text{and} \quad B_{J\pm1}=b_\pm(2pk/s_0)^{J\pm1}.$$

Substitution of these expressions into Eq. (6.4) gives

$$R_+=a(2pk/s_0)^J(p/M)+b_+k(2pk/s_0)^{J+1},$$
$$R_-=b_-(2pk/s_0)^{J-1}k+a(p/M)(2pk/s_0)^J. \quad (6.6)$$

Near the s-channel threshold only the lowest power in the momenta is dominant, and it follows from Eq. (6.6) that

$$R_-\approx c_-(2pk/s_0)^{J-1}k,$$
$$R_+\approx c_+(2pk/s_0)^{J-1}k(p/M)^2. \quad (6.7)$$

This is consistent with the assumed threshold behavior in the previous section. We note that the physical region of the t channel is not very close to the s-channel threshold. We shall assume that even in the t-channel physical region, the threshold behavior (6.7) is valid, as is done in the scattering of spinless particles. In the application of this model to the analysis of experimental data, the threshold behavior (6.7) should be used.

In vector-meson production, the analog of Eq. (6.1) is

$$T=\sum_m O^m A^m, \quad (6.8)$$

where the basis $\{O^m\}$ is

$$O^1=\gamma_5(\gamma\cdot\epsilon), \quad O^4=O^1(\gamma\cdot\Delta),$$
$$O^2=\gamma_5(p\cdot\epsilon), \quad O^5=O^2(\gamma\cdot\Delta),$$
$$O^3=\gamma_5(\Delta\cdot\epsilon), \quad O^6=O^3(\gamma\cdot\Delta). \quad (6.9)$$

This differs from Ref. 32 in replacing the meson momentum by Δ_μ. We find this convenient for the same reason as in Sec. IV. In the general case, the situation is more complicated and requires modification of the basis in order to arrive at a form for the helicity amplitude that can be easily compared with the L-S coupling scheme. For simplicity we shall consider only the part of the amplitude which represents the $N\bar{N}$ system in the singlet state. Since $\bar{u}_{\lambda_1}\gamma_5 v_{\lambda_2}$ transforms like a singlet $N\bar{N}$ system, the part of the amplitude T_1, corresponding to $N\bar{N}$ in the singlet state, is given by

$$T_1=A^{(2)}\gamma_5(p,\epsilon)+A^{(3)}\gamma_5(\Delta,\epsilon). \quad (6.10)$$

If we maintain the basis (6.8), then $\bar{u}\gamma_5\gamma_\mu v$ contains a singlet component, and therefore, (6.10) represents the amplitude with $N\bar{N}$ in the singlet state only if we remove the singlet part from the other basis. We assume that this has been done. According to Sec. V, the exchange of isovector trajectories with the quantum numbers $J^{PG}=0^{--}, 1^{++}$ will contribute only to T_1. Now we may repeat the previous reasoning for T_1 to study the threshold behavior and the kinematical factors involved in the exchange of these trajectories.

The helicity amplitude of T_1 is then given by

$$F_{01}=A^{(2)}(p_0/M)pd_{10}^1(z),$$
$$F_{00}=(p_0/M)[k_0(V)/m_V][A^{(2)}p+A^{(3)}k]d_{00}^1(z),$$
$$F_{0-1}=A^{(2)}(p_0/M)pd_{10}^1(z). \quad (6.11)$$

From Eq. (6.10) it follows that the kinematical factors are

$$c_{10}=c_{-10}=(p_0/M), \quad \text{and} \quad c_{00}=(p_0/M)[k_0(V)/m_V],$$

which are the same as those obtained from the calculation of the corresponding Born term. The reduced residues of this class of trajectories, expressed in terms of $A^{(2)}$ and $A^{(3)}$, are given by

$$R_+=[1/(2J+1)]^{1/2}kA_J^{(3)}+pA_{J+1}^{(2)},$$
$$R_-=[1/(2J+1)]^{1/2}kA_J^{(3)}+pA_{J-1}^{(2)}. \quad (6.12)$$

By the analyticity of $A^{(2)}$ and $A^{(3)}$, as discussed before, we arrive at the threshold behavior.

$$R_+=a_+k^2p(2pk/s_0)^{J-1}, \quad (6.13a)$$
$$R_-=a_-p(2pk/s_0)^{J-1}. \quad (6.13b)$$

In the expression for the Regge-pole contribution, Eqs. (6.13) are multiplied by the asymptotic form of $P_\beta^{(b,a)}(z)$. For the case $\lambda_i=1$, $\lambda_f=0$ we obtain the form

$$p(2pk/b_0)^{J-1}[(2z)^{J-1}+c(2z)^{J-3}+\cdots]$$

for the contribution from $L=J-1$, where $pkz=2t+s-\sum_i m_i^2$. The second- and higher-order term of the asymptotic expansion will lead to singularities at $s=0$, a problem which has been solved by Freedman, Wang,[11] and Jones[33] by introducing the daughter trajectories. The daughter trajectories required for the present mass configuration[34] are $\alpha_d=\alpha_p-2n$ ($n=$integer), where $\alpha_p=$principal trajectory. Near $s=0$, the daughter trajectories remove the singularities arising from the higher-order terms in the asymptotic expansion. This amounts to considering the first term of the asymptotic expansion. However, at larger momentum transfer, the daughter effect is proportional to t^{-2} times that of the principal trajectory, and therefore it can be neglected. For the contribution from $L=J+1$ to this amplitude there is a pole at $s=0$, due to the factor k^2 in the expression for R_+ in Eq. (6.13). This contradicts the analyticity assumption for $A^{(2)}$ and $A^{(3)}$, because from Eq. (6.11) and the analyticity of $A^{(2)}$ and $A^{(3)}$, it follows that the helicity amplitude cannot have a pole at $s=0$. This

[30] M. Foissart, Invited paper at the La Jolla Conference on Weak and Strong Interactions, 1961 (unpublished); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 41, 1962 (1961) [English transl.: Soviet Phys.—JETP 14, 1395 (1962)].

[31] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 42, 1260 (1962); [English transl.: Soviet Phys.—JETP 15, 837 (1962)]; cf. also C.A.M., p. 151.

[32] L. S. Liu and P. Singer, Phys. Rev. 135, B1017 (1964).

[33] D. Z. Freedman, C. E. Jones, and J. M. Wang, University of California Radiation Laboratory Report UCRL 17113 (unpublished).

[34] We wish to thank Professor Freedman for a discussion on this point.

contradiction arises from the extrapolation of the threshold behavior (6.13a) from the region near the s-channel threshold to the t-channel physical region. A simple possibility that retains the threshold behavior of the $L=J+1$ contribution near the s-channel threshold and that does not lead to poles at $s=0$, is to take $a_+=b_+s$. Then the threshold behavior (6.13a) should be replaced by

$$R_+=b_+k^2sp(2pk/s_0)^{J-1}. \tag{6.13c}$$

These examples show that from the dispersion-theoretic ansatz we obtain the same kinematical factors as those obtained from the Born term. The analyticity of the invariant amplitudes imposes an additional restriction (6.13a). Presumably, these statements are also valid for the exchange of other classes of trajectories. The results for the threshold behavior of the reduced residues and for the kinematical factors of the other trajectories in V production are as follows.

The threshold behavior of the reduced residues for the exchange of the trajectories with $T-1$, $J^{PG}=1^{+-}$, 2^+ is

$$R_-=a_-{}^1k(2pk/s_0)^{J-1},$$
$$R_+=a_+{}^1k(2pk/s_0)^{J-1}(p/M)^2, \tag{6.14}$$

with the kinematical factors

$$c_{01}=-c_{0,-1}=\sqrt{s},$$
$$c_{11}=-c_{1,-1}=s/2M,$$
$$c_{00}=c_{01}=0; \tag{6.15}$$

and the threshold behavior of the reduced residue of the exchange of trajectories with $T=1$, $J^{PG}=1^{-+}$, 2^{+-} is

$$R_-=a_-{}''p(2pk/s_0)^{J-1},$$
$$R_+=a_+{}''p(2pk/s_0)^{J-1}(k^2s/m_0{}^3), \tag{6.16}$$

with the kinematical factors

$$c_{0,\lambda_i}=0,\quad c_{1,\pm1}=1,\quad c_{10}=k_0(V)/m_V.$$

The expressions for R_+ are to be substituted into the corresponding equation for $R_{\lambda_f\lambda_i}$ found in the previous section, to obtain the Regge-pole contribution.

VII. CONCLUSION

We have developed a Regge-pole model with L-S coupling to describe the coupling of a trajectory, with definite quantum numbers J^{PG}, to the particles involved in the reaction. This scheme determines the structure of the s-channel amplitude, which in turn characterizes the decay density matrix of the produced resonance.

The characteristics of the present model are as follows: The reduced residues, in the sense of L-S coupling, are helicity-independent. These residues are specified by the orbital angular momentum L and the total spin S of the particles coupled to the trajectory. The possible orbital angular momenta L and total spins S are determined by the selection rules from the known J^{PG} of the trajectory. This fact has been used to classify the trajectories in Sec. V. The orbital angular momenta L and the total spins S are meaningful only in the s-channel c.m. system. From the P conservation it follows that for $PN\to P'N$ and $PN\to VN$, the orbital angular momenta are $L=J$ at one vertex and $L=J\pm1$ at the other.

When a particle is Reggeized, we expect the helicity amplitude to change its structure as the total angular momentum changes. For example, if 0^{--} and 2^{--} occur in nature, then on the mass shell of 2^{--}, the helicity amplitude must have a nonvanishing helicity-flip component. However, on the mass shell of its recurrence at 0^{--}, this helicity-flip component must vanish, since a spin-zero particle does not carry any angular momentum to induce a spin flip. Such a change of structure is a direct consequence of the L-S scheme.

The kinematical factors have been studied using the dispersion-theoretic and the Born-term approach. Examples indicate that both lead to the same result. Using the analyticity assumption, the threshold behavior of the reduced residues is studied. We find that for $L=J\pm1$, the threshold behavior for the equal-mass case is consistent with that obtained from the Born term. For the unequal-mass case, the threshold of the reduced residue with $L=J+1$ may lead to a pole at $s=0$, and to comply with the analyticity assumption this pole has to be removed by multiplying with s. In a sequel to this paper, we will apply our model to πN charge-exchange scattering, η production, and ω production.

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APPENDIX A. ANALYTIC PROPERTIES OF ROTATION MATRICES

For $\lambda_1-\lambda_2>0$ and $\lambda_1+\lambda_2>0$, the rotation matrix $d_{\lambda_1\lambda_2}{}^J(z)$ is defined by[35]

$$d_{\lambda_1\lambda_2}{}^J(z)=N_{\lambda_1\lambda_2}(J)[\tfrac{1}{2}(1+z)]^{\frac{1}{2}(\lambda_1-\lambda_2)}\times[\tfrac{1}{2}(1-z)]^{\frac{1}{2}(\lambda_1-\lambda_2)}P_{J-\lambda_1}{}^{(\lambda_1-\lambda_2,\,\lambda_1+\lambda_2)}(z),$$

with

$$N_{\lambda_1\lambda_2}(J)=\left[\frac{\Gamma(J+\lambda_1+1)\Gamma(J-\lambda_1+1)}{\Gamma(J+\lambda_2+1)\Gamma(J-\lambda_2+1)}\right]^{1/2}; \tag{A1}$$

and $P_m{}^{(a,b)}(z)$ is the Jacobi function defined below.

[35] M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964).

In other cases of λ_1 and λ_2, it is related to (A1) by the symmetry properties

$$d_{\lambda_1\lambda_2}{}^J(z)=(-1)^{\lambda_1-\lambda_2}d_{\lambda_2\lambda_1}{}^J(z)=(-1)^{\lambda_1-\lambda_2}d_{-\lambda_1,\,-\lambda_2}{}^J(z)=d_{-\lambda_2,\,-\lambda_1}{}^J(z). \quad \text{(A2)}$$

The Jacobi function is given by

$$P_n{}^{(a,b)}(z)=\frac{\Gamma(n+a+1)}{\Gamma(n+1)\Gamma(a+1)}\times F(-n,\,n+a+b+1,\,a+1;\,\tfrac{1}{2}(1-z)). \quad \text{(A3)}$$

Analytic Properties in J Plane

From (A3) it follows that $P_n{}^{(a,b)}$ is an entire function in the n plane; therefore, $P_{J-\lambda_1}{}^{(\lambda_1-\lambda_2,\,\lambda_1+\lambda_2)}(z)$ does not lead to any singularity in the J plane. In order to exhibit the singularities in the J plane, (A1) is written as

$$d_{\lambda_1\lambda_2}{}^J(z)=n_{\lambda_1\lambda_2}(J)p(a,b;\,z), \quad \text{(A4)}$$

where $p(a,b;\,z)=F(-n,\,n+a+b+1,\,a+1;\,\frac{1}{2}(1-z)$, and from (A1) and (A3),

$$n_{\lambda_1\lambda_2}(J)=\frac{1}{\Gamma(\lambda_1-\lambda_2+1)}\left[\frac{\Gamma(J+\lambda_1+1)\Gamma(J+\lambda_2+1)}{\Gamma(J-\lambda_1+1)\Gamma(J+\lambda_2+1)}\right]^{1/2}.$$

The singularities in the J plane are completely determined by $n_{\lambda_1\lambda_2}(J)$, which comes from the normalization factor $N_{\lambda_1\lambda_2}(L)$. Relevant $n_{\lambda_1\lambda_2}(J)$ for V production are $n_{00}(J)=n_{11}(J)=1$, $n_{10}(J)=[J(J+1)]^{1/2}$, and $n_{b-1}(J)=\frac{1}{2}J(J+1)$. Only $n_{10}(J)$ leads to kinematical cuts in the J plane.

Analytic Properties in the z Plane

Equation (A3) is valid in the circle $|1-z|<2$, where the hypergeometric series converges. A representation valid for $\mathrm{Re}z>0$ is

$$p(a,\,b;\,z)=\tfrac{1}{2}(1+z)^n F(-n,\,-n+b;\,a+b;\,(z-1)/(z+1)). \quad \text{(A5)}$$

This representation was used in Ref. 6, when we calculated the Regge-pole contribution for which $|z|$ is not very large, before the daughter trajectory was discovered.

For noninteger n, the Jacobi function has a cut on the real axis, where $x<-1$. The behavior of this function near the cut can be expressed in terms of the Jacobi function of the second kind, defined by

$$Q_n{}^{(a,b)}(z)=\frac{1}{2}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(2n+a+b+2)}\times\left(\frac{z-1}{2}\right)^{-n-a-1}\left(\frac{z+1}{2}\right)^{-b}\times F(n+1,\,n+a+1;\,2n+a+b+2;\,2/(1-z)). \quad \text{(A6)}$$

It can be shown that as $z\rightarrow x\pm i0$, for x on the cut, the equation

$$P_n{}^{(a,b)}(x\pm i0)\,|_{x<-1}=e^{\pm i\pi n}P_n{}^{(b,a)}(-x)-(-1)^{\lambda_1-\lambda_2}(2/\pi)\sin\pi n Q_n{}^{(b,a)}(-x). \quad \text{(A7)}$$

holds.

The asymptotic form for $P_n{}^{(a,b)}(z)$ can be found by expressing the Jacobi functions in terms of the Legendre functions and using the asymptotic expansion of the latter. These relations are to be found in the Appendix of Ref. (36).

APPENDIX B. COUPLING COEFFICIENTS

A. $S_1=0\quad S_2=1$

λ	$L=J$	$L=J-1$	$L=J+1$
+1	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\frac{1}{\sqrt{2}}\left(\frac{J}{2J+1}\right)^{1/2}$
0	0	$\left(\frac{J}{2J+1}\right)^{1/2}$	$-\left(\frac{J+1}{2J+1}\right)^{1/2}$
−1	$+\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\frac{1}{\sqrt{2}}\left(\frac{J}{2J+1}\right)^{1/2}$

B. $S_1=S_2=\frac{1}{2}$

	$S=0$	$S=1$		
λ	$L=J$	$L=J$	$L=J-1$	$L=J+1$
0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}\left(\frac{J}{2J+1}\right)^{1/2}$	$-\frac{1}{\sqrt{2}}\left(\frac{J+1}{2J+1}\right)^{1/2}$
+1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\frac{1}{\sqrt{2}}\left(\frac{J}{2J+1}\right)^{1/2}$

[36] M. Gell-Mann, M. L. Goldberger, F. W. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).