where

$$\alpha=\frac{1-x_1^2-y^2-2x_1^3y\pm[(x_1^2+y)^2-4x_1^2(x_1-y)]^{1/2}}{x_1(x_1-y)},$$

$$\beta=\left(\frac{1-x_1^2}{1-y^2}\right)^{1/2},$$

$$\gamma=\frac{x_1^2-x_1y}{1-x_1^2},$$

$$\xi=\frac{\alpha(x-x_1^2)-1+x_1^2}{\alpha(1-x_1^2)-x+x_1^2}.$$

Evidently, this expression for $\rho_{10}(s,t)$ is not a zero function. In order that $\rho_{10}(s,t)\equiv 0$ for all x and y, it is required that the coupling constant g for the 3ϕ vertex be zero. As a consequence, there can be no pole term and no box-diagram singularity either. This means that $\rho_0(s,t)\equiv 0$. Therefore, the entire double density function is also zero, $\rho(s,t)=\rho_0(s,t)+\rho_1(s,t)=0$. From this point on, the usual procedure will lead us to the conclusion that there can be no scattering at all, provided there are no production processes in the s channel.

ACKNOWLEDGMENT

I am indebted to Professor John S. Toll for his continued encouragement and for many discussions.

PHYSICAL REVIEW VOLUME 166, NUMBER 5 25 FEBRUARY 1968

Gribov-Pomeranchuk Phenomenon in N/D Approach

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(Received 28 July 1967)

A detailed study of the left-hand function of the Froissart-Gribov representation enables the integral equation for the N function to be reduced to a form studied by Tamarkin. This enables us to show explicitly that the N function develops an essential singularity and causes an infinite number of Regge poles to accumulate at $l=-1$. When moving cuts are introduced, the integral equation, when again reduced to Tamarkin form, gives conditions on the discontinuity across the cuts for eliminating the essential singularity. The technique in the present form, however, is applicable only to the right-most singularity in the complex angular momentum plane. Extensions of these techniques for π-N scattering (in the s channel) are given as an example of the inclusion of spin effects.

1. INTRODUCTION

Several interesting relations have recently been derived on the basis of superconvergence of strong-interaction amplitudes.[1] These have also been recognized as necessary, at least for the amplitudes involving large helicity flip, in order to satisfy the Froissart bound[2] (a consequence of direct-channel unitarity) for the total amplitude. The asymptotic behavior of these amplitudes has been known for some time to be related to the analyticity properties in the angular momentum plane of the crossed channel. The superconvergence is a consequence of a holomorphy domain in the angular momentum plane larger than that suggested by the Froissart-Gribov (F-G) representation. It was first observed by Gribov and Pomeranchuk[3] that if one attempts to continue the partial-wave amplitude defined by the F-G representation, the presence of fixed poles of the Q_J function conflicts with elastic unitarity and causes an accumulation of an infinite number of Regge poles of positive signature at $l=-1$. A pole at $l=-1$ normally does not contribute to the asymptotic behavior, owing to the wrong signature factor, but, if it becomes an essential singularity, will prevent analytic continuation beyond this point and necessarily force an asymptotic behavior $1/s^{1-\epsilon}$ for any amplitude. Thus, in order to ensure superconvergence, it is necessary to take the mechanism that prevents essential singularities more seriously, as it implies restrictions imposed by unitarity in the direct as well as the crossed channels.

Mandelstam[4] has shown that certain sets of diagrams can produce moving branch points in the complex angular momentum plane, and that by the inclusion of

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[1] V. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966); B. Sakita and K. C. Wali, Phys. Rev. Letters **18**, 29 (1967); L. K. Pande, Nuovo Cimento Letters **48**, 839 (1967); R. Ramachandran, Phys. Rev. **166**, 1528 (1968).

[2] M. Froissart, Phys. Rev. **123**, 1053 (1961); Y. Hara, *ibid.* **136**, B507 (1964).

[3] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters **2**, 239 (1962).

[4] S. Mandelstam, Nuovo Cimento **30**, 1113 (1963); **30**, 1127 (1963); **30**, 1148 (1963); V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. **139**, B184 (1965); J. C. Polkinghorne, J. Math. Phys. **6**, 1960 (1965).

their effects the essential singularity can be avoided. The Gribov-Pomeranchuk phenomenon is a result of the conflict between the singularities in the generalized potential and unitarity, and therefore a formulation such as the N/D technique is more suitable for such a study. This has been recognized by Jones and Teplitz[5] in a recent interesting paper, and they attempt to discuss the existence of the essential singularity and the Regge cut that shields it, using the conventional N/D techniques. They have argued that the kernel of the integral equation for $N(l,s)$ has a pole at $l=-1$, and that this leads to an essential singularity of N. However, this proof is not yet conclusive, since, as has been pointed out by Mandelstam and Wang,[6] it may well be possible to choose the inhomogeneous term in such a way as to avoid the essential singularity. In this paper we will use methods parallel to those of Jones and Teplitz, and study in detail the left-hand function. We will then be able to reduce the integral equation for N into an equation of the Tamarkin form.[7] Tamarkin has studied a certain class of integral equations with regard to their analyticity in the parameter plane, and has shown the conditions necessary for the pole in the kernel to imply an essential singularity in the resolvent of the kernel. We will then utilize a theorem applicable to a symmetric nondegenerate kernel in establishing the conditions necessary for N to develop an essential singularity. In particular, the proof is crucially dependent on the threshold behavior, and the conjectures of Jones and Teplitz appear valid only for the *right-most singularity* in every process.

In Sec. 2, we consider equal-mass spinless particle scattering. The left-hand function is studied, in particular, with respect to its singularity in the neighborhood of $l=-1$. This is solely dependent on the existence of the third double spectral function. In Sec. 3 the appropriate N/D equations are then formulated; utilizing our full knowledge of the left-hand function, the integral equation for N is reduced to Tamarkin's form. An infinite number of Regge poles are found to accumulate at $l=-1$. Section 4 is devoted to the study of the same integral equation in the presence of moving cuts. As l approaches -1, those cuts that emerge through the inelastic threshold into the physical sheet move towards the elastic threshold and blanket the unitarity cut. Using again the Tamarkin form, we obtain the conditions on the discontinuity across the moving cut required to avoid the essential singularity. As an example of similar effects in higher-spin scattering, we consider the partial-wave amplitude in the direct-channel π-N process. In a fermion process, the Gribov-Pomeranchuk phenomenon shifts to $J=-\frac{1}{2}$.[8] The left-hand function possesses a fixed pole at $J=-\frac{1}{2}$, whose residue is again an integral over the third double spectral functions of the invariant amplitudes. In Sec. 5, the integral equation for the appropriate N is converted into Tamarkin form and we observe the development of a similar essential singularity. The conditions on the discontinuity across the moving cut needed to shield this singularity are similar to those found in the spinless case. We again find limitations imposed by the threshold factor, in that it is possible to carry out this proof only for the right-most singularity at $J=-\frac{1}{2}$. In Sec. 6 we discuss the consequence of the absence of a fixed pole and its connections to the bilinear unitarity condition.

In Appendix A, we have collected together the properties of the left-hand function in π-N scattering needed for the discussion of the integral equation. It may be observed that the positive definiteness of the third double spectral function is crucial for the existence of the poles,[3] at least in the imaginary part of the left-hand function. In Appendix B, we have explicitly considered a scattering involving 2 units of helicity flip ("ρ"$+\sigma\to$"ρ"$+\sigma$) to demonstrate such a property, at least in the neighborhood of the boundary of the third double spectral function.

2. STUDY OF THE LEFT-HAND FUNCTION

We begin with the Froissart-Gribov representation for the partial-wave amplitude of equal-mass spinless scattering:

$$a_{\pm}(l,s)=\frac{2}{\pi(s-4m^2)^{l+1}}\left[\int_{4m^2}^{\infty}Q_l\left(1+\frac{2t}{s-4m^2}\right)A_t(s,t)dt\right.$$
$$\left.\pm\int_{4m^2}^{\infty}Q_l\left(1+\frac{2u}{s-4m^2}\right)A_u(s,u)du\right],\quad(1)$$

where

$$A_t(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{\rho_{st}(s',t)}{s'-s}+\frac{1}{\pi}\int_{4m^2}^{\infty}du'\frac{\rho_{tu}(t,u')}{u'-u},\quad(2)$$

$$A_u(s,u)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{\rho_{su}(s',u)}{s'-s}+\frac{1}{\pi}\int_{4m^2}^{\infty}dt'\frac{\rho_{tu}(t',u)}{t'-t}.\quad(3)$$

It is well known[9] that the discontinuity across the left-hand cut of (1) and also the left-hand function are defined in the complex l plane, and the left-hand function has fixed poles at negative integer values of l. We will exploit the knowledge of the singularity structure of these functions in obtaining the consequences of the

[5] C. E. Jones and V. L. Teplitz, Phys. Rev. **159**, 1271 (1967).
[6] S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967).
[7] J. D. Tamarkin, Ann. Math. **28**, 127 (1927). In particular, see Theorem II and the corresponding example on p. 152. It may be noted that Tamarkin's example refers to a definite symmetric kernel. However, in our case it is sufficient that the kernel be nondegenerate and symmetric.

[8] Ya. I. Azimov, Phys. Letters, **3**, 195 (1963).
[9] Haridas Banerjee, Phys. Rev. **131**, 1832 (1963).

unitarity condition through the conventional N/D methods. To this end, we define

$$F_{\pm}(l,s)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{\Delta a_{\pm}(l,s')}{s'-s}ds', \tag{4}$$

where subtractions are implied where necessary. Explicitly, using the techniques given in an earlier reference,[10,11]

$$F_{\pm}(l,s)=F_{\pm}{}^{1}(l,s)\pm F_{\pm}{}^{2}(l,s),$$

with

$$F_{\pm}{}^{1}(l,s)=\frac{2}{\pi^2}\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}ds'\frac{\rho_{st}(s',t)}{s'-s}\left[\frac{1}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s-4m^2}\right)-\frac{1}{(s'-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s'-4m^2}\right)\right]$$
$$\pm\int_{4m^2}^{\infty}du\int_{4m^2}^{\infty}ds'\frac{\rho_{su}(s',u)}{s'-s}\times\left[\frac{1}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2u}{s-4m^2}\right)-\frac{1}{(s'-4m^2)^{l+1}}Q_l\left(1+\frac{2u}{s'-4m^2}\right)\right], \tag{5}$$

and

$$F_{\pm}{}^{2}(l,s)=\frac{2}{\pi^2}\frac{1}{(s-4m^2)^{l+1}}\left[\int_{4m^2}^{\infty}dt\,Q_l\left(1+\frac{2t}{s-4m^2}\right)\int_{4m^2}^{\infty}du'\frac{\rho_{tu}(t,u')}{u'-4m^2+s+t}\right.$$
$$\left.\pm\int_{4m^2}^{\infty}du\,Q_l\left(1+\frac{2u}{s-4m^2}\right)\int_{4m^2}^{\infty}dt'\frac{\rho_{tu}(t',u)}{t'-4m^2+s+u}\right]. \tag{6}$$

From the above equations it is obvious that $F(l,s)$ has only the left-hand cut $(-\infty\leqslant s\leqslant 0)$ and is a holomorphic function of l, except for the poles of Q_l functions at the negative integer values of l. The residues of such poles in $F_{+}{}^{2}(l,s)$ $(F_{-}{}^{2}(l,s))$ are associated with the existence of the third double spectral function ρ_{tu}, and survive only for odd (even) negative integer values of l. However, there is a further simplification for the rightmost singularity at $l=-1$, in that the contributions arising from $F_{\pm}{}^{1}(l,s)$ vanish identically:

$$\lim_{l\to-1}\left[\frac{1}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s-4m^2}\right)-\frac{1}{(s'-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s'-4m^2}\right)\right]_{(\text{Pole})}=0. \tag{7}$$

Therefore, the residue $\chi_{+}(s)$ of the fixed pole at $l=-1$ of $F_{+}(l,s)$ is given (apart from some irrelevant constants) by

$$\chi_{+}(s)=\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}du'\frac{\rho_{tu}(t,u')}{u'-4m^2+s+t}, \tag{8}$$

and similarly for $F_{-}(l,s)$ the residue $\chi_{-}(s)=0$. In the event that ρ_{tu} is not zero everywhere, the residue survives at least in the imaginary part of $\chi_{+}(s)$, as was first noted by Gribov and Pomeranchuk. In what follows we shall confine our attention to the study of the amplitude $a_{+}(l,s)$, since a parallel analysis for $a_{-}(l,s)$ is quite similar. In the neighborhood of $l=-1$, therefore,

$$F(l,s)\approx\frac{\chi(s)}{l+1}. \tag{9}$$

3. FORMULATION OF N/D

The amplitude $a(l,s)$ can be written as $N(l,s)/D(l,s)$, where $D(l,s)$ carries the right-hand cut of $a(l,s)$, and $N(l,s)$ the left-hand cut. With the knowledge of the left-hand function, and assuming elastic unitarity, an integral equation for $N(l,s)$ can be written

$$N(l,s)=F(l,s)+\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{F(l,s')-F(l,s)}{s'-s}\times\lambda(s')N(l,s'), \tag{10}$$
$$\lambda(s)=(s-4m^2)^{l+1/2}/\sqrt{s}.$$

The $D(l,s)$ function is then given by

$$D(l,s)=1-\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{\lambda(s')}{s'-s}N(l,s'). \tag{11}$$

In order to study these integral equations in the neighborhood of $l=-1$, we utilize the information we obtained about $F(l,s)$ and write

$$N(l,s)=\frac{\chi(s)}{l+1}+\frac{1}{\pi(l+1)}\int_{4m^2}^{\infty}ds'\frac{\chi(s')-\chi(s)}{s'-s}\lambda(s')N(l,s'). \tag{12}$$

Substituting

$$\psi(l,s)=(l+1)N(l,s)(\sqrt{\lambda}(s)), \tag{13}$$

[10] For a detailed discussion on the subtraction in Eq. (4) and the extension of the domain of validity of the representation of $F(l,s)$ see Appendix C of Ref. 9.

[11] Haridas Banerjee and G. C. Joshi, Phys. Rev. **137**, B1576 (1965).

we have

$$\psi(l,s)=(\sqrt{\lambda(s)})\chi(s)+\frac{1}{\pi(l+1)}\int_{4m^2}^{\infty}ds'(\sqrt{\lambda(s)})\times\frac{\chi(s')-\chi(s)}{s'-s}(\sqrt{\lambda(s')})\psi(l,s)\,, \quad (14)$$

together with

$$\int_{4m^2}^{\infty}ds\,\lambda(s)|\chi(s)|^2<\infty\,, \quad (15)$$

$$\int_{4m^2}^{\infty}\int_{4m^2}^{\infty}dsds'\lambda(s)\left|\frac{\chi(s')-\chi(s)}{s'-s}\right|^2\lambda(s')<\infty\,. \quad (16)$$

The integral equation [note that it is not Fredholm type because of the pole in the parameter $(l+1)$] is now of the form studied in detail by Tamarkin.[7] As in the example given by Tamarkin, consider the integral equation

$$u(x,\xi)=f(x)+\frac{1}{\lambda}\int_a^b K(x,\xi)u(\xi)d\xi\,, \quad (17)$$

whose kernel is symmetric nondegenerate. We will now use the theorem (see, for example, Pogorzelski[12]): A symmetric kernel has infinitely many eigenvalues if and only if it is nondegenerate. The characteristic values of the equation have zero as the limiting point, so that $\lambda=0$ is an essential singularity for the resolvent of $K(x,\xi)/\lambda$. We now show that the kernel of Eq. (14) is nondegenerate. Using Eq. (8), the kernel $K(s,s')$ has the form

$$(\sqrt{\lambda(s)})\frac{\chi(s')-\chi(s)}{s'-s}(\sqrt{\lambda(s')})=-2[\lambda(s)\lambda(s')]^{1/2}\times\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}du'\frac{\rho_{tu}(t,u')}{(u'-4m^2+s+t)(u'-4m^2+s'+t)}.$$

Since it is obviously not possible to express this in the form

$$\sum_{i=1}^{N}f_i(s)g_i(s)\,,$$

where N is finite, this kernel is indeed nondegenerate; it then follows that $\psi(l,s)$ and, therefore, $N(l,s)$ have an essential singularity at $l=-1$. We should like to emphasize here that it is not possible to conclude directly from Eq. (10) that $N(l,s)$ must have an essential singularity at $l=-1$, because it may happen that the resolvent of a meromorphic kernel $K(s,s',l)$ is analytic in the whole l plane. For instance, if $K(s,s',l)$ is the resolvent of another analytic kernel $K'(s,s',l)$, then the resolvent of $K(s,s',l)$ coincides with $K'(s,s',l)$ and is analytic. These possibilities have been discussed by Tamarkin, and from his discussion it follows that $N(l,s)$ has an essential singularity only because one can reduce the integral equation to the form of Eq. (17). That this is possible in the present case is due to the precise form of $F(l,s)$.

Now, following Jones and Teplitz[5], one can show that as $s\rightarrow\infty$, the infinity many poles of $N(l,s)$ imply through Eq. (11) the accumulation of an infinite number of Regge poles at $l=-1$. As such, this implies the existence of a Gribov-Pomeranchuck[3] essential singularity at $l=-1$.

Similar G-P phenomena are known to occur at the other negative integer values of l. However, we find that our proof of essential singularity cannot be carried out for the other fixed poles. It may be noted that where $l\leqslant-\frac{3}{2}$, the conditions (15) and (16) cease to be valid because of the threshold factor. Similar difficulties have been encountered by Mandelstam[13] in connection with analytic continuation[11] of partial-wave amplitudes with the help of an auxiliary function. The functions $(s-4m^2)^{l+1/2}$ in (15) and (16) give in the lower limit $(s-4m^2)^{l+3/2}$, which diverges for $l\leqslant\frac{3}{2}$. Thus the inhomogeneous term is not square integrable and the kernel is not square summable. Since, these are the two essential conditions in Tamarkin's proof, the essential singularity can be established by this technique only for the right-most singularity at $l=-1$.

It may be noted that we have so far ignored the question of the convergence at the upper limit of the integration in (15) and (16), since the only relevant point is that elastic unitarity holds over a finite region. In a general situation the validity of (15) and (16) at the upper limit of integration is not guaranteed. However, we may introduce an auxiliary function and impose certain constraints on the asymptotic behavior of the amplitude such that the boundedness conditions are satisfied. The details concerning this procedure have been given in Ref. 11.

Just as in the Gribov-Pomeranchuk arguments, the manifestation of the essential singularity is crucially dependent on the existence of a positive definite third double spectral function ρ_{tu}. Indeed, if this were absent, as Jones and Teplitz also observed, the left-hand function would have no poles at $l=-1$.

4. INTEGRAL EQUATION IN THE PRESENCE OF MOVING CUTS

In order to avoid the essential singularity, Mandelstam[4] has conjectured the presence of moving cuts. In contrast to the diagram techniques, the present formalism is more transparent in following the consequences of introducing such cuts. In essence, according to Mandelstam's arguments, as l approaches -1, the moving branch cut envelopes the elastic unitarity cut,

[12] W. Pogorzelski, *Integral Equations and their Applications* (PWN-Polish Scientific Publishers, Warszawa, Poland, 1966), Vol. 1, p. 132. We are grateful to Professor R. L. Warnock for clarification of this point.

[13] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).

thus preventing the fixed pole of Q_l function from becoming an essential singularity.

Using methods parallel to those of Jones and Teplitz, we will now reduce the integral equation for $N(l,s)$ in the presence of cuts to Tamarkin form and examine its consequences. Instead of Eqs. (10) and (11) we now have

$$N(l,s)=F(l,s)+\frac{1}{\pi}\int_{4m^2}^{S_c(l)} ds'\frac{F(l,s')-F(l,s)}{s'-s}\lambda(l,s')N(l,s')$$
$$-\frac{1}{\pi}\int_{S_c(l)}^{\infty} ds'\frac{F(l,s')-F(l,s)}{s'-s}\Delta\left(\frac{1}{a(l,s')}\right)N(l,s'), \quad (18)$$

and

$$D(l,s)=1-\frac{1}{\pi}\int_{4m^2}^{S_c(l)}\frac{N(l,s')\lambda(l,s')}{s'-s}$$
$$+\frac{1}{\pi}\int_{S_c(l)}^{\infty} ds'\frac{N(l,s')}{s'-s}\Delta\left(\frac{1}{a(l,s')}\right). \quad (19)$$

$S_c(l)$ is the reflection of the moving branch point in the l plane, and it coincides with the elastic threshold as $l\to -1$. Then, only the second integrals in both (18) and (19) survive.

Let us now assume the discontinuity across both elastic and moving cuts of the inverse amplitude to be given by

$$\Delta\left(\frac{1}{a(l,s)}\right)=-g(s) \quad \text{at} \quad l=-1; \quad (20)$$

then

$$\psi(l,s)=\chi(s)(\sqrt{g(s)})+\frac{1}{\pi(l+1)}\int_{4m^2}^{\infty} ds'(\sqrt{g(s')})$$
$$\times\frac{\chi(s')-\chi(s)}{s'-s}(\sqrt{g(s)})\psi(l,s'), \quad (21)$$

where now

$$\psi(l,s)=(l+1)N(l,s)(\sqrt{g(s)}). \quad (22)$$

With Eq. (21), Tamarkin's arguments again become applicable, and there will again exist an essential singularity in $\psi(l,s)$, and therefore in $N(l,s)$, at $l=-1$. In order to avoid this trouble, the only other possibility is that the total discontinuity across the right-hand cuts must vanish at least as fast as $(l+1)$, when $l\to -1$. Specifically,

$$\Delta\left(\frac{1}{a(l,s)}\right)=\Delta_c\left(\frac{1}{a(l,s)}\right)-\lambda(s), \quad (23)$$

where Δ_c is the discontinuity across the moving cut. If we assume for $\Delta(1/a(l,s))$ the form

$$\Delta\left(\frac{1}{a(l,s)}\right)=-\lambda(s)(l+1), \quad (24)$$

then

$$\Delta_c\left(\frac{1}{a(l,s)}\right)=-\lambda(s)l. \quad (25)$$

In particular, the discontinuity across the moving cut has precisely the same behavior as on the elastic cut.[14] We may note, in passing, that $g(s)$, being the same as $\lambda(s)$, satisfies the requirements of square integrability and square summability of the kernel at $l=-1$ (since $l\geqslant -\frac{3}{2}$). Clearly these arguments cannot be applied to the cuts shielding the poles at $l=-3, -5\cdots$. It is also clear that the vanishing of $\Delta(1/a(l,s))$ by a higher power like $(l+1)^m$ prevents a Regge pole from passing through $l=-1$.

Thus, with the properties of the cut assumed, the integral Eq. (21) reduces to an ordinary Fredholm type, so that $N(l,s)$ has a fixed pole at $l=-1$ due to $F(l,s)$, and $D(l,s)$ has no such pole. Thus $a(l,s)$ contains this fixed pole whose residue, we may now recall, is related to an integral involving the third double spectral functions. This pole, of course, does not contribute to the asymptotic behavior, $l=-1$ being the nonsense point in the wrong-signature amplitude. Since the origin of this pole is foreign to potential scattering, the residue has a peculiar left-hand cut $(-4m^2\geqslant s\geqslant -\infty)$, in contrast to Regge poles, which are believed to have only a right-hand cut.

5. π-N SCATTERING

That the Gribov-Pomeranchuk phenomenon occurs in the scattering of particles with spin has been noted by Azimov[8] and Mandelstam.[4] It is well known that substantially the same effects occur at $j=\sigma_1+\sigma_2-1$, where σ_1 and σ_2 are the spins of the particles. Using the general methods introduced for the spinless scattering case, we will now show the accumulation point at $J=-\frac{1}{2}$, in the πN partial-wave amplitude, as an explicit example of an application to situations with spin.

We shall study the even-signature, odd-parity amplitude and define the form appropriate for an N/D formulation.[15]

$$h^e(J-\tfrac{1}{2}, W)=\frac{1}{(4q^2)^{J-1/2}}a^e(J-\tfrac{1}{2}, W)\frac{16\pi W}{E+M}. \quad (26)$$

Here, the left-hand function, which is somewhat more complicated, has fixed poles due to $Q_{J-1/2}$ functions at $J=-\frac{1}{2}, -\frac{5}{2}\cdots$. In the neighborhood of the right-most singularity, as in the spinless case, we could write, in the notation of a paper[16] by one of us (G.C.J.) (see Appendix A for details),

$$F(J-\tfrac{1}{2}, W)=\frac{\chi(W)}{J+\frac{1}{2}}+R(W), \quad (27)$$

[14] R. Oehme, Phys. Rev. Letters **18**, 1222 (1967).
[15] S. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).
[16] G. C. Joshi, Phys. Rev. **141**, 1471 (1966).

where

$$\chi(W)=\frac{4}{\pi}\left\{\int_4^\infty dt\int_{(M+1)^2}^\infty du' \times\frac{\rho_{tu}(t,u')+(W-M)\bar\rho_{tu}(t,u')}{u'-\Sigma+s+t}+\int_{(M+1)^2}^\infty du\int_4^\infty dt' \times\frac{\rho_{tu}(t',u)+(W-M)\bar\rho_{tu}(t',u)}{t'-\Sigma+s+u}+(W-M)+c\right\}, \quad (28)$$

and the precise form of $R(W)$ is irrelevant for our purposes. It is obvious that $\chi(W)$ cannot vanish in the presence of the third double spectral functions $\rho_{t\mu}$ and $\bar\rho_{t\mu}$ of the invariant amplitudes of A and B. Furthermore, since $\chi(W)$ has a unique sign, at least in the neighborhood of the boundary of the double-spectral function (as can be shown from the lowest possible box diagrams[17]), there is no possibility that the integrals in (28) vanish. These are then sufficient conditions to write N/D equations analogous to (10). Defining

$$h^e(J-\tfrac{1}{2},W)=N(J,W)/D(J,W)\,, \quad (29)$$

we have

$$N(J,W)=F(J,W)+\frac{1}{\pi}\left[\int_{-(M+1)}^{-\infty}+\int_{(M+1)}^{\infty}\right]dW' \times\frac{F(J,W')-F(J,W)}{W'-W}\lambda(J,W')N(J,W')\,, \quad (30)$$

with

$$\lambda(J,W)=\frac{E+M}{16\pi W}(4q^2)^{J-1/2}q$$

and

$$D(J,W)=1-\frac{1}{\pi}\left[\int_{-(M+1)}^{-\infty}+\int_{(M+1)}^{\infty}\right] \times dW'\frac{\lambda(J,W')N(J,W')}{W'-W}. \quad (31)$$

In the neighborhood of $J=-\frac{1}{2}$, Eq. (30) can be converted into one with a symmetric kernel in the form of Eq. (17). With

$$\psi(J,W)=(J+\tfrac{1}{2})N(J,W)(\surd\lambda(W))\,,$$

we get

$$\psi(J,W)=\chi(W)(\surd\lambda(W))+\frac{1}{\pi(J+\frac{1}{2})} \times\left[\int_{-(M+1)}^{-\infty}+\int_{(M+1)}^{\infty}\right]dW'\frac{\chi(W')-\chi(W)}{W'-W} \times[\lambda(W')\lambda(W)]^{1/2}\psi(J,W')\,. \quad (32)$$

As before, by similar arguments, N develops an essential singularity, and consequently an infinite number of Regge poles accumulate at $J=-\frac{1}{2}$ as $W\to\pm\infty$.

To shield this essential singularity, it is again necessary to introduce two Regge cuts such that at $J=-\frac{1}{2}$ their branch points coincide with the elastic threshold $\pm(M+1)$. In addition, the combined discontinuity across the Regge as well as the unitarity cut must vanish:

$$\Delta\left(\frac{1}{h^e(J-\frac{1}{2},W)}\right)=\Delta_c\left(\frac{1}{h^e(J-\frac{1}{2},W)}\right)-\lambda(W) =-(J+\tfrac{1}{2})\lambda(W) \quad (33)$$

or

$$\Delta_c\left(\frac{1}{h^e(J-\frac{1}{2},W)}\right)=-(J-\tfrac{1}{2})\lambda(W)\,. \quad (34)$$

The threshold behavior of the two cuts is identical. Just as in the π-π case, the extension of this proof to the other negative half-odd-integer J seems impossible on account of the breakdown of square integrability for $J\leqslant-1$, due to the threshold behavior $(q^2)^J$.

A similar analysis for the right-most singularity in every process is now straightforward, and depends solely upon the existence of the third double spectral function.

6. DISCUSSION

To summarize, using the detailed knowledge of the left-hand function, it is always possible to reduce the integral equation for $N(J,s)$ in the neighborhood of its right-most singularity in the J plane into Tamarkin form. In this form it is transparent that $N(J,s)$ develops an essential singularity unless appropriate moving cuts are introduced. This mechanism then eliminates the essential singularity and substitutes a wrong-signatured fixed pole in the amplitude. This restores the possibility of superconvergence of the maximum helicity-flip amplitude.[18]

Though such fixed poles do not contribute to the asymptotics, their absence leads to some interesting consequences. For example, they give rise to superconvergence relations, valid separately for the right- and left-hand cuts, as has been noted by Schwarz.[19] That these relations are not found valid is an indication of the presence of the fixed poles, which are to be expected as a result of the third double spectral function in relativistic scattering.[6]

The poles of Q_J function that give the fixed pole of our amplitude are easily seen to be related to the possibility of writing a decoupled bilinear form of the unitarity condition. Indeed, if the scattering amplitude is decomposed instead in terms of Khuri amplitudes,[20]

[17] J. Charap, E. Lubkin, and A. Scotti, Ann. Phys. (N. Y.) **21**, 143 (1963).

[18] However, in order to satisfy the Froissart bound, a higher order of superconvergence is necessary for the amplitude with higher helicity flips.

[19] J. H. Schwarz, Phys. Rev. **159**, 1269 (1967); **162**, 1671 (1967).

[20] N. Khuri, Phys. Rev. **132**, 914 (1963).

the corresponding left-hand discontinuity is an entire function in the ν plane. Since the representation corresponding to the F-G representation for Khuri amplitudes is given by

$$c(\nu,s)=\frac{1}{\pi}\int_{4m^2}^{b} dt\, t^{-\nu-1}A_t(s,t)\,, \tag{35a}$$

$$b(\nu,s)=\frac{1}{\pi}\int_{4m^2}^{\infty} du\, u^{-\nu-1}A_u(s,u)\,, \tag{35b}$$

the left-hand discontinuity ($s<0$) is

$$\Delta c(\nu,s)=\frac{1}{\pi}\int_{4m^2}^{-s} dt\, t^{-\nu-1}\rho_{tu}(t,\, 4m^2-s-t)\,, \tag{36a}$$

$$\Delta b(\nu,s)=\frac{1}{\pi}\int_{4m^2}^{-s} du\, u^{-\nu-1}\rho_{tu}(4m^2-s-u,\, u)\,. \tag{36b}$$

The integrals being finite, just as in the F-G representation, the left-hand discontinuities are defined everywhere in the ν plane. Furthermore, they are analytic in the entire ν plane. The absence of poles in the left-hand discontinuity is, however, accompanied by the absence of a simple decoupled bilinear unitarity condition. The presence of fixed poles in the partial-wave amplitude as a dynamical consequence of the left-hand function seems observable only in the presence of a simple unitarity condition.

The study of the G-P phenomenon and the cut mechanism through the formalism of N/D techniques is particularly suitable for introducing unitarity effects. However, in the present form, it is confined to the *right-most* singularity alone, as a consequence of limitations imposed by threshold behavior. This limitation, in fact, first appeared in the work of Mandelstam,[11,13] where the threshold behavior prevents the strip-by-strip analytic continuation of the partial-wave amplitude beyond $l=-\frac{3}{2}$.

ACKNOWLEDGMENTS

We are grateful to Professor D. Amati and Dr. R. Jengo for useful discussions in the early stages of this work. It is a pleasure to thank Professor R. L. Warnock for many valuable discussions and for a critical reading of the manuscript. We are also grateful to Professor J. C. Polkinghorne for reading the manuscript and for helpful comments. Finally we would like to acknowledge the hospitality extended to us by Professor A. Salam and Professor P. Budini and the IAEA at the International Center for Theoretical Physics, Trieste.

APPENDIX A: GENERALIZED POTENTIAL FOR π-N SCATTERING

In this Appendix we calculate the left-hand function of $h^e(J-\frac{1}{2},\, W)$, and then study its behavior in the neighborhood of $J=-\frac{1}{2}$.

The left-hand function is defined by[16]

$$F^e(J-\tfrac{1}{2},\, W)=\frac{1}{\pi}\int_{(M^2+2)^{1/2}}^{-(M^2-1)/M} dW'\frac{\alpha_1(J-\frac{1}{2},\, W')}{W'-W}+\frac{1}{\pi}\int_{(M^2-1)/M}^{(M^2+2)^{1/2}} dW'\frac{\alpha_2(J-\frac{1}{2},\, W')}{W'-W}$$
$$+\frac{1}{\pi}\int_{-(M-1)}^{(M-1)} dW'\frac{\alpha_3(J-\frac{1}{2},\, W')}{W'-W}+\frac{\gamma}{\pi}\int_0^{2\pi} id\varphi\frac{\alpha_4(J-\frac{1}{2},\, \varphi)}{We^{-1\varphi}-\gamma}+\frac{1}{\pi}\int_{-\infty}^{+\infty} dy\frac{\alpha_5(J-\frac{1}{2},\, y)}{y+iW}. \tag{A1}$$

Following Ref. 16, we obtain

$$F^e(J-\tfrac{1}{2},\, W)=\sum_{K=1}^{8} F_K{}^e(J-\tfrac{1}{2},\, W)\,, \tag{A2}$$

where

$$F_1(J-\tfrac{1}{2},\, W)=\frac{4}{\pi}\int_4^\infty dt\int_{(M+1)^2}^\infty ds'\frac{\rho_{st}(s',t)}{s'-s}\Big[\chi(s)Q_{J-1/2}(z(s,t))-\chi(s')Q_{J-1/2}(z(s',t))\Big]$$
$$+4\int_4^\infty dt\, Q_{J-1/2}(z(s,t))\varphi_2(t,\, \Sigma-s-t)\chi(s)\,, \tag{A3}$$

$$F_2(J-\tfrac{1}{2},\, W)=\frac{4}{\pi}\int_{(M+1)^2}^\infty du\int_{(M+1)^2}^\infty ds\frac{\rho_{su}(s',u)}{s'-s}[\chi(s)Q_{J-1/2}(z'(s,u))-\chi(s')Q_{J-1/2}(z'(s',u))]$$
$$+4\int_{(M+1)^2}^\infty Q_{J-1/2}(z'(s,u))\psi_2(\Sigma-s-u,\, u)\chi(s)du\,, \tag{A4}$$

$$F_3(J-\tfrac{1}{2},\, W)=\frac{4}{\pi}\int_4^\infty dt\int_{(M+1)^2}^\infty ds'\frac{\bar{\rho}_{st}(s',t)}{s'-s}[\chi'(s)Q_{J-1/2}(z(s,t))-\chi'(s')Q_{J-1/2}(z(s',t))]$$
$$+4\int_4^\infty dtQ_{J-1/2}(z(s,t))\bar{\varphi}_2(t,\, \Sigma-s-t)\chi'(s)\,, \tag{A5}$$

$$F_4(J-\tfrac{1}{2},W)=\frac{4}{\pi}\int_{(M+1)^2}^{\infty}du\int_{(M+1)^2}^{\infty}ds'\frac{\bar{\rho}_{su}(s',u)}{s'-s}[\chi'(s)Q_{J-1/2}(z'(s,u))-\chi'(s')Q_{J-1/2}(z'(s',u))]$$
$$+4\int_{(M+1)^2}^{\infty}du\,Q_{J-1/2}(z(s,u))\bar{\psi}(\Sigma-s-u,u)\chi'(s)\,,\quad\text{(A6)}$$

where

$$\chi(s)=\frac{1}{[4q^2(s)]^{J+1/2}}=\frac{\chi'(s)}{W-M}=\frac{q^2(s)}{[E(s)-M]^2}\chi''(s)=\frac{q^2(s)}{[E(s)-M]^2}\frac{\chi'''(s)}{(W+M)}.\quad\text{(A7)}$$

The explicit forms of $(F_5$–$F_8)$ are not needed for our discussion and are given in Ref. 15 [with $\chi(s)$'s defined as in (A7)]. It should be noted that by taking arbitrarily large subtractions in (A1), the domain of validity of $F(J-\frac{1}{2},W)$ in the J plane can be extended to an arbitrary large domain.[9] We consider the behavior of the above representation at $J=-\frac{1}{2}$; in particular, we take Eq. (A4). In this equation, for $J=-\frac{1}{2}$, the first integral does not contain any pole terms, because

$$\lim_{J\to-1/2}[\chi(s)Q_{J-1/2}(z(s,t))-\chi(s')Q_{J-1/2}(z(s',t))]_{(\text{Pole})}=0;\quad\text{(A8)}$$

and the second term has a pole whose residue is given by

$$4\int_4^{\infty}dt\,\varphi_2(t,\Sigma-s-t)\,.\quad\text{(A9)}$$

We may use similar arguments for all the other F's (F_5 to F_8) and finally obtain

$$F(J-\tfrac{1}{2},W)=R(W)+\frac{\chi(W)}{J+\frac{1}{2}},\quad\text{(A10)}$$

where $R(W)$ is analytic in $(-\infty\leqslant W\leqslant-(M+1))$ and $((M+1)\leqslant W\leqslant\infty)$, and whose precise form is not needed for our discussions. $\chi(W)$ is given by

$$\chi(W)=4\left\{\int_4^{\infty}dt[\varphi_2(t,\Sigma-s-t)+(W-M)\bar{\varphi}_2(t,\Sigma-s-t)]+\int_{(M+1)^2}^{\infty}du[\psi_2(\Sigma-s-u,u)+(W-M)\bar{\psi}_2(\Sigma-s-u,u)]+[(W-M)/\pi]+c\right\},\quad\text{(A11)}$$

where c is an irrelevant constant and

$$\varphi_2(t,\Sigma-s-t)=\frac{1}{\pi}\int_{(M+1)^2}^{\infty}\frac{\rho_{tu}(t,u')}{u'-\Sigma+s+t}du',\quad\text{(A12)}$$

$$\psi_2(\Sigma-s-u,u)=\frac{1}{\pi}\int_4^{\infty}\frac{\rho_{tu}(t',u)}{t'-\Sigma+s+u}dt',\quad\text{(A13)}$$

with similar expressions for $\bar{\varphi}_2$ and $\bar{\psi}_2$. It should be noted that in getting Eq. (A11) we have also taken into account the nucleon pole term (which is in F_8), which also has a fixed pole at $J=-\frac{1}{2}$.

APPENDIX B: DOUBLE SPECTRAL FUNCTION FOR "ρ"$+\sigma\to$"ρ"$+\sigma$

It is well known that in the case of $\pi\pi$ scattering and NN scattering, the double spectral functions (d.s.f.) of the relevant invariant amplitude, where they exist, have a unique sign.[17] We shall demonstrate here a similar property when a spin-1 particle is involved, as for instance in $\rho\pi$ scattering.

The simplest diagram that contributes to the d.s.f. is a box diagram. We may avoid some inessential complications arising from the negative G parity of the π meson by discussing instead $\rho\sigma$ scattering. The fixed poles, etc., in the t channel $(\sigma+\sigma\to\rho+\rho)$ are related to the third double spectral function $\rho_{su}(s,u)$, the contribution to which is obtained typically from a diagram such as in Fig. 1. With $P=\frac{1}{2}(p_1+p_2)$, $K=\frac{1}{2}(q_1+q_2)$, and $\Delta=(p_2-p_1)=(q_1-q_2)$, we define the amplitude[21]

$$T_{\mu\nu}=A_1P_\mu P_\nu+A_2P_\mu K_\nu+A_3P_\mu\Delta_\nu+B_1K_\mu P_\nu+B_2K_\mu K_\nu+B_3K_\mu\Delta_\nu+C_1\Delta_\mu P_\nu+C_2\Delta_\mu K_\nu+C_3\Delta_\mu\Delta_\nu+Dg_{\mu\nu}\,,\quad\text{(B1)}$$

where A_1 and B_2 are the invariant amplitudes with two units of helicity flip and with no helicity flip, respectively. The projection operators for A_1 and B_2 are obtained quite easily (especially since we consider the particular case when $q_1^2=q_2^2=\mu^2$ and $p_1^2=p_1^2=m^2$) and

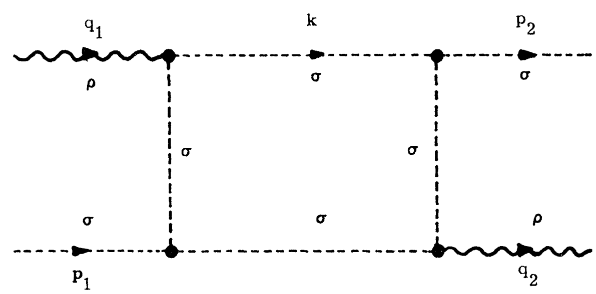

FIG. 1. The box diagram contributing to ρ_{su} for "ρ"$+\sigma\to$"ρ"$+\sigma$.

[21] S. Fubini, Nuovo Cimento 43, 475 (1966).

are given by

$$\mathcal{P}_{\mu\nu}{}^{A_1}=\frac{K^2}{\lambda}\left[\left(-g_{\mu\nu}+\frac{K_\mu K_\nu}{K^2}\right)+R_{\mu\nu}\right] \qquad \text{(B2)}$$

and

$$\mathcal{P}_{\mu\nu}{}^{B_2}=\frac{P^2}{\lambda}\left[\left(-g_{\mu\nu}+\frac{P_\mu P_\nu}{P^2}\right)+R_{\mu\nu}\right], \qquad \text{(B3)}$$

where

$$R_{\mu\nu}=2\left[-\frac{K\cdot P}{\lambda}(K_\mu P_\nu+P_\mu K_\nu)+g_{\mu\nu}+\frac{K^2P_\mu P_\nu+P^2K_\mu K_\nu}{\lambda}-\frac{\Delta_\mu\Delta_\nu}{\Delta^2}\right]$$

and

$$\lambda=(P\cdot K)^2-P^2K^2,$$

such that

$$\mathcal{P}_{\mu\nu}{}^{A_1}T_{\mu\nu}=A_1,$$
$$\mathcal{P}_{\mu\nu}{}^{B_2}T_{\mu\nu}=B_2.$$

Since the contribution from diagram 1 to $T_{\mu\nu}$ is given by

$$T_{\mu\nu}=\int d^4k(2k-q_1)_\mu(2k-q_2-2p_2)_\nu \times\Big[(m^2-k^2)(m^2-(k-p_2)^2)(m^2-(k-p_2-q_2)^2)\times(m^2-(k-q_1)^2)\Big]^{-1}, \qquad \text{(B4)}$$

it is easily seen that

$$\rho_{su}{}^{A_1,B_2}=\int d^4k\ \mathcal{P}_\mu{}^{A_1,B_2}(2k-q_1)_\mu(2k-q_2-2p_2)_\nu \times\delta(m^2-k)\delta^2(m^2-(k-p_2)^2)\delta(m^2-(k-p_2-q_2)^2)\times\delta(m^2-(k-q_1)^2). \qquad \text{(B5)}$$

The transformation from the k integration to the integrations over the arguments of the delta functions gives

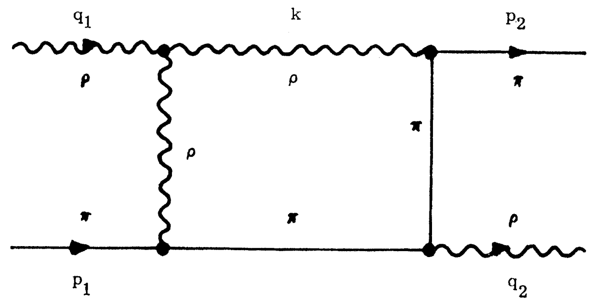

FIG. 2. The box diagram contributing to ρ_{su} for "ρ"+π → "ρ"+π.

a Jacobian J:

$$J=\{[su-(m^2-\mu^2)^2][(s-4m^2)(u-4m^2)-(3m^2-\mu^2)^2]\}^{-1/2}. \qquad \text{(B6)}$$

Using the projection operators and the δ functions, the integrations in (B5) are trivially performed, resulting in

$$\rho_{su}{}^{A_1}(s,u)=\frac{J(t-4\mu^2)}{[su-(m^2-\mu^2)^2]}\left[2m^2+\mu^2+\frac{(s-u)^2\mu^2}{t(t-4\mu^2)}\right] \qquad \text{(B7)}$$

and

$$\rho_{su}{}^{B_2}(s,u)=\frac{J(t-4\mu^2)}{[su-(m^2-\mu^2)^2]}\left[2m^2+\mu^2-\frac{s+u}{4}-\frac{(s-u)^2}{4t}\right]. \qquad \text{(B8)}$$

The boundary of the region where the double spectral function does not vanish is also given by the locus of the zeros of J. In (B7) the terms within the square bracket are positive for $t<0$, so $\rho_{su}{}^{A_1}$ obviously has a unique sign, wherever it exists. Similarly, $\rho_{su}{}^{B_2}$ has a unique, sign when $m^2<\mu^2$ (which is necessary in order to avoid complications arising from anomalous thresholds).

For the real $\rho\pi$ scattering, a typical box diagram that could contribute to ρ_{su} is shown in Fig. 2. The complications due to the inclusion of isospin, however, do not alter the arguments for a unique sign of the d.s.f.