Approach to Equal-Time Commutators in Quantum Field Theory

RICHARD A. BRANDT*
Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland

(Received 9 December 1966; revised manuscript received 5 September 1967)

Equal-time current commutators $[J(x),J'(x')]$ should be calculated as suitable equal-time limits of ordinary current commutators. Since this calculation is usually ambiguous or impractical, we propose instead to define them as limits of $[J(x;\xi),J'(x';\xi')]$ for $\xi,\xi'\to 0$, where $J(x;\xi)$ is a suitable nonlocal expression in the fields which converges to $J(x)$ for $\xi\to 0$. This alternative should be more reliable than the usual ones, such as taking equal-time limits inside of spectral representations or taking limits of time-ordered products from positive and negative time differences. The former procedure is invalid when the spectral function is nonintegrable, and the latter when equal-time δ functions are present. An analysis of two-point functions is presented which illustrates the above effects. In this connection, it is shown that the commutator $\langle 0|[j_k,j_4]|0\rangle$ in electrodynamics has a $\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')$ term in addition to the usual $\partial_k\delta(\mathbf{x}-\mathbf{x}')$ term. Our definition is shown to give correct results in a number of soluble models. It is then used to calculate commutators for electrodynamics in all orders of perturbation theory. The main new result is that, contrary to previous assertions, the commutator $[j_k(x),j_4(x')]$ is a q-number—essentially $e^4{:}\mathbf{A}^2{:}\partial_k\delta(\mathbf{x}-\mathbf{x}')$ in the Gupta-Bleuler gauge. This result, together with a similar one for $[j_k(x),\dot{A}_l(x')]$, is shown to be consistent with gauge invariance and to be suitable for use in equal-time commutators which arise in reduction formulas. Finally, reduction formulas are used to explicitly establish the correctness of our results in fourth order.

1. INTRODUCTION

THE recent successful calculations based on ETCCR's[1] strongly support the usefulness of Gell-Mann's idea[2] that such relations should be abstracted from field-theoretic models and postulated as correct in the real world. The situation is analogous to the previous suggestion that dispersion relations, etc., should be similarly abstracted from quantum field theory. A serious problem arises, however, in that these ETCCR's cannot in general be correctly calculated directly from the canonical ET field CR's. Thus, which, if any, ETCCR's are valid in the perturbation solutions of a particular field theory is not *a priori* known. There will in general be extra noncanonical dynamical terms present in addition to the canonical ones.[3,4] This is analogous to, and related to, the question of subtractions in dispersion relations. In this paper, we shall propose, investigate, and illustrate a simple method for calculating these extra terms to all orders in any renormalizable field theory.[5]

A. Standard Commutator

Let us consider a particular matrix element of the commutator of two relatively local Wightman field operators $A(x)$ and $B(y)$:

$$C_{\alpha\beta}{}^{AB}(x;y)\equiv\langle\alpha|[A(x),B(y)]|\beta\rangle. \tag{1.1}$$

The most natural and reliable way of investigating the ET properties of this distribution is to smear it with a testing function $f_n(x_0-y_0)$, where $\{f_n\}$ is a sequence of functions in $\mathcal{S}$ [6] which converges to the δ function in the topology of $\mathcal{S}'$:

$$f_n(\tau)\xrightarrow[n\to\infty]{\mathcal{S}'}\delta(\tau). \tag{1.2}$$

If the limit (considering C as a distribution in $x+y$ and $x-y$)

$$\lim_{n\to\infty}\int dy_0\, f_n(x_0-y_0)C_{\alpha\beta}{}^{AB}(x;y)\equiv E_{\alpha\beta}{}^{AB}(\mathbf{x},\mathbf{y};x_0) \tag{1.3}$$

exists for each sequence $\{f_n\}$ and has a value independent of the sequence used, then it is clear that the ET limit of (1.1) is well defined and is given by (1.3). We shall refer to this as the *orthodox* definition of an ETC, and to (1.3) as the *orthodox value* of the ETC.

Although theoretically appealing, the definition (1.3) is practically inappropriate for (at least) the following three reasons.

(i) The limit (1.3) will, in most cases of interest, depend on the chosen sequence $\{f_n\}$. It is then not clear which, if any, sequence to use in order to define the ETC. It might be argued that in this case the ETC simply does not exist. Since ETC's do seem to exist in nature, however, we shall take the point of view that there should exist criteria for the choice of a suitable sequence. These criteria might arise, for example, from a consideration of how the ETC's are to be employed.[7] One should, of course, choose a sequence

* Supported in part by the U. S. Air Force under Contract No. AFOSR-500-66.
[1] We use the following abbreviations: equal time (ET), equal-time commutator (ETC), equal-time current commutator (ETCC), commutation relation (CR), current commutation relation (CCR), vacuum expectation value (VEV).
[2] M. Gell-Mann, Physics **1**, 63 (1964).
[3] T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955).
[4] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).
[5] A preliminary account of this work appeared as University of Maryland Technical Report No. 643, 1966 (unpublished).
[6] For definitions of $\mathcal{S}$ and $\mathcal{S}'$, see, e.g., I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. I.
[7] The situation seems analogous to that existing in perturbation theory before renormalization was introduced. There, ambiguities arose which were later resolved by employment of observability and invariance criteria.

such that the limit will exist and will be antisymmetric under interchange of A and B.[8] A large amount of arbitrariness can, however, still remain. Consider, for example, the simple but important case

$$C(x,y)=\delta(x-y)+R(x,y)\,,\quad R(\mathbf{x},t,\mathbf{y},t)=0\,. \quad (1.4)$$

Then

$$E(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})\lim_{n\to\infty} f_n(0)\,. \quad (1.5)$$

Now $\lim_{n\to\infty} f_n(0)$ can be anything in the range $(-\infty,+\infty]$, so that (1.5) is completely ambiguous. One might argue that in this case one really has $C(x,y)=\infty\times\delta(\mathbf{x}-\mathbf{y})$, so that the ETC does not exist in the usual sense. The point is that there may be a criterion for choosing a particular $\lim f_n(0)<\infty$. The definition (1.3), however, does not seem to lead to one.

(ii) The explicit distributions (1.1) needed to evaluate (1.3) will not in general be known. In perturbation theory, one is given matrix elements of time-ordered products:

$$T_{\alpha\beta}{}^{AB}(x;y)\equiv\langle\alpha|T(A(x)B(y))|\beta\rangle\,. \quad (1.6)$$

These, however, are usually ambiguous at $x=y$, just the point of interest. One can add suitable distributions with support at $x=y$ to (1.6) without changing the content of the theory. This is related to an aspect of renormalization invariance and to the arbitrariness of off-mass-shell extrapolations of the S matrix.[9–11] Mathematically, it corresponds to the fact that the product $\theta(x-y)A(x)B(y)$ is generally ill-defined. Even if one chooses a definite renormalization prescription and takes (1.6) to be given by the corresponding sum of renormalized Feynman diagrams, there is no simple method to determine (1.1).[12] Furthermore, even if a method were chosen, one could not hope to have explicit enough knowledge of (1.1) for all matrix elements in all orders to calculate (1.3). Thus operator relations valid in all orders could not be deduced.

(iii) Because of distribution-theoretic subtleties present in (1.1), the calculation (1.3) can be rather intricate. We shall see, in fact, that even for the simple case of vacuum expectation values of Wick products of free fields, these subtleties are such as to render most of relevant considerations in the literature incomplete.

B. New Commutator

In an effort to overcome these difficulties, we shall propose a new method for calculating ETC's of current operators. To all orders in the perturbative solution of any renormalizable field theory, each renormalized current operator $A(x)$ can be explicitly written as a weak limit

$$A(x)=\lim_{\xi\to 0} A(x;\xi)\,, \quad (1.7)$$

where $A(x;\xi)$ is a function of the renormalized local operators $\chi_i(x)$, $\chi_j(x+\xi)$ of the theory evaluated at time x_0.[13–15] Our proposal is, roughly, to define[16]

$$[A(x),B(x')]\equiv\lim_{\xi\to 0,\xi'\to 0}[A(x;\xi),B(x';\xi')] \quad (1.8)$$

and to evaluate the right side by using the known ETCR's satisfied by the fields χ_i. We shall make Eq. (1.8) more precise in the following, but let us emphasize here the fact that it is a definition. We shall not prove (1.8) in this paper but will only show that it leads to simple, reasonable, and relatively unambiguous results.

Let us indicate how the definition (1.8) removes each of the difficulties (i)–(iii) mentioned above.

(i′) Our definition (1.8) amounts to a natural way of choosing a sequence $\{f_n\}$. Specifically, we require that the sequence be chosen so that the usual field ETCR's are valid and so that the interchange of $x_0'\to x_0$ and $\xi\to 0$ limits implicit in (1.8) is allowed.[17] As evidence for the reliability and effectiveness of this choice, we shall find that it defines an ETC with the following desirable properties: (a) It agrees with the orthodox ETC in cases when the latter leads to unambiguous results; (b) it gives correct results in low orders of perturbation theory; (c) it is consistent with the relevant field equations and invariance properties; (d) it is suitable for ET commutators arising in formal manipulations with reduction formulas. This latter property is especially important since it is in this context that ETCR's are employed in practice. Consider, for example, the amplitude

$$T(k)=\int dx\, e^{-ik\cdot x}\langle\alpha|TA(x)B(0)|\beta\rangle\,. \quad (1.9)$$

Formally, one finds

$$k_0T(k)=i\int dx\, e^{-ik\cdot x}\langle\alpha|\{T\dot{A}(x)B(0) + \delta(x_0)[A(\mathbf{x},0),B(0)]\}|\beta\rangle\,. \quad (1.10)$$

When the orthodox ET commutator does not exist, it is not clear which definition of ET commutator to use

[8] With ETC's defined by (1.3), or any similar limiting process, the Jacobi identity will only be valid provided an interchange of limits is possible. This will be discussed in Sec. 6.

[9] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

[10] M. C. Polivanov, in *Proceedings of the ICTP Lectures, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965).

[11] A detailed discussion of these matters will be given in Sec. 6.

[12] D. Ruelle, Nuovo Cimento **19**, 356 (1961); O. Steinmann, Helv. Phys. Acta **36**, 90 (1963).

[13] R. A. Brandt, Ann. Phys. (N. Y.) **44**, 221 (1967).

[14] R. A. Brandt, University of Maryland Technical Report No. 673 (unpubished). We shall refer to this work as I.

[15] W. Zimmermann, Commun. Math. Phys. **6**, 161 (1967).

[16] Throughout this paper, we shall take $x_0=x_0'$.

[17] There is, of course, no guarantee that such a sequence will exist. This will be discussed below.

in (1.10).[18] We shall show that it is consistent to use our definition (1.8).

(ii′) The operator structures of the current operators (1.7) are explicitly known both (exactly) for the explicitly soluble models[19–27] and (to all orders of perturbation theory) for the renormalizable field theories.[13–15] Thus the calculation of (1.8) can be simply and explicitly performed in these theories. For the soluble models, (1.8) will be seen to agree with the known results. In perturbation theory, in view of the ambiguities discussed in (ii) above, we must in general be content with the fact that (1.8) possesses the desirable properties mentioned in (i′) above. In fourth-order quantum electrodynamics, however, we shall be able to explicitly verify our results.

(iii′) In cases where a comparison can be made, (1.8) will be seen to reproduce all the results of (1.3) with much less effort. Furthermore, exact operator expressions are obtained, rather than simply the matrix elements of such provided by (1.3). Thus (1.8) does not appear to discard any of the distribution-theoretic aspects of the formalism. These aspects are, in fact, made considerably more natural and transparent through use of (1.8).

In view of (i′)–(iii′), we feel that (1.8) is a sensible replacement for (1.3) in places where the latter is impractical or ambiguous. It is very possible, of course, that (1.8) may also entail some ambiguities. The value obtained could, for example, depend upon the net $\{A(x;\xi)\}$ chosen to represent $A(x)$ or upon the way in which the limits $\xi\rightarrow 0$, $\xi'\rightarrow 0$ are taken. Both of these difficulties will actually arise in some perturbation-theoretic examples we shall consider. The former is, however, easily resolved by an appeal to covariance properties, and the latter by imposing the requirement that the commutator reproduce the well-defined free-field results in the limit of vanishing coupling constant. We shall establish these free-field Wick product results in Sec. 3.

A more serious difficulty which could arise is the possibility that the orthodox commutator (1.3) might exist (in the sense that it is sequence-independent) and be different from (1.8). This is possible in view of the unjustified interchanges of limits and use of weak convergence involved in (1.8). Again we appeal to (i′)–(iii′) for evidence that this does not occur. Another possibility is that (1.3) might be sequence-dependent, but there exists no sequence $\{f_n\}$ which is such that it reproduces (1.8). We feel that this situation is unlikely, since we expect, by analogy with the fact that a suitable rearrangement of a conditionally convergent series can make it converge to any desired sum, that if (1.3) is sequence-dependent, there should exist a sequence appropriate to (1.8). In any case, any serious descrepancy between a sequence-dependent (1.3) and (1.8) would probably require a detailed investigation of questions concerning applicability before its significance could be understood.

We therefore feel that (1.8) provides a suitable method for calculating ETCR's in quantum field theory. These relations could then be abstracted and incorporated with those suggested by Gell-Mann in order to make the latter consistent with perturbation theory. The added information should allow a far wider range of applicability. Our results will of course be model-dependent, so that a suitable model from which to abstract must also be specified. An interesting possibility is the renormalizable quark model suggested by Johnson and Low.[28] In this paper, however, we shall be concerned with illustrating and justifying our method rather than with seriously suggesting commutators to be used for practical applications.

In addition to the possibility of abstraction, there are numerous other reasons for wanting to possess ETCR's valid in perturbation theory. Such relations would contain valuable information, useful in deriving sum rules, low-energy theorems, Ward identities, etc. Even in ordinary quantum electrodynamics, the proof of the generalized Ward identity based on ETCR's is far simpler than a rigorous direct proof. Furthermore, such relations are useful for actually specifying current operators[29] and for concisely expressing principles such as gauge invariance. Some concrete realizations of these possibilities will be discussed in Sec. 6.

There are, of course, in addition to (1.8), a variety of other possible definitions of ETC's. We shall give a critical discussion of these in Sec. 2 and compare them to (1.3) and (1.8). We shall conclude that (1.8) appears to be the simplest and most reliable alternative to (1.3).

C. Summary

In Sec. 2, we give a more precise version of our proposal (1.8) and compare it with other possible alternatives to (1.3). The methods of taking ET limits inside of spectral representations and of taking ET limits of time-ordered products from positive and negative time differences are criticized. The former procedure is invalid when the spectral function is nonintegrable and the latter when ET δ functions are present. In Sec. 3, we present a careful analysis of the ET behavior of spectral

[18] The point is that (1.9) can be well defined but not the individual terms in (1.10). In other words, (1.9) will not depend on the sequence $\{\theta_n(t)\}$ converging to $\theta(t)$, whereas each term in (1.10) can have such a dependence.

[19] R. Haag and G. Luzzato, Nuovo Cimento **13**, 415 (1959).

[20] P. Federbush, Progr. Theoret. Phys. (Kyoto) **26**, 148 (1961).

[21] K. Johnson, Nuovo Cimento **20**, 773 (1961).

[22] C. Sommerfield, Ann. Phys. (N. Y.) **26**, 1 (1963).

[23] L. S. Brown, Nuovo Cimento **29**, 617 (1963).

[24] W. Thirring and J. E. Wess, Ann. Phys. (N. Y.) **27**, 331 (1964).

[25] B. Klaiber, Nuovo Cimento **36**, 165 (1965).

[26] F. Schwabl, W. Thirring, and J. Wess, Ann. Phys. (N. Y.) **44**, 200 (1967).

[27] C. R. Hagen, Nuovo Cimento **51A**, 1033 (1967).

[28] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppls. **37**, 74 (1966); **38**, 74 (1966).

[29] I, Sec. IX.

representations and illustrate both of the effects. The ETCC $\langle 0|[j_k,j_4]|0\rangle$ in quantum electrodynamics is shown to have, in all orders of perturbation theory, a $\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')$ term in addition to the usual $\partial_k\delta(\mathbf{x}-\mathbf{x}')$ term. We also discuss other calculations of ETCCR's, including the perturbative calculations of Johnson and Low and of Langerholc.

We test our proposal in Sec. 6 by using it to calculate known ETC's in the extended Thirring model of Sommerfield, the four-dimensional derivative coupling model, and the free-field Wick product model. We obtain perfect agreement, except that the coefficient of the above-mentioned $\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')$ term in the latter model is not quite specified.

The field ETCR's and current operators valid in all orders of quantum electrodynamics (Gupta-Bleuler gauge) are exhibited in Sec. 5 and used, after some simplifications are introduced, to compute ETCR's by our method (1.8). We find, for example, the usual result $[j_4(x),\psi(x')]=-ie\psi(x)\delta(\mathbf{x}-\mathbf{x}')$ and the new result

$$12\pi^2Z_3{}^2e^{-2}[j_k(x),j_4(x')]=-\infty\,\partial_k\delta(\mathbf{x}-\mathbf{x}')-\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')+e^2[:\mathbf{A}^2:\partial_k+2:A_k\mathbf{A}\cdot\nabla:]\delta(\mathbf{x}-\mathbf{x}')\,. \quad (1.11)$$

This expression was made unique by the requirement that it reproduce the correct free-field result for $e\to 0$. It is exact in fourth order but may require an over-all constant multiplier in higher orders.

Section 6 is devoted mainly to a discussion of (1.11). We show that it is consistent with gauge invariance and is suitable for use in reduction formulas. Then, assuming reduction formulas, we show explicitly that it is correct in fourth order. There the A^2 term corresponds to the usual finite subtraction required by the photon-photon scattering amplitude. Previous suggestions that the commutator is a c number are criticized.

2. DEFINITIONS OF ETC'S

In this section, we shall state more precisely our proposed definition of ETC's and compare it with other possible definitions. The current operators A, B are defined by field equations,[30] e.g.,

$$(\Box-\mu^2)a(x)=gA(x)\,, \quad (2.1)$$

$$(\Box-\nu^2)b(y)=hB(y)\,. \quad (2.2)$$

Some current operators are actually defined by their coupling to other (leptonic) currents, but this is irrelevant for our considerations. The currents can be represented as the weak limits

$$A(x)=\lim_{\xi\to 0}A(x;\xi)\,, \quad (2.3)$$

$$B(y)=\lim_{\eta\to 0}B(y;\eta)\,, \quad (2.4)$$

where ξ and η are spacelike vectors, say $\xi=(0,\boldsymbol{\xi})$, $\eta=(0,\boldsymbol{\eta})$. Now, $A(x)$ is an operator-valued distribution in x, and $A(x;\boldsymbol{\xi})$ is one in x and $\boldsymbol{\xi}$, so that (2.3) really means

$$\int dx\,\phi(x)\langle\alpha|A(x)|\beta\rangle=\lim_{r\to\infty}\int d\boldsymbol{\xi}dx\,\kappa_r(\boldsymbol{\xi})\phi(x)\langle\alpha|A(x;\boldsymbol{\xi})|\beta\rangle \quad (2.5)$$

for any suitable states $|\alpha\rangle$, $|\beta\rangle$, for any testing function $\phi(x)$, and for any sequence $\{\kappa_r(\boldsymbol{\xi})\}$ of testing functions converging to $\delta(\boldsymbol{\xi})$ in $\mathcal{S}'$. Similarly,

$$\int dy\,\psi(y)\langle\alpha|B(y)|\beta\rangle=\lim_{s\to\infty}\int d\boldsymbol{\eta}dy\,\lambda_s(\boldsymbol{\eta})\psi(y)\langle\alpha|B(y;\boldsymbol{\eta})|\beta\rangle\,, \quad (2.6)$$

with $\lambda_s(\eta)\to\delta(\boldsymbol{\eta})$.

Now the quantity

$$\int dxdy\,\phi(x)\psi(y)\langle\alpha|[A(x),B(y)]|\beta\rangle$$

is well defined, and the orthodox value of the corresponding ETC is obtained by setting

$$\phi_m(x)=\phi(\mathbf{x})\phi_m(x_0-t)\,,\quad \psi_n(y)=\psi(\mathbf{y})\psi_n(y_0-t)\,, \quad (2.7)$$

where $\phi_m(\tau)\to\delta(\tau)$ and $\psi_n(\tau)\to\delta(\tau)$, and calculating

$$\lim_{m,n}\int dxdy\,\phi_m(x)\psi_n(y)\langle\alpha|[A(x),B(y)]|\beta\rangle\,. \quad (2.8)$$

We note that this is necessarily equal to[31]

$$\lim_{m,n}\lim_{r,s}\int d\boldsymbol{\xi}d\boldsymbol{\eta}\,dxdy\,\kappa_r\lambda_s\phi_m\psi_n\times\langle\alpha|[A(x;\boldsymbol{\xi}),B(y;\boldsymbol{\eta})]|\beta\rangle\,. \quad (2.9)$$

Our definition of ETC is obtained by interchanging the two limits in (2.9):

$$\lim_{r,s}\lim_{m,n}\int d\boldsymbol{\xi}d\boldsymbol{\eta}\,dxdy\,\kappa_r\lambda_s\phi_m\psi_n\times\langle\alpha|[A(x;\boldsymbol{\xi}),B(y;\boldsymbol{\eta})]|\beta\rangle\,. \quad (2.10)$$

We thus see that going from the orthodox definition (2.8) to our definition (2.10) involves an unjustified interchange of spacelike and timelike limits. This operation, however, might be expected to have validity in a relativistic theory.

In any case, as emphasized in the Introduction, (2.10) appears to resolve the ambiguities and other difficulties

[30] Notation: Greek indices are summed from 1 to 4, Latin indices from 1 to 3. Spatial vectors are written in boldface. Thus $p^2=p_ip_i+p_4p_4=\mathbf{p}^2+p_4{}^2=\mathbf{p}^2-p_0{}^2$. Also $\Box=\partial_\mu\partial_\mu=\Delta+\partial_4{}^2=\Delta-\partial_0{}^2$, $\Delta=\partial_i\partial_i=\nabla^2$. We write $\partial/\partial x_\mu F(x)=\partial_\mu F(x)=F_{,\mu}(x)$, $\delta^{(4)}(x)=\delta(x)$, $\delta^{(3)}(\mathbf{x})=\delta(\mathbf{x})$, $\int d^4x=\int dx$, and $\int d^3x=\int d\mathbf{x}$.

[31] This assumes that the states are suitable. We shall ignore such domain questions in this paper.

connected with (2.8) and to be much more manageable. There are, of course, other possible alternatives to (2.8), and in the remainder of this section, we shall critically consider some of those which have been proposed.

Let us first consider the possibility of making use of the fact that the Wightman functions are boundary values of analytic functions.[32] In this way, distributions need not be explicitly considered. Wightman[32] has shown that the free-field canonical commutation relations are equivalent to a certain relation between the n- and $(n-2)$-point Wightman functions involving a contour integral. In more singular cases, however, this is not appropriate, and direct consideration of the singularities involved is required. In realistic cases, moreover, the relevant calculations seem to be more involved and to require more information than the orthodox one (2.8).

Another possibility for reducing the effort involved in computing the orthodox ETC (2.8) is to introduce the Jost-Lehmann-Dyson (JLD) representation[33] for the matrix element of the commutator. We shall consider in detail the simplest case of the vacuum expectation value and the corresponding Källén-Lehmann representation.[34–36] We note that in practice one is only interested in the "truncated" commutator

$$[A,B]_T \equiv [A,B] - \langle 0|[A,B]|0\rangle. \tag{2.11}$$

Furthermore, from a theoretical point of view, the ET limit of (2.11) is much more likely to exist than that of the ordinary commutator $[A,B]$. We shall nevertheless explicitly consider the VEV, both because it simply illustrates the properties of more interesting matrix elements and because, even when its ET limit does not exist, it contains well-defined information which will be of considerable use to us later on.

We write

$$\langle 0|[A(x),B(y)]|0\rangle = \sum_{i=1}^{N} \int d\rho_i(a)\; O_i(\partial)\Delta(x-y;a)\,, \tag{2.12}$$

where the $d\rho_i(a)$ are measures, the $O_i(\partial)$ are polynomials in ∂ with vector and/or spinor indices corresponding to those of AB, and $\Delta(x;a)$ is the usual commutator function with mass $\sqrt{a}$. Now, the $O_i(\partial) \times\Delta(x-y;a)$ have simple and easily computable ET properties, and it is tempting to assume that the ET limit of (2.12) could be taken inside the integral. Consider, for example, the usual case where one has

$$\int d\rho(a)\; \partial_0\Delta(x-y;a)\,, \tag{2.13}$$

with ρ a positive measure bounded by a power.[32] The integrand has a well-defined (orthodox) ET limit

$$\partial_0\Delta(x;a)|_{x_0=0} = -\delta(\mathbf{x})\,. \tag{2.14}$$

Thus one often concludes that the ET limit of (2.13) is $-\delta(\mathbf{x})\int d\rho(a)$. This is only correct, however, provided $\int d\rho(a)<\infty$. In the alternative case, the conclusion that the ET limit of (2.13) is $-\infty\,\delta(\mathbf{x})$ is incorrect for two reasons. It is misleading[32] [since the correct infinite coefficient of $\delta(\mathbf{x})$ actually arises in a completely different way], and it is incomplete [since higher derivatives of $\delta(\mathbf{x})$ may be present]. We shall illustrate these remarks in Sec. 3.

A similar situation exists with regard to the JLD representation. Assuming suitable spectral functions, it has been used to present a rigorous derivation of the Adler-Weisberger relation,[37] to derive mass relations,[38] and also to study possible noncanonical terms in commutators.[39,40] Here, however, there need be no simple connection between bad spectral-function behavior and nonexistence of ET limits.[41]

As another means for resolving in perturbation theory some of the ambiguities described in (i)-(iii) of Sec. 1, we mention the possibility of explicitly performing the intermediate sum implicit in (1.1).[42] Each of the factors $\langle\alpha|A(x)|\gamma\rangle$ and $\langle\gamma|B(y)|\beta\rangle$ is well defined for all x and y, and ambiguities occur only in connection with the possible divergence of the sum over intermediate states $|\gamma\rangle$. These can be handled, moreover, in analogy with the procedure in conventional renormalization theory. Unfortunately, this method is quite tedious and becomes effectively impossible to perform in higher orders.

In certain circumstances, some of the difficulties described in (i)-(iii) of Sec. 1 may not be present. Let us suppose that difficulty (i) does not exist, so that (1.3) is sequence-independent and, furthermore, that the same is true for the ET limits of each of the Wightman functions

$$W_{\alpha\beta}{}^{AB}(x,y) = \langle\alpha|A(x)B(y)|\beta\rangle \tag{2.15}$$

and

$$\tilde{W}_{\alpha\beta}{}^{AB}(x,y) = \langle\alpha|B(y)A(x)|\beta\rangle\,. \tag{2.16}$$

It then follows that much of difficulty (ii) can be avoided. Indeed, choosing suitable sequences $\{f_n{}^{(\pm)}\}$ which satisfy

$$f_n{}^{(\pm)}(\tau)\to\delta(\tau)\,,\quad f_n{}^{(\pm)}(\mp\tau)=0 \quad \text{for}\quad \tau\geq 0\,, \tag{2.17}$$

[32] A. S. Wightman, Phys. Rev. **101**, 860 (1956).
[33] R. Jost and H. Lehmann, Nuovo Cimento **5**, 1598 (1957); F. J. Dyson, Phys. Rev. **110**, 1460 (1958).
[34] G. Källén, Helv. Phys. Acta **25**, 417 (1952).
[35] H. Lehmann, Nuovo Cimento **11**, 342 (1954).
[36] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).
[37] B. Schroer and P. Stichel, Commun. Math. Phys. **3**, 258 (1966).
[38] A. H. Völkel, University of Pittsburgh Report No. NYO-3829-6 (unpublished).
[39] J. L. Gervais and M. LeBellac, Nuovo Cimento **47A**, 822, (1967).
[40] J. W. Meyer and H. Suura, Phys. Rev. **160**, 1366 (1967).
[41] B. Schroer and P. Stichel, Phys. Rev. **162**, 1394 (1967).
[42] R. A. Brandt and Y. S. Kim, Phys. Rev. **161**, 1473 (1967).

we can write

$$E(\mathbf{x},\mathbf{y};x_0)=\lim_n \int dy_0 \,[W(x,y)f_n^{(+)}(x_0-y_0) - \tilde{W}(x,y)f_n^{(-)}(x_0-y_0)]$$

$$=\lim_n \int dy_0\, T(x,y) \times[f_n^{(+)}(x_0-y_0)-f_n^{(-)}(x_0-y_0)]. \quad (2.18)$$

Here only the time-ordered product (1.6) appears and, furthermore, in a way which is independent of its ambiguities at $x=y$.

Whether or not the above situation prevails, however, can only be determined after computing the limits (1.3) and corresponding ones involving (2.15) and (2.16). Thus difficulties (i) and (ii) are not really avoided. One might, of course, choose to define ETC's by (2.18). This need not correspond to choosing a definite sequence in (1.3) but is nevertheless an attractive possibility in view of its avoidance of much of difficulty (ii). We feel, however, that this definition neglects important aspects of the ET behavior of the commutator and need not correspond to a meaningful notion of ETC.

Namely, if the commutator function (1.1) contains distributions with support at $x_0=y_0$, these will be completely overlooked by (2.18). When this is the case, the result of the orthodox calculation (1.3) will involve a dependence on the numbers $f_n^{(r)}(0)$ for some r's, as in Eq. (1.5). We shall then say that (1.1) has "discrete ET support" or contains "discrete ET singularities." This situation is far from academic and, as we shall see, frequently occurs. If such terms are present in (1.1), then they apparently constitute a significant aspect of its ET behavior which should play a role in its ET limit. Thus, in this case, (2.18) does not really seem to correspond to the ETC of A and B.

We would like to emphasize that the existence of a discrete ET singularity does not mean that a meaningful ETC will not exist, although the definition (1.3) will clearly not lead to one for a *general* sequence. The point is that a definite regularization procedure, dictated, say, by the way in which the ETCR's are to be used, can be employed. This could correspond to a definite choice for the sequence $\{f_n\}$, which would assign definite values to the $\lim_n f_n^{(r)}(0)$. Equation (2.18) does, of course, amount to a regularization procedure which completely disposes with all discrete ET singularities. We shall see in Sec. 6, however, that (2.18) is not always suitable for use in reduction formulas. We take this to imply that the associated regularization is not useful.

A related consequence of the definition (2.18) is that it cannot take all the effects of renormalization into account. Indeed, many renormalization subtractions are proportional to $\delta(x-y)=\delta(\mathbf{x}-\mathbf{y})\delta(x_0-y_0)$ and its derivatives and therefore do not contribute to (2.18).

Often in the literature, one finds ETC's defined by the formal counterpart of (2.18):

$$E(\mathbf{x},\mathbf{y};x_0)=\lim_{y_0\to x_0-} W(x,y)-\lim_{y_0\to x_0+}\tilde{W}(x,y) =(\lim_{y_0\to x_0-}-\lim_{y_0\to x_0+})T(x,y). \quad (2.19)$$

This definition only makes sense when W is a *function* for y_0 in some open interval $(x_0,x_0+\epsilon)$, and $\tilde{W}$ is a function for y_0 in some $(x_0-\epsilon',x_0)$. This is not even the case for free scalar fields. Thus the calculation of (2.14) from the behavior

$$\Delta_F(x;a)\approx-(1/4\pi)\delta(x^2),\quad x^2\approx 0 \quad (2.20)$$

by means of (2.19) is completely ambiguous, whereas (2.18) immediately gives (2.14), as does (1.3) from

$$\Delta(x;a)\approx-(1/2\pi)\epsilon(x_0)\delta(x^2),\quad x^2\approx 0. \quad (2.21)$$

Moreover, even when (2.19) does make sense, it still possesses the inadequacies mentioned above when discrete ET singularities are present. In some simple model field theories, however, the singularities of the Wightman functions are mild enough to permit application of (2.19). We shall comment further on this in Sec. 3.

As a final example of an approach to ETC's, we shall consider the functional differentiation formalism. Let us specifically consider quantum electrodynamics. Here it is argued[43,44] that, as a consequence of gauge invariance,

$$-i[j^0(x),j^l(x')]=\partial_k(\delta_3'j^l(x')/\delta A_k(x)), \quad (2.22)$$

and that, as a consequence of Lorentz covariance,

$$\delta_3'j^l(x')/\delta A_k(x)=-\delta(\mathbf{x}-\mathbf{x}')\rho^{lk}(x) \quad (2.23)$$

for some symmetric operator ρ. Now, although the three-dimensional functional derivations in (2.22) are rather formal operations, this equation appears to constitute a definition of ETC which, at least in quantum electrodynamics, possesses many of the advantages of the one which we have proposed. Thus (2.22) can be computed directly from our electric-current operator, assuming that the differentiation can be interchanged with the local limit $\xi\to 0$. Although, as we shall see, (2.22) and (2.23) are violated in second-order perturbation theory (i.e., for free fields), (2.22) appears at least to reproduce the correct form of the ETC.

3. CALCULATIONS OF ETCR'S

A. Introduction

In this section, we shall discuss the calculations of ETC's which have appeared in the literature. We shall also use the orthodox definition to calculate the vacuum

[43] J. Schwinger, Phys. Rev. **130**, 406 (1963).
[44] L. S. Brown, Phys. Rev. **150**, 1338 (1966).

expectation values of ETC's of some Wick products of free fields. The results of this calculation will be useful in assessing the validity of some of the approaches (including our own) summarized in Sec. 2.

The simplest examples of ETC's with noncanonical values are provided by the soluble "relativistic" two-dimensional models.[45] In most of these models, the formal expression (2.19) gives results in agreement with the orthodox definition (1.3) of ETC. In Sec. 4, we shall show that our definition (1.8) also gives correct results. We shall do likewise for the four-dimensional derivative coupling model.

A matrix element of an ETC in the Lee model has been calculated using (2.19).[46] This commutator was found to have its canonical value when a cutoff was present but to differ from that value by a multiplicative constant for the theory with no cutoff. This corresponds to a δ-function noncanonical term. The same results are obtained[47] using the orthodox definition (1.3). However, in applying these results to derive low-energy theorems, the orthodox method[47] must be employed in order to avoid ambiguities which arise from formal manipulations. We shall not apply our definition (1.8) to this model, since we do not expect the associated space and time interchanges to be valid in nonrelativistic situations.

The first example which exhibited noncanonical terms in ETC's was given by Goto and Imamura[3] in 1955. They considered essentially the spectral representation

$$F(x)=\int_0^\infty da\,\pi(a)\partial_0\Delta(x;a)\,, \tag{3.1}$$

with $\pi(a)$ a positive function,[48] and brought the ET limit inside the integral to obtain

$$F(\mathbf{x},0)=-\int da\,\pi(a)\delta(\mathbf{x})\,, \tag{3.2}$$

whereas the canonical CR's give $F(\mathbf{x},0)=0$.

Wightman[32] pointed out in 1956 that the conclusion (3.2) is valid only for $\int da\,\pi(a)<\infty$. He showed that for $\pi(a)=1/\sqrt{a}$, (3.2) is misleading. We shall see below that for $\pi(a)=1$, (3.2) is incomplete, since a term $\Delta\delta(\mathbf{x})$ must then also be present. Nevertheless, variations of the above formal argument have often subsequently appeared in the literature.[49–54] In most of these cases, however, the authors were probably concerned only with the case $\int da\,\pi(a)<\infty$.

B. Spectral Representations

General Results

The quantity $\int da\,\pi(a)$ is almost always infinite in renormalized perturbation theory, and so we shall undertake a detailed study of the ET behavior of (3.1) in this situation. We shall consider the case for tempered fields in which $\pi(a)$ is positive and bounded by a power[55]

$$\int_0^\lambda da\,\pi(a)\leq C(1+\lambda^L) \tag{3.3}$$

for some C and L. For $\phi(\mathbf{x})\in\mathcal{S}(R^3)$ and $f_n(t)\in\mathcal{S}(R^1)=\mathcal{S}$, we have

$$\langle F(x),\phi(\mathbf{x})f_n(t)\rangle=\frac{1}{(2\pi)^4}\int da\,\pi(a)\int dp\,(-ip_0)\times\hat{\Delta}(p;a)\hat{\phi}(-\mathbf{p})\hat{f}_n(-p_0)\,, \tag{3.4}$$

where carets denote Fourier transforms, so that

$$\hat{\Delta}(p;a)=\int dx\,e^{-ip\cdot x}\Delta(x;a)=-2\pi i\delta(p^2+a)\epsilon(p)\,. \tag{3.5}$$

Thus

$$\frac{1}{(2\pi)^4}\int dp\,(-ip_0)\hat{\Delta}(p;a)\hat{\phi}(-\mathbf{p})\hat{f}_n(-p_0)=-\frac{1}{(2\pi)^3}\int d\mathbf{p}\,\hat{\phi}(-\mathbf{p})\hat{f}_n{}^e((\mathbf{p}^2+a)^{1/2})\,, \tag{3.6}$$

where

$$\hat{f}_n{}^e(\nu)\equiv\tfrac{1}{2}[\hat{f}_n(\nu)+\hat{f}_n(-\nu)]\,. \tag{3.7}$$

Now we set

$$d\mathbf{p}=\mathbf{p}^2d|\mathbf{p}|d\Omega=\tfrac{1}{2}|\mathbf{p}|d\mathbf{p}^2d\Omega\,,\quad b=\mathbf{p}^2 \tag{3.8}$$

and

$$\phi(b)\equiv-\frac{1}{(2\pi)^3}\int d\Omega\,\tfrac{1}{2}b^{1/2}\hat{\phi}(-\mathbf{p})\,, \tag{3.9}$$

so that (3.4) becomes

$$\langle F(x),\phi(\mathbf{x})f_n(t)\rangle=\int_0^\infty da\int_0^\infty db\,\pi(a)\phi(b)\hat{f}_n{}^e((a+b)^{1/2})\,. \tag{3.10}$$

For any positive integer R, we can write

$$\hat{f}_n{}^e((a+b)^{1/2})=\sum_{r=0}^{R}\frac{b^r}{r!}\left(\frac{\partial}{\partial a}\right)^r\hat{f}_n{}^e(a^{1/2})+g_n{}^{(R)}(a,b)\,, \tag{3.11}$$

[45] For a complete and useful critical survey of these models, with references to the original literature, the reader is referred to the lectures of Wightman (Ref. 55).
[46] J. S. Bell, Nuovo Cimento **47A**, 616 (1967).
[47] R. A. Brandt and C. A. Orzalesi, Phys. Rev. **162**, 1747 (1967).
[48] In the cases we shall consider, $\pi(a)$ will be a measurable function or a δ function. In the general case $da\pi(a)$ can represent a more complicated measure.
[49] K. Johnson, Nucl. Phys. **25**, 431 (1961).
[50] G. Pócsik, Nuovo Cimento **43A**, 541 (1966).
[51] S. Okubo, Nuovo Cimento **44A**, 1015 (1966).
[52] L. S. Brown, Phys. Rev. **150**, 1388 (1966), Appendix A.
[53] D. G. Boulware, Phys. Rev. **151**, 1024 (1966).
[54] D. G. Boulware and S. Deser, Phys. Rev. **151**, 1278 (1966).
[55] A. S. Wightman, in *High Energy Electromagnetic Interactions and Field Theory*, edited by M. Lèvy (Gordon and Breach Science Publishers, Inc., New York, 1966).

where

$$g_n^{(R)}(a,b)=\frac{1}{R!}\int_0^1 d\tau(1-\tau)^R\left(\frac{\partial}{\partial\tau}\right)^{R+1} \times\hat{f}_n^e((a+b\tau)^{1/2}). \quad (3.12)$$

We now let $\{f_n(t)\}$ [and hence $\{f_n^e(t)\}$] be a sequence of functions in $\mathcal{S}$ converging to $\delta(t)$ in $\mathcal{C}'$. Then $\{\hat{f}_n^e(\nu)\}$ is a sequence of functions in $\mathcal{S}$ converging to 1 in $\mathcal{S}'$ and also pointwise. From this it is easy to see (from integration by parts) that for $R>L$, the term $g_n^{(R)}(a,b)$ in (3.11) gives a vanishing contribution to (3.10) in the limit $n\to\infty$. Hence the distribution $\bar{F}(x)$ defined by

$$\langle\bar{F}(x),\phi(\mathbf{x})f_n(t)\rangle\equiv\int da\int db\,\pi(a)\phi(b)g_n^{(R)}(a,b), \quad (3.13)$$

with $R>L$, vanishes for $t=0$ in the orthodox sense:

$$\bar{F}(\mathbf{x},0)=0. \quad (3.14)$$

That is,

$$\lim_{n\to\infty}\langle\bar{F}(x),\phi(\mathbf{x})f_n(t)\rangle=0 \quad (3.15)$$

for each $\phi\in\mathcal{S}(R^3)$ and for each sequence $\{f_n\}$.

We now write (3.10) as

$$\langle F(x),\phi(\mathbf{x})f_n(t)\rangle=-\sum_{r=0}^{R}\frac{(-1)^r}{r!}\int da\,\pi(a)\left(\frac{\partial}{\partial a}\right)^r\hat{f}_n^e(a^{1/2}) \times\frac{1}{(2\pi)^3}\int d\mathbf{p}(-\mathbf{p}^2)^r\hat{\phi}(-\mathbf{p})+\langle\bar{F}(x),\phi(\mathbf{x})f_n(t)\rangle, \quad (3.16)$$

or

$$F(x)=\sum_{r=0}^{R}\mathfrak{F}_r(t)\Delta^r\delta(\mathbf{x})+\bar{F}(x), \quad (3.17)$$

where

$$\mathfrak{F}_r(t)=-\frac{1}{r!}\int d\nu\,e^{-i\nu t}|\nu|\pi^{(r)}(\nu^2) \quad (3.18)$$

in the sense of distribution theory. The $t\sim0$ behavior of $\mathfrak{F}_r(t)$ is learned by taking the limit $n\to\infty$ in (3.16). Formally defining $F(\mathbf{x},0)$ by

$$\langle F(\mathbf{x},0),\phi(\mathbf{x})\rangle=\lim_{n\to\infty}\langle F(x),\phi(\mathbf{x})f_n(t)\rangle, \quad (3.19)$$

we obtain

$$F(\mathbf{x},0)=\sum_{r=0}^{R}K_r\Delta^r\delta(\mathbf{x}), \quad (3.20)$$

where

$$K_r=\lim_{n\to\infty}K_{r,n}, \quad (3.21)$$

with

$$K_{r,n}=-\frac{(-1)^r}{r!}\int da\,\pi(a)\left(\frac{\partial}{\partial a}\right)^r\hat{f}_n^e(a^{1/2}). \quad (3.22)$$

Note that we can put $R=\infty$ in (3.20), since $K_r=0$ for $r>R$:

$$F(\mathbf{x},0)=\sum_{r=0}^{\infty}K_r\Delta^r\delta(\mathbf{x}). \quad (3.23)$$

Examples

We shall now discuss the existence and uniqueness of (3.23) for some specific spectral functions $\pi(a)$. We note first that when $\int da\,\pi(a)<\infty$, Eqs. (3.21) and (3.22) give $K_0=-\int da\,\pi(a)$ and $K_r=0$ for $r>0$, so that (3.20) agrees with (3.2).

Let us next consider the case $\pi(a)=1/\sqrt{a}$ for which $\int da\,\pi(a)=\infty$. Then

$$\begin{aligned}K_{0,n}&=-\int_0^\infty da\,a^{-1/2}\hat{f}_n^e(a^{1/2})=-2\int_0^\infty d\nu\,\hat{f}_n^e(\nu)\\&=-2\int_{-\infty}^\infty d\nu\,\hat{f}_n^e(\nu)\theta(\nu)\\&=-4\pi\int_{-\infty}^\infty dt\,f_n^e(t)\left[\frac{i}{2\pi t}+\tfrac{1}{2}\delta(t)\right]\\&=-2\pi f_n^e(0),\end{aligned} \quad (3.24)$$

$$\begin{aligned}K_{1,n}&=+\int_0^\infty da\,a^{-1/2}\frac{\partial}{\partial a}\hat{f}_n^e(a^{1/2})\\&=\int_0^\delta da\,a^{-1/2}\frac{\partial}{\partial a}\hat{f}_n^e(a^{1/2})-\frac{1}{\sqrt{\delta}}\hat{f}_n^e(\delta^{1/2})\\&\quad+\frac{1}{2}\int_\delta^\infty da\,a^{-3/2}\hat{f}_n^e(a^{1/2}),\end{aligned} \quad (3.25)$$

etc. We see that $K_r=0$ for $r>0$, whereas K_0 is ambiguous (i.e., sequence-dependent) and exhibits a discrete ET singularity of the type discussed earlier. Indeed, the corresponding (well-defined) four-dimensional distribution can be written as

$$F(x)=-2\pi\delta(x)+\bar{F}(x), \quad (3.26)$$

where

$$\bar{F}(\mathbf{x},0)=0, \quad (3.27)$$

which involves the distribution $\delta(t)$ with discrete ET support. The orthodox ET limit, consequently, is ill defined:

$$F(\mathbf{x},t)=-2\pi[\lim_{n\to\infty}f_n(0)]\delta(\mathbf{x}). \quad (3.28)$$

For $f_n(0)\to\infty$, the result (3.2) is reproduced, but in a completely different way. The definition (2.18), on the other hand, gives $F(\mathbf{x},0)=0$.

A general class of spectral functions containing similar discrete ET singularities is given by

$$\pi_{2k+1}(\nu^2)=\nu^{2k+1},\quad k=0,1,2,\cdots, \quad (3.29)$$

These give

$$K_{0,n}{}^{(2k+1)}=-2\int_0^\infty d\nu\, \nu^{2k+2}\hat{f}_n{}^e(\nu)$$
$$=-4\pi(-i)^{2k+2}\int dt\, f_n{}^{e(2k+2)}(t)\left[\frac{i}{2\pi t}+\tfrac{1}{2}\delta(t)\right]$$
$$=2\pi(-1)^k f_n{}^{e(2k+2)}(0)\,, \qquad (3.30)$$

etc. Thus (3.20) contains $2k+1$ terms with ambiguous coefficients of the $\Delta^r\delta(\mathbf{x})$. The corresponding four-dimensional distribution $F_{2k+1}(x)$ involves $\delta^{(2k+2)}(t)\delta(\mathbf{x})$, $\cdots$, $\delta(t)\Delta^{k+1}\delta(\mathbf{x})$. We see here higher-order derivatives of $\delta(\mathbf{x})$ not predicted by (3.2). For the $k=0$ case, we have

$$K_{0,n}{}^{(1)}=2\pi f_n{}^{e(2)}(0)\,,$$
$$K_{1,n}{}^{(1)}=\int_0^\infty d\nu\, \nu\partial\hat{f}_n{}^e(\nu)=-\int_0^\infty d\nu\, \hat{f}_n{}^e(\nu)=\pi f_n{}^e(0)\,,$$
$$K_{2,n}{}^{(1)}=-\tfrac{1}{4}\int_0^\infty d\nu\, \partial^2\hat{f}_n{}^e(\nu)=\tfrac{1}{4}\hat{f}_n{}^{e\prime}(0)\,, \qquad (3.31)$$

etc., so that

$$F_1(x)=2\pi\delta''(t)\delta(\mathbf{x})+\pi\delta(t)\Delta\delta(\mathbf{x})+\bar{F}_1(x)\,, \qquad (3.32)$$

where $\bar{F}_1(\mathbf{x},0)=0$, and, correspondingly,

$$F_1(\mathbf{x},0)=2\pi[\lim_{n\to\infty} f_n{}^{(2)}(0)]\delta(\mathbf{x}) +\pi[\lim_{n\to\infty} f_n(0)]\Delta\delta(\mathbf{x}) \qquad (3.33)$$

is ambiguous.

Another simple class of spectral functions is given by

$$\pi_{2k}(\nu^2)=\nu^{2k},\quad k=0,1,2,\cdots. \qquad (3.34)$$

These give

$$K_{0,n}{}^{(2k)}=-2\int_0^\infty d\nu\, \nu^{2k+1}\hat{f}_n{}^e(\nu)$$
$$=4\pi i(-1)^k\int dt\, f_n{}^{e(2k+1)}(t)\left[\frac{i}{2\pi t}+\tfrac{1}{2}\delta(t)\right]$$
$$=-2(-1)^k\int dt\, f_n{}^{e(2k+1)}(t)(1/t)\,, \qquad (3.35)$$

etc. Here the principal values $1/t$ contribute rather than $\delta(t)$, so that discrete ET singularities are not present. The $K_r{}^{(2k)}$ are still ambiguous for $r\leq 2k$. For the case $k=0$, we find

$$K_{0,n}{}^{(0)}=-2\int dt\, f_n{}^{e\prime}(t)(1/t)\,, \qquad (3.36)$$

$$K_{1,n}{}^{(0)}=\int_0^\infty d\nu\, \partial\hat{f}_n{}^e(\nu)=-\hat{f}_n{}^e(0)\,. \qquad (3.37)$$

Thus $K_0{}^{(0)}$ is ambiguous, $K_1{}^{(0)}=-1$, and $K_r{}^0=0$ for $r>1$. We note the finiteness and uniqueness of $K_1{}^{(0)}$, which will be of use to us later on. The corresponding four-dimensional distribution has the form

$$F_0(x)=2\partial_0(1/t)\delta(\mathbf{x})-\Delta\delta(\mathbf{x})+\bar{F}_0(x)\,, \qquad (3.38)$$

where $\bar{F}_0(\mathbf{x},0)=0$. The orthodox ET limit gives

$$F_0(\mathbf{x},0)=(?)\delta(\mathbf{x})-\Delta\delta(\mathbf{x})\,. \qquad (3.39)$$

The $\Delta\delta(\mathbf{x})$ term is unique and unambiguous, e.g., in the sense that

$$\lim_{t\to 0}\int d\mathbf{x}\, \mathbf{x}^2F_0(x)=-\int d\mathbf{x}\, \mathbf{x}^2\Delta\delta(\mathbf{x})=-6 \qquad (3.40)$$

for any approach to $t=0$. Nevertheless, it is still not predicted by (3.2).

Quantum Electrodynamics: General Formalism

Discussions analogous to the above could be given for more general $\pi(a)$ in (3.1) and for more general representations (2.12). Rather than becoming involved with unnecessary generalities, we shall consider only the representation (2.12) for the commutator $[j_\mu,j_\nu]$ involving the conserved electric current in quantum electrodynamics and the corresponding $\pi(a)$ to all orders in perturbation theory. We have[34]

$$F_{\mu\nu}(x-y)\equiv\langle 0|[j_\mu(x),j_\nu(y)]|0\rangle$$
$$=i\int da\, \pi(a)(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\Delta(x-y;a)\,, \qquad (3.41)$$

where $\pi(a)$ vanishes for $a\leq 4m^2$, with m the electron mass. The relation with our previous notation is

$$F_{k0}(x)=-i\partial_k F(x)\,. \qquad (3.42)$$

Proceeding as above, we find

$$\langle F_{kl}(x),\phi(\mathbf{x})f_n(t)\rangle=\sum_r\frac{(-1)^r}{r!}\int da\, \pi(a)\left(\frac{\partial}{\partial a}\right)^r$$
$$\times[\hat{f}_n{}^o(a^{1/2})a^{-1/2}]\frac{1}{(2\pi)^3}\int d\mathbf{p}\,(-\mathbf{p}^2)^r$$
$$\times(p_kp_l-\delta_{kl}\mathbf{p}^2)\hat{\phi}(-\mathbf{p})+\cdots\,, \qquad (3.43)$$

$$\langle F_{00}(x),\phi(\mathbf{x})f_n(t)\rangle=-\sum_r\frac{(-1)^r}{r!}\int da\, \pi(a)\left(\frac{\partial}{\partial a}\right)^r$$
$$\times[\hat{f}_n{}^o(a^{1/2})a^{-1/2}]\frac{1}{(2\pi)^3}\int d\mathbf{p}\,(-\mathbf{p}^2)^{r+1}$$
$$\times\hat{\phi}(-\mathbf{p})+\cdots\,, \qquad (3.44)$$

$$\langle F_{k0}(x),\phi(\mathbf{x})f_n(t)\rangle = i\sum_r \frac{(-1)^r}{r!}\int da\,\pi(a)\left(\frac{\partial}{\partial a}\right)^r \hat{f}_n{}^e(a^{1/2}) \times\frac{1}{(2\pi)^3}\int d\mathbf{p}\,(-ip_k)(-\mathbf{p}^2)^r \times\hat{\phi}(-\mathbf{p})+\cdots, \quad (3.45)$$

where the omitted terms vanish for $n\to\infty$ (i.e., for $t\to 0$), and

$$\hat{f}^o(\nu)=\tfrac{1}{2}[\hat{f}(\nu)-\hat{f}(-\nu)]. \quad (3.46)$$

The ET limits are

$$F_{kl}(\mathbf{x},0)=\sum_{r=0}^{\infty}\tilde{K}_r(\delta_{kl}\Delta-\partial_k\partial_l)\Delta^r\delta(\mathbf{x}), \quad (3.47)$$

$$F_{00}(\mathbf{x},0)=\sum_{r=0}^{\infty}\tilde{K}_r\Delta^{r+1}\delta(\mathbf{x}), \quad (3.48)$$

$$F_{k0}(\mathbf{x},0)=-i\sum_{r=0}^{\infty}K_r\partial_k\Delta^r\delta(\mathbf{x}), \quad (3.49)$$

where

$$\tilde{K}_r=\lim_{n\to\infty}\tilde{K}_{r,n},\quad K_r=\lim_{n\to\infty}K_{r,n}, \quad (3.50)$$

with

$$\tilde{K}_{r,n}=-\frac{(-1)^r}{r!}\int da\,\pi(a)\left(\frac{\partial}{\partial a}\right)^r[\hat{f}_n{}^0(a^{1/2})a^{-1/2}], \quad (3.51)$$

$$K_{r,n}=-\frac{(-1)^r}{r!}\int da\,\pi(a)\left(\frac{\partial}{\partial a}\right)^r\hat{f}_n{}^e(a^{1/2}). \quad (3.52)$$

Let us next investigate the behavior of $\pi(a)$ in perturbation theory. We shall use the fact that in all orders the leading singularities are independent of the mass m and hence are given by dimensional arguments.[56,13,14] The currents $j_\mu(x)$ have dimension (in inverse mass units) three $[\dim j=\dim\bar{\psi}\psi=\dim\bar{\psi}+\dim\psi=\frac{3}{2}+\frac{3}{2}=3]$, so that the leading singularity of $j_\mu(x)j_\nu(0)$ for $x\sim 0$ (with all components of order ζ, $\zeta\sim 0$) behaves as ζ^{-6} within logarithmic factors. Since $\Delta(x;a)\sim\zeta^{-2}$, we see from (3.41) that $\pi(a)$ can at worst grow logarithmically for large a (recall that a has units of mass squared). The same conclusion can be reached from consideration of (3.47)–(3.52). Thus (3.48) requires that $\tilde{K}_r=0$ for $r>0$ and (3.49) requires that $K_r=0$ for $r>1$. Comparison with (3.50)–(3.52) then again requires $\pi(a)$ to be logarithmically bounded.

This behavior of $\pi(a)$ is, of course, well known. It corresponds to the logarithmic divergence of the renormalization constant $Z_3{}^{-1}$,

$$Z_3{}^{-1}=1+\int da\,\pi(a)a^{-1}, \quad (3.53)$$

and to the finiteness of the quantity

$$k^2\int da\,\pi(a)a^{-1}(k^2+a)^{-1},$$

which is directly related to the polarization tensor $\Pi_{\mu\nu}(k)$.

The above information enables us to reduce (3.49) to

$$F_{k0}(\mathbf{x},0)=-K_0\partial_k\delta(\mathbf{x})-iK_1\partial_k\Delta\delta(\mathbf{x}), \quad (3.54)$$

where

$$K_0=-\lim_{n\to\infty}\int da\,\pi(a)\hat{f}_n{}^e(a^{1/2}) \quad (3.55)$$

is (within logs) at worst quadratically divergent, and

$$K_1=\lim_{n\to\infty}\int da\,\pi(a)\frac{\partial}{\partial a}\hat{f}_n{}^e(a^{1/2}) \quad (3.56)$$

is at worst logarithmically divergent. Formally, we have

$$K_0=-\int da\,\pi(a). \quad (3.57)$$

Furthermore, we can explicitly evaluate K_1 as follows. We have

$$K_1=-\lim_{n\to\infty}\int da\,\hat{f}_n{}^e(a^{1/2})\frac{\partial}{\partial a}\pi(a) = -\lim_{n\to\infty}\int d\nu\,\hat{f}_n{}^e(\nu)\frac{\partial}{\partial\nu}\pi(\nu^2). \quad (3.58)$$

Now $\partial\pi(\nu^2)/\partial\nu$ is square integrable, and hence so is its Fourier transform[57]

$$v(t)\equiv\lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{N}d\nu\,e^{-i\nu t}\frac{\partial}{\partial\nu}\pi(\nu^2). \quad (3.59)$$

Thus

$$K_1=-\lim_{n\to\infty}2\pi\int dt\,f_n{}^e(t)v(t) = -2\pi\lim_{t\downarrow 0}\tfrac{1}{2}[v(t)+v(-t)]=-\lim_{N\to\infty}\pi(N^2) \quad (3.60)$$

for any sequence $\{f_n(t)\}$.

We can now formally write (3.54) as

$$F_{k0}(\mathbf{x},0)=i\int da\,\pi(a)\partial_k\delta(\mathbf{x})+i\pi(\infty)\partial_k\Delta\delta(\mathbf{x}). \quad (3.61)$$

The second term, which is present in all orders of perturbation theory, has heretofore been overlooked in treatments of electrodynamics. For rigorous analysis, of course, Eq. (3.61) should be replaced by

$$F_{k0}(\mathbf{x},t)=-2\pi i w^e(t)\partial_k\delta(\mathbf{x}) +2\pi i v^e(t)\partial_k\Delta\delta(\mathbf{x})+\bar{F}_{k0}(\mathbf{x},t), \quad (3.62)$$

[56] K. Wilson (unpublished).

[57] The limit is taken in the L^2 norm.

where

$$\bar{F}_{k0}(\mathbf{x},0)=0\,, \tag{3.63}$$

and where $w(t)$ is the (distribution-theoretic) Fourier transform of $-2\nu\pi(\nu^2)$.

Quantum Electrodynamics: Low Orders

We shall conclude our study of the ET structure of the two-point commutator function by working out (3.54) for the explicit spectral functions $\pi(a)$ in second- and fourth-order perturbation theory. In second order, the $j_\mu(x)$ in (3.41) is the free-field Wick product

$$j_\mu(x)=ie\lim_{\xi\to0}[\bar{\psi}(x)\gamma_\mu\psi(x+\xi) -\langle0|\bar{\psi}(x)\gamma_\mu\psi(x+\xi)|0\rangle] \tag{3.64}$$

$$=\tfrac{1}{2}ie\lim_{\xi\to0}[\bar{\psi}(x)\gamma_\mu\psi(x+\xi)-\gamma_\mu\psi(x)\bar{\psi}(x+\xi)]\,. \tag{3.65}$$

The corresponding spectral function $\pi^{(2)}(a)$ is

$$\pi^{(2)}(a)=\frac{e^2}{12\pi^2}\left(1+\frac{\alpha}{2a}\right)\left(1-\frac{\alpha}{a}\right)^{1/2}\theta(a-\alpha)\,, \tag{3.66}$$

where

$$\alpha\equiv4m^2. \tag{3.67}$$

Equation (3.66) can be written

$$\pi^{(2)}(a)=\pi_0^{(2)}(a)+\pi_1^{(2)}(a)\,, \tag{3.68}$$

where

$$\pi_0^{(2)}(a)=(e^2/12\pi^2)\theta(a-\alpha)=\pi^{(2)}(\infty)\theta(a-\alpha)\,, \tag{3.69}$$

and

$$\pi_1^{(2)}(a)=\frac{e^2}{12\pi^2}\left[\left(1+\frac{\alpha}{2a}\right)\left(1-\frac{\alpha}{a}\right)^{1/2}-1\right]\theta(a-\alpha)\,. \tag{3.70}$$

We see that $\pi_1^{(2)}(a)\sim a^{-2}$ for large a, so that it contributes to K_0 the finite term $-\int da\,\pi_1^{(2)}(a)$. According to (3.60), the term $\pi_0^{(2)}(a)$ contributes to K_1 the finite term $-e^2/12\pi^2$. Its contribution to K_0 is the ambiguous expression

$$K_{00}^{(2)}=-\frac{e^2}{12\pi^2}\lim_{n\to\infty}\int_\alpha^\infty da\,\hat{f}_n^{\,e}(a^{1/2})$$

$$=\frac{e^2\alpha}{12\pi^2}-\frac{e^2}{6\pi^2}\int dt\,f_n^{\,e\prime}(t)\frac{1}{t}\,. \tag{3.71}$$

Thus

$$F_{k0}^{(2)}(\mathbf{x},0)=-i\left[\frac{e^2\alpha}{12\pi^2}-\frac{e^2}{6\pi^2}\lim_{n\to\infty}\int dt\,f_n^{\,e\prime}(t)\frac{1}{t} -\int da\,\pi_1^{(2)}(a)\right]\partial_k\delta(\mathbf{x})+\frac{ie^2}{12\pi^2}\partial_k\Delta\delta(\mathbf{x})\,. \tag{3.72}$$

We see that the previously neglected $\partial_k\Delta\delta(\mathbf{x})$ term is finite and unambiguous [in the sense of Eq. (3.40)]. The $\partial_k\delta(\mathbf{x})$ term is, however, ambiguous. It is divergent for a smooth sequence such as

$$f_n(t)=(n/\pi)^{1/2}e^{-nt^2}. \tag{3.73}$$

We observe that the result (3.72) is not given by the functional differential formalism (2.22) and (2.23).

Next we consider the fourth-order spectral function $\pi^{(4)}(a)$. It has been explicitly calculated by Källén and Sabry[58] and can be written

$$\pi^{(4)}(a)=(e^4/72\pi^4)(\ln a/\alpha)\theta(a-\alpha)+O(1)\,. \tag{3.74}$$

The $O(1)$ term contributes exactly as did $\pi^{(2)}(a)$ above. The first term in (3.74) gives an ambiguous contribution to K_0. Its contribution to K_1 diverges, but in a well-defined way as specified by Eq. (3.60):

$$K_1^{(4)}=-\lim_{N\to\infty}\pi^{(4)}(N^2)=-(e^4/72\pi^4)\lim_{N\to\infty}\ln N^2/\alpha\,. \tag{3.75}$$

This fact will be of use to us later on.

C. Free Fields

All of the above behaviors exhibited for the vacuum expectation values can also be derived for other matrix elements of CC's by using the JLD representation with suitable spectral functions. Here, however, the situation is somewhat more complicated,[41] and far less general statements can be made without explicit knowledge of the spectral functions.

For the model in which the currents are simply free-field Wick products, on the other hand, complete results can be obtained. Langerholc[59] has considered the Wick products corresponding to the currents

$$J(\Gamma_A;x)=\bar{\psi}(x)\Gamma_A\psi(x)\,, \tag{3.76}$$

where $\Gamma_A=\lambda_a\Lambda_\alpha$, with λ_a an internal symmetry matrix and Λ_α a Dirac matrix. He has investigated the behavior of

$$\int dz\,dX[J(\Gamma_A;x),J(\Gamma_B;y)]f(\mathbf{z},X)g(X_0)\phi(z_0)\,, \tag{3.77}$$

where $z=x-y$, $X=\frac{1}{2}(x+y)$, f and g are arbitrary testing functions, and $\phi(z_0)\to\delta(z_0)$. The expected result

$$[J(\Gamma_A;x),J(\Gamma_B;y)] -\langle0|\cdots|0\rangle\underset{z_0\to0}{\longrightarrow}i\delta(\mathbf{z})J([\Gamma_A,\Gamma_B]_4;x)\,, \tag{3.78}$$

where

$$[\Gamma_A,\Gamma_B]_4\equiv\Gamma_A\gamma_4\Gamma_B-\Gamma_B\gamma_4\Gamma_A\,, \tag{3.79}$$

was obtained. Combined with our treatment above of the VEV, this gives one a complete understanding of the ET behavior of free-field Wick products. In Sec. 4, we shall reproduce these results using our definition (1.8).

---

[58] G. Källén and A. Sabry, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd, **29**, No. 17 (1955).

[59] J. Langerholc, DESY Report No. 66/24 (unpublished).

D. Perturbation Theory

In the absence of more realistic field-theoretic models, the best way of going beyond the above results is apparently with the use of perturbation theory. An important step in this direction has been taken by Johnson and Low.[28] These authors used the definition (2.19) of ETC's and assumed that the time-ordered products (1.6) are given by the appropriate Feynman rules for timelike separation of x and y. In view of our remarks above concerning discrete ET singularities, however, we prefer to formulate results based on Eq. (2.19) as follows: either (i) discrete ET singularities exist (in which case, orthodox ET limits do not exist), or (ii) the ETC is given by (2.19).

It is formally shown in Ref. 28 that (2.19) is given by

$$E_{\alpha\beta}(\mathbf{x},\mathbf{y};x_0)=(2\pi)^{-4}e^{i\mathbf{q}\cdot(\mathbf{x}+\mathbf{y})}e^{-2ix_0q_0} \times\int d\mathbf{p}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\oint_C dp_0\, M_{\alpha\beta}(p+q,p-q)\,, \quad (3.80)$$

where C is an infinite clockwise contour, and

$$(2\pi)^4\delta(p_\alpha+k_1-p_\beta-k_2)M_{\alpha\beta}(k_1,k_2) =\int dx\, dy\langle\alpha|T(A(x)B(y))|\beta\rangle e^{-ik_1\cdot x+ik_2\cdot y}. \quad (3.81)$$

It should be remarked that since the derivation of (3.80) from (2.19) is only formal, (3.80) should be considered as the Johnson-Low definition of the ETC.

Johnson and Low use (3.80) to calculate the ETC's of the quark currents (3.76) in a model with interaction Lagrangian

$$\mathfrak{L}_I=g_s\bar{\psi}\psi\phi_s+g_p\bar{\psi}\gamma_5\psi\phi_p+g_v\bar{\psi}\gamma_\mu\psi A_\mu\,, \quad (3.82)$$

where ϕ_s, ϕ_p, and A_μ are, respectively, scalar, pseudoscalar, and vector fields. They calculate the vacuum-one-meson matrix elements of the commutators in third order and find the expected terms (with divergent coefficients, however) as well as extra (finite) terms in all commutators except $[V_0,V_0]$, $[A_0.A_0]$, $[S,S]$, $[S,P]$, and $[P,P]$. They also find that the Jacobi identity does not in general hold.

We have argued above that these results are probably precisely valid only when (1.1) has no ET singularities. The calculation, furthermore, required the manipulation of infinite quantities and a momentum-space cutoff. This involved an interchange of limits which might not be justified. Related considerations have been advanced by Hamprecht[60] and by Polkinghorne.[61] These authors argued that the extra terms found in Ref. 28 depend on the way in which the cutoff is introduced, and that they only arise from divergent integrals. In the absence of a justification for the choice of a particular cutoff procedure, they concluded that the presence of the extra terms has not been established.

Finally, we mention that the prescription (3.80) has been used[62] to calculate the c-number Schwinger terms. The results do not include all the terms predicted above by the orthodox method.

In view of these facts, the above results should perhaps be taken as indicative of the complicated nature of ETCR's rather than as precise statements about the values of ETC's.

An interesting application of the orthodox definition of ETC to the calculations of ETCR's in perturbation theory is given in the paper of Langerholc[59] referred to above. Langerholc explicitly constructed the renormalized current operators corresponding to (3.76) and (3.82) to fourth order (i.e., to second order in $\mathfrak{L}_I$) and again investigated (3.77). In first order, the expected (finite) right side of (3.78) again appeared, as well as numerous extra terms which were often divergent for any choice of the time-smearing sequence.[63] These terms were, however, in general quite different from those found in Ref. 28. This would be expected if discrete ET singularities were present. Similar conclusions were reached for the ETC of the second-order current operators.

4. SOLUBLE MODELS

In this section, we begin our detailed investigation of our proposal (1.8) by using it to calculate some known ETCR's in the soluble field-theoretic models. All these models have unit S matrix but nevertheless pose nontrivial tests on (1.8).

A. Extended Thirring Model

We shall begin by considering Sommerfield's[22] solution to the extended two-dimensional Thirring[45] model, in which the current j_μ of a massless fermion field ψ is coupled to itself and also to a massive vector boson field $B_{\mu\nu}$. Thus the Lagrangian is[64]

$$\mathfrak{L}=\tfrac{1}{2}i[\bar{\psi}\gamma\cdot\partial\psi-\partial\bar{\psi}\cdot\gamma\psi]-\tfrac{1}{2}[\partial A\cdot B-A\partial\cdot B] -\tfrac{1}{2}\mu^2A^2+\tfrac{1}{4}B^2+gj\cdot A+\tfrac{1}{2}\sigma j\cdot j\,, \quad (4.1)$$

where

$$\mu=0,1\,,\quad g^{11}=-g^{00}=1\,,\quad \gamma^0=\sigma_2\,,\quad \gamma^1=i\sigma_1. \quad (4.2)$$

Couplings with external fields are also considered as a device for solving the model. The assumed ETCR's satisfied by the canonical fields ψ, $\psi^\dagger$, A_1, and $B\equiv B_{01}$ are the standard canonical ones, so that the only non-

[60] B. Hamprecht, Nuovo Cimento **47A**, 770 (1967).

[61] J. C. Polkinghorne, Nuovo Cimento **52A**, 351 (1967).

[62] B. Hamprecht, Nuovo Cimento **50A**, 449 (1967).

[63] A p-space cutoff was employed to obtain these results. A cutoff is not used in the later DESY Report No. 67/26 (unpublished). See also B. Schroer and P. Stichel, University of Pittsburgh Report No. NYD-3829-11 (unpublished).

[64] We use here the notation of Ref. 22.

vanishing commutators are[16]

$$\{\psi_\alpha(x),\psi_\beta^\dagger(x')\}=\delta_{\alpha\beta}\delta(x_1-x_1') \quad (4.3)$$

and

$$[A_1(x),B(x')]=i\delta(x_1-x_1'). \quad (4.4)$$

The current $j^\mu(x)$ is defined, following Johnson,[21] as the limit of a nonlocal product of fermion fields in spacelike *and* timelike directions.

In addition, a noncovariant single-time current $S_\mu(x)$ is defined as

$$S_\mu(x)=\tfrac{1}{2}\lim_{\xi\to 0}[\bar\psi(x)\gamma_\mu\psi(x+\xi)-\gamma_\mu\psi(x)\bar\psi(x+\xi)] \quad (4.5)$$

$$=\lim_{\xi\to 0}[\bar\psi(x)\gamma_\mu\psi(x+\xi)-J_\mu(\xi)]\equiv\lim_{\xi\to 0}S_\mu(x;\xi), \quad (4.6)$$

where

$$\xi_0=0, \quad \bar\psi=\psi^\dagger\gamma_0,$$

and

$$J_\mu(\xi)=-i\,\mathrm{tr}\gamma_\mu G(\xi), \quad (4.7)$$

with

$$G(\xi)=i\langle 0|T\psi(x+\xi)\bar\psi(x)|0\rangle. \quad (4.8)$$

We can apply our proposal only to this current.

The currents j and S are found to satisfy a relationship involving the external fields. When j is considered as depending explicitly on these external fields, a consistent explicit solution to the model is found. For example, one has

$$G(\xi)=(1/2\pi)\gamma\cdot\xi[\xi^2+i\epsilon]^{-1}. \quad (4.9)$$

Equation (2.19) can be used to compute ETCR's in this model. The relations (4.3) and (4.4) are found to be valid in the solution. Sommerfield also finds, by explicit consideration of matrix elements, that the following ETCR's involving S_μ are valid:

$$[S_0(x),\psi(x')]=\psi(x)\delta(x_1-x_1'), \quad (4.10)$$

$$[S_1(x),\psi(x')]=-\gamma^0\gamma^1\psi(x)\delta(x_1-x_1'), \quad (4.11)$$

$$[S_0(x),A_1(x')]=[S_0(x),B(x')]=[S_1(x),A_1(x')]=[S_1(x),B(x')]=0, \quad (4.12)$$

$$[S_0(x),S_0(x')]=[S_1(x),S_1(x')]=0, \quad (4.13)$$

$$[S_0(x),S_1(x')]=(i/\pi)\partial_1\delta(x_1-x_1'). \quad (4.14)$$

Each of these CR's is precisely that obtained by taking the corresponding commutator inside the $\xi\to 0$ limit defining S_μ. This is immediately clear for (4.10)–(4.12). For example,

$$\lim_{\xi\to 0}[S_1(x;\xi),\psi(x')]=-\lim_{\xi\to 0}\gamma^0\gamma^1\psi(x+\xi)\delta(x_1-x_1')$$
$$=-\gamma^0\gamma^1\psi(x)\delta(x_1-x_1')$$
$$=[S_1(x),\psi(x')]. \quad (4.15)$$

The same is true for (4.13) and (4.14). Explicitly, we have

$$[S_1(x;\xi),S_0(x';\xi')]=\bar\psi(x)\gamma_1\psi(x'+\xi')\delta(x_1-x_1'+\xi_1)-\bar\psi(x')\gamma_1\psi(x+\xi)\delta(x_1-x_1'-\xi_1'). \quad (4.16)$$

Thus

$$\lim_{\xi'\to 0}[S_1(x;\xi),S_0(x';\xi')]$$
$$=\bar\psi(x)\gamma_1\psi(x+\xi)\delta(x_1-x_1'+\xi_1)-\bar\psi(x)\gamma_1\psi(x+\xi)\delta(x_1-x_1')$$
$$=\bar\psi(x)\gamma_1\psi(x+\xi)[\xi_1\partial_1\delta(x_1-x_1')+O(\xi_1^2)]. \quad (4.17)$$

Now we use Eq. (4.6) and the fact that

$$\lim_{\xi\to 0}S_\mu(x;\xi)\xi_1^2=0 \quad (4.18)$$

to write

$$\lim_{\xi\to 0}\bar\psi(x)\gamma_1\psi(x+\xi)\xi_1=\lim_{\xi\to 0}J_1(\xi)\xi_1=i/\pi \quad (4.19)$$

and

$$\lim_{\xi\to 0}\bar\psi(x)\gamma_1\psi(x+\xi)\xi_1^2=\lim_{\xi\to 0}J_1(\xi)\xi_1^2=0, \quad (4.20)$$

where we used (4.7) and (4.9). Hence

$$\lim_{\xi\to 0}\lim_{\xi'\to 0}[S_1(x;\xi),S_0(x';\xi')]=(i/\pi)\partial_1\delta(x_1-x_1')=[S_1(x),S_0(x')], \quad (4.21)$$

the last equality following from (4.14).

Thus our proposal (1.8) is seen to be correct in this case. We can likewise easily derive (4.13). We conclude that our definition of ETC is the same as the usual one for this model. Sommerfield[22] remarked that (4.14) could not be computed directly from the canonical commutation rules but must be inferred by evaluating its matrix elements. Using our definition (1.8), however, (4.14) can be derived from the canonical rules plus the properties (4.6) and (4.9) of the solution. That is precisely the effectiveness of our proposal.

B. Derivative Coupling Model

The next model we shall consider is that with vector derivative coupling between a spinor and a scalar field in four dimensions. This model is nontempered and has no scattering, but is nevertheless interesting for our purposes, since it possesses a nontrivial current operator and field ETCR. We shall use the careful and complete formulation of Klaiber.[25] Earlier references can be found in Ref. 55. The notation is that of Bogoliubov and Shirkov.[9]

The renormalized field operators ψ and ϕ satisfy the renormalized field equations

$$(\square-m^2)\phi(x)=0 \quad (4.22)$$

and

$$(i\gamma\cdot\partial-M)\psi(x)=h(x), \quad (4.23)$$

Thus $\phi(x)$ is a free field. The current operator $h(x)$ is given by

$$h(x)=\lim_{\xi\to 0} h(x;\,\xi)\,,\quad \xi_0=0\,, \tag{4.24}$$

with

$$h(x;\,\xi)=g\gamma^\mu[\psi(x)\partial_\mu\phi(x+\xi)-g\partial_\mu D^+(\xi)\cdot\psi(x)]. \tag{4.25}$$

The fields obey the following ETCR's:

$$[\phi(x),\phi(x')]=\{\psi(x),\psi(x')\}=[\phi(x),\psi(x')]=[\dot\phi(x),\dot\phi(x')]=0\,, \tag{4.26}$$

$$[\phi(x),\dot\phi(x')]=i\delta(\mathbf{x}-\mathbf{x}')\,, \tag{4.27}$$

$$\{\psi(x),\psi^\dagger(x')\}=Z^{-1}\delta(\mathbf{x}-\mathbf{x}')\,, \tag{4.28}$$

$$[\dot\phi(x),\psi(x')]=-g\psi(x)\delta(\mathbf{x}-\mathbf{x}'). \tag{4.29}$$

The constant Z^{-1} is ill defined, so that (4.28) is only a formal equation and will not be used.

We want to see if the ETC $[h(x),\phi(x')]$ can be correctly computed by our method of interchanging the commutation and $\xi\to 0$ limit. That this is the case is by no means obvious, in view of the relatively complicated nature of (4.23)–(4.29).

The commutator in question can be correctly calculated from the field equation (4.23) and the field CR's (4.26) and (4.29). It follows from (2.46) that[65]

$$0=\partial_0[\psi(x),\phi(x')]=[\dot\psi(x),\phi(x')]+[\psi(x),\dot\phi(x')]. \tag{4.30}$$

Thus, from (4.29), we find

$$[\dot\psi(x),\phi(x')]=-g\delta(\mathbf{x}-\mathbf{x}')\psi(x). \tag{4.31}$$

By (4.23), (4.31), and (4.26), we have

$$[h(x),\phi(x')]=[i\gamma^0\dot\psi(x)+i\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\psi(x)-M\psi(x),\phi(x')]=-ig\gamma^0\psi(x)\delta(\mathbf{x}-\mathbf{x}'). \tag{4.32}$$

Now, from (4.25), (4.26), and (4.27), we get

$$[h(x;\,\xi),\phi(x')]=-ig\gamma^0\psi(x)\delta(\mathbf{x}-\mathbf{x}'+\boldsymbol{\xi})\,, \tag{4.33}$$

so that

$$\lim_{\xi\to 0}[h(x;\,\xi),\phi(x)]=[h(x),\phi(x')]. \tag{4.34}$$

We therefore see that our definition of ETC agrees with the usual one in this case.

C. Free Fields, Truncated Commutator

The final model which we shall consider is that in which the current operators are Wick products of free spinor fields. Although this model is simpler than the previous two, new features will arise because we will work in four dimensions rather than in two as for the Thirring model, and we shall be considering more involved commutators than we did for the derivative coupling model.

[65] We are assuming here that ∂_0 acts on the ETC as a derivation. This need not be the case in general.

The currents, corresponding to (3.76), can be defined as

$$j(\Gamma_A;x)=\lim_{\xi\to 0} j'(\Gamma_A;x;\,\xi)\,,\quad \xi_0=0 \tag{4.35}$$

where

$$j'(\Gamma_A;x;\,\xi)=i\bar\psi(x)\Gamma_A\psi(x+\xi)-iJ^{(0)}(\Gamma_A:\,\xi)\,, \tag{4.36}$$

with

$$J^{(0)}(\Gamma_A;\,\xi)=\langle 0|\bar\psi(x)\Gamma_A\psi(x+\xi)|0\rangle. \tag{4.37}$$

An equivalent definition is

$$j(\Gamma_A;x)=\lim_{\xi\to 0} j(\Gamma_A;x;\,\xi)\,,\quad \xi_0=0 \tag{4.38}$$

where

$$j(\Gamma_A;x;\,\xi)=\tfrac{1}{2}i[\bar\psi(x)\Gamma_A\psi(x+\xi)-\Gamma_A\psi(x)\bar\psi(x+\xi)]=\tfrac{1}{2}i[\bar\psi(x)\Gamma_A\psi(x+\xi)+\bar\psi(x+\xi)\Gamma_A\psi(x)]\,, \tag{4.39}$$

the two forms being equivalent by spacelike anticommutivity. The fields satisfy the canonical ETCR's

$$\{\bar\psi(x),\psi(x')\}=\gamma_4\delta(\mathbf{x}-\mathbf{x}'). \tag{4.40}$$

The ETCCR's are given by Eq. (3.78), together with equations analogous to (3.72) for the VEV's.

We shall now calculate these ETCCR's by our method (1.8). Both j' and j give the same results for the truncated commutator in this case, but we shall work with j for future convenience. Using (4.40) to evaluate

$$[j(\Gamma_A;x;\,\xi),j(\Gamma_B;x';\,\xi')]\,,$$

we obtain a well-defined function of ξ and ξ', with a well-defined limit as $\xi'\to 0$:

$$\lim_{\xi'\to 0}[j(\Gamma_A;x;\,\xi),j(\Gamma_B;x';\,\xi')]$$
$$=-\tfrac{1}{2}\{\bar\psi(x)\Gamma\psi(x')\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}')-\bar\psi(x')\tilde\Gamma\psi(x+\xi)\delta(\mathbf{x}-\mathbf{x}')+\bar\psi(x+\xi)\Gamma\psi(x')\delta(\mathbf{x}-\mathbf{x}')-\bar\psi(x')\tilde\Gamma\psi(x)\delta(\mathbf{x}+\boldsymbol{\xi}-\mathbf{x}')\}\,, \tag{4.41}$$

where we wrote

$$\Gamma=\Gamma_A\gamma_4\Gamma_B\,,\quad \tilde\Gamma=\Gamma_B\gamma_4\Gamma_A. \tag{4.42}$$

Writing

$$\delta(\mathbf{x}-\mathbf{x}'+\boldsymbol{\xi})=\delta(\mathbf{x}-\mathbf{x}')+\boldsymbol{\xi}\cdot\boldsymbol{\partial}\delta(\mathbf{x}-\mathbf{x}')+\tfrac{1}{2}(\boldsymbol{\xi}\cdot\boldsymbol{\partial})^2\delta(\mathbf{x}-\mathbf{x}')+\tfrac{1}{6}(\boldsymbol{\xi}\cdot\boldsymbol{\partial})^3\delta(\mathbf{x}-\mathbf{x}')+O(\xi^4)\,, \tag{4.43}$$

Eq. (4.41) becomes

$$\lim_{\xi'\to 0}[j(\Gamma_A;x;\,\xi),j(\Gamma_B;x';\,\xi')]$$
$$=i[j(\Gamma;x;\,\xi)-j(\tilde\Gamma;x;\,\xi)]\delta(\mathbf{x}-\mathbf{x}')-\tfrac{1}{2}[\bar\psi(x)\Gamma\psi(x+\xi)-\bar\psi(x+\xi)\tilde\Gamma\psi(x)][\boldsymbol{\xi}\cdot\boldsymbol{\partial}\delta(\mathbf{x}-\mathbf{x}')+\tfrac{1}{2}(\boldsymbol{\xi}\cdot\boldsymbol{\partial})^2\delta(\mathbf{x}-\mathbf{x}')+\tfrac{1}{6}(\boldsymbol{\xi}\cdot\boldsymbol{\partial})^3\delta(\mathbf{x}-\mathbf{x}')+O(\xi^4)]. \tag{4.44}$$

Since $j'(\Gamma_A;x;\xi)$ has the finite limit $j(\Gamma_A;x)$ for $\xi\to 0$, it follows from (4.36) that

$$\lim_{\xi\to 0}\bar\psi(x)\Gamma_A\psi(x+\xi)\xi=\lim_{\xi\to 0}J^{(0)}(\Gamma_A;\,\xi)\xi.$$

Using this and (4.38) in (4.44), we obtain

$$\lim_{\xi\to0}\lim_{\xi'\to0}[j(\Gamma_A;x;\xi),j(\Gamma_B;x';\xi')]=ij([\Gamma_A,\Gamma_B]_4;x)$$
$$\times\delta(\mathbf{x}-\mathbf{x}')-\tfrac{1}{2}\lim_{\xi\to0}[J^{(0)}(\Gamma;\xi)-J^{(0)}(\tilde{\Gamma};-\xi)]$$
$$\times[\xi\cdot\boldsymbol{\partial}\delta(\mathbf{x}-\mathbf{x}')+\tfrac{1}{2}(\xi\cdot\boldsymbol{\partial})^2\delta(\mathbf{x}-\mathbf{x}')$$
$$+\tfrac{1}{6}(\xi\cdot\boldsymbol{\partial})^3\delta(\mathbf{x}-\mathbf{x}')+O(\xi^4)]. \quad (4.45)$$

Since $J^{(0)}$ is a c number, we see that our definition (1.8) of ETC reproduces the result (3.78) obtained from the orthodox definition. We have the rigorous result

$$\lim_{\xi\to0}\lim_{\xi'\to0}\{[j(\Gamma_A;x;\xi),j(\Gamma_B;x';\xi')]-\langle0|\cdots|0\rangle\}$$
$$=ij([\Gamma_A,\Gamma_B]_4;x)\delta(\mathbf{x}-\mathbf{x}')$$
$$=[j(\Gamma_A;x),j(\Gamma_B;y)]_T, \quad (4.46)$$

where T denotes the "truncated" commutator defined by Eq. (2.11).

D. Free Fields, Vacuum Expectation Value

Since the truncated commutator is all that is required in practice, we could stop here—content with our result. It will be very convenient for later purposes, however, to explicitly consider the VEV. We first make the incidental observation that for the $V-A$ algebra ($\Gamma=\gamma_4\lambda$ or $\gamma_4\gamma_5\lambda$), the $J^{(0)}(\Gamma;\xi)$ vanish identically (since $\xi_4=0$), so that the simple equations

$$[j(\Gamma_A;x),j(\Gamma_B;y)]=ij([\Gamma_A,\Gamma_B]_4;x)\delta(\mathbf{x}-\mathbf{x}')$$
$$(V-A\text{ algebra}) \quad (4.47)$$

are valid in this case. Thus the free-field Wick products form a representation of Gell-Mann's ET $V-A$ current algebra.[2]

For $\Gamma_A=\gamma_k$, $\Gamma_B=\gamma_4$, however, the $J^{(0)}$ do not vanish. This case will occupy us for the remainder of this section. We write

$$ej(\gamma_\mu;x;\xi)=j_\mu(x;\xi),\quad ej'(\gamma_\mu;x;\xi)=j_\mu'(x;\xi), \quad (4.48)$$

and

$$J^{(0)}(\gamma_k;\xi)=J_k^{(0)}(\xi)=J^{(0)}(\xi^2)\xi_k=-J_k^{(0)}(-\xi), \quad (4.49)$$

so that (4.45) becomes

$$\lim_{\xi,\xi'\to0}[j_k(x;\xi),j_4(x';\xi')]=-e^2\lim_{\xi\to0}J_k^{(0)}(\xi)$$
$$\times[\xi\cdot\boldsymbol{\partial}\delta(\mathbf{x}-\mathbf{x}')+\tfrac{1}{2}(\xi\cdot\boldsymbol{\partial})^2\delta(\mathbf{x}-\mathbf{x}')$$
$$+\tfrac{1}{6}(\xi\cdot\boldsymbol{\partial})^3\delta(\mathbf{x}-\mathbf{x}')+O(\xi^4)]. \quad (4.50)$$

Let us investigate the properties of (4.50) by using the known behavior of $J_k^{(0)}(\xi)$. We can put

$$J_k^{(0)}(\xi)=\mathrm{tr}\gamma_k S_F(\xi), \quad (4.51)$$

where

$$S_F(\xi)=\langle0|T\psi(x+\xi)\bar{\psi}(x)|0\rangle. \quad (4.52)$$

Only the leading singularity of $S_F(\xi)$ for $\xi\sim0$,

$$S_F(\xi)\sim\frac{1}{2\pi^2}\frac{\gamma\cdot\xi}{\xi^4}, \quad (4.53)$$

will contribute to (4.50).[66] Thus we can use

$$J_k^{(0)}(\xi)\sim\frac{2}{\pi^2}\frac{\xi_k}{\xi^4}. \quad (4.54)$$

If we formally define

$$J_{kl}^{(0)}=\lim_{\xi\to0}J_k^{(0)}(\xi)\xi_l=\lim_{\xi\to0}\frac{2}{\pi^2}\frac{\xi_k\xi_l}{\xi^4}, \quad (4.55)$$

$$J_{klm}^{(0)}=\lim_{\xi\to0}J_k^{(0)}(\xi)\xi_l\xi_m=\lim_{\xi\to0}\frac{2}{\pi^2}\frac{\xi_k\xi_l\xi_m}{\xi^4}, \quad (4.56)$$

$$J_{klmn}^{(0)}=\lim_{\xi\to0}J_k^{(0)}(\xi)\xi_l\xi_m\xi_n=\lim_{\xi\to0}\frac{2}{\pi^2}\frac{\xi_k\xi_l\xi_m\xi_n}{\xi^4}, \quad (4.57)$$

then (4.50) becomes

$$\lim_{\xi,\xi'\to0}[j_k(x;\xi),j_4(x';\xi')]=-e^2J_{kl}^{(0)}\partial_l\delta(\mathbf{x}-\mathbf{x}')$$
$$-\tfrac{1}{2}e^2J_{klm}^{(0)}\partial_l\partial_m\delta(\mathbf{x}-\mathbf{x}')$$
$$-\tfrac{1}{6}e^2J_{klmn}^{(0)}\partial_l\partial_m\partial_n\delta(\mathbf{x}-\mathbf{x}'). \quad (4.58)$$

We see that our method has reproduced the essential form of the correct result (3.72) for this ETC, but our coefficient (4.57) of the $\partial^3\delta(\mathbf{x}-\mathbf{x}')$ term is ambiguous, whereas the correct coefficient is unique. Indeed, (4.57) has a (finite) value which depends on how the limit $\xi\to0$ is taken. What has happened is that, although the limit of $j_\mu(x;\xi)$ for $\xi\to0$ gives $j_\mu(x)$ no matter how the limit is taken, the result of (4.58) *does* depend on how the limit is taken. This type of ambiguity is typical of the type which occurs when space and time limits are interchanged.[67] We shall see in Sec. 6 that it is directly related to the usual amgibuity which arises when an invariant amplitude is decomposed into the sum of a T product of currents and an ETC.

Since all limits give the same current operator, we must find some other criterion for choosing among the possible limits to use in (4.58). If no such criterion existed, then our method would not be useful in this case. Fortunately, we have the exact expression for the correct commutator given in (3.72) and we should, of course, take the limits in (4.58) in such a way that (3.72) is reproduced. We cannot then claim to have derived (3.72) by our method, but we shall see in Sec. 5 that the above ambiguity is the only one that occurs in *all* orders of perturbation theory. Thus resolving the ambiguity here will enable us to uniquely

[66] Actually, another term in $S_F(\xi)$ will contribute a finite quantity to the first term in (4.50), but this is negligible compared to the infinite contribution of (4.53).

[67] See, for example, Ref. 37, footnote 9.

determine the commutator in all orders—namely, by the requirement that it reproduce the unambiguous part of the known commutator (3.72) for $e^2\to 0$ (dividing out the trivial e^2 factor on each side).

Let us first note that, independently of knowledge of (3.72), the limits (4.55)-(4.57) should be taken in a rotationally invariant way in order to maintain spatial symmetry. Thus we write

$$J_{kl}^{(0)}=\infty\,\delta_{kl}\,,\quad J_{klm}^{(0)}=0 \tag{4.59}$$

and

$$J_{klmn}^{(0)}=(2/\pi^2)C\chi_{klmn}\,, \tag{4.60}$$

where

$$\chi_{klmn}=\delta_{kl}\delta_{mn}+\delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm}\,. \tag{4.61}$$

Then (4.58) becomes

$$[j_k(x),j_4(x')] =-e^2\infty\,\partial_k\delta(\mathbf{x}-\mathbf{x}')-(e^2/\pi^2)C\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')\,. \tag{4.62}$$

Comparison with (3.72) (using $ij_0=j_4$) now gives

$$C=\tfrac{1}{12}\,. \tag{4.63}$$

That is, we are to resolve the ambiguity in (4.57) by setting

$$\lim_{\xi\to 0}\frac{\xi_k\xi_l\xi_m\xi_n}{\xi^4}=\tfrac{1}{12}\chi_{klmn}\,. \tag{4.64}$$

Thus

$$[j_k(x),j_4(x')] =-e^2\infty\,\partial_k\delta(\mathbf{x}-\mathbf{x}')-(e^2/12\pi^2)\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')\,. \tag{4.65}$$

We note that (4.64) is not quite the result obtained by averaging with a spatially symmetric function with unit integral, which would give $(1/15)\chi_{klmn}$. Nevertheless, (4.64) is what is required to reproduce the orthodox result (3.72). We shall explicitly show in Sec. 6 that (4.64) is also correct in fourth order. We remark finally that, given (4.64), our method reproduces all of the correct free-field Wick product ETCR's with far greater ease than that required by the orthodox method. The $\partial\Delta\delta(\mathbf{x}-\mathbf{x}')$ term, in particular, arises in a very natural way, as in (4.44).

5. PERTURBATION THEORY

We are now ready to use our definition (1.8) to derive ETCR's in all orders of perturbation theory. All of our examples will be taken from quantum electrodynamics, whose electric-current operator, after all, is the model for all the currents of current algebra. In this section, we shall only calculate the commutators. Section 6 will be devoted to showing that they are consistent, useful, and explicitly valid in low orders of perturbation theory.

A. Field Commutation Relations

The field ETCR's that we want to use are essentially those given by Källén[16]:

$$\{\psi_\alpha(x),\psi_\beta(x')\}=\{\bar\psi_\alpha(x),\bar\psi_\beta(x')\}=0\,, \tag{5.1}$$

$$\{\bar\psi_\alpha(x),\psi_\beta(x')\}=\gamma_{4\beta\alpha}Z_1^{-1}\delta(\mathbf{x}-\mathbf{x}')\,, \tag{5.2}$$

$$[A_\mu(x),\psi_\alpha(x')]=0\,, \tag{5.3}$$

$$[A_\mu(x),A_\nu(x')]=0\,, \tag{5.4}$$

$$[\partial_4A_\mu(x),A_\nu(x')]=-[Z_3^{-1}\delta_{\mu\nu}-(Z_3^{-1}-1)\,\delta_{\mu4}\delta_{\nu4}]\times\delta(\mathbf{x}-\mathbf{x}')\,, \tag{5.5}$$

$$[\partial_4A_\mu(x),\partial_4'A_\nu(x')]=-(Z_3^{-1}-1)(\delta_{\mu4}\partial_\nu+\partial_{\nu4}\partial_\mu)\times\delta(\mathbf{x}-\mathbf{x}')\,. \tag{5.6}$$

Ignoring for a moment the divergences of Z_1^{-1} and Z_3^{-1}, let us discuss the validity of these CR's. We first must emphasize that they are specifically supposed to hold only in the Gupta-Bleuler[68] gauge, which is covariant and such that the vacuum state is a true no-particle state. This gauge will be used throughout this paper. It is the one for which the dimensional arguments given below Eq. (3.52) are valid.[69] We shall furthermore require that

$$\langle 0|A_\mu(x)|0\rangle=0\,. \tag{5.7}$$

Thus our field equations are

$$\Box A_\mu(x)=-j_\mu(x) \tag{5.8}$$

and

$$(\gamma\cdot\partial+m)\psi(x)=f(x)\,, \tag{5.9}$$

where

$$\partial_\mu A_\mu^{(+)}(x)|\Phi\rangle=0$$

for physical states $|\Phi\rangle$.

Equations (5.1)-(5.6) can be formally derived from consideration of the spectral representations[34]

$$\langle 0|[A_\mu(x),A_\nu(y)]|0\rangle=i\int_0^\infty da\left\{\left[\delta(a)+\frac{\pi(a)}{a}\right]\delta_{\mu\nu}-\left[\frac{\pi(a)}{a^2}-2M\delta(a)\right]\partial_\mu\partial_\nu\right\}\Delta(x-y;a) \tag{5.10}$$

and

$$\langle 0|\{\psi_\alpha(x),\bar\psi_\beta(y)\}|0\rangle =-i\int_{-\infty}^{\infty}d\kappa\,[\delta(\kappa-m)+\sigma(\kappa)]S_{\alpha\beta}(x-y;\kappa)\,, \tag{5.11}$$

where

$$M=\frac{1}{2}\int da\frac{\pi(a)}{a^2}\,, \tag{5.12}$$

$\sigma(\kappa)=0$ for $\kappa\in[-m,+m]$, and

$$S(x;\kappa)=(\gamma\cdot\partial-\kappa)\Delta(x;\kappa^2)\,. \tag{5.13}$$

---

[68] S. N. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950); K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[69] The point is that if, e.g., $\tilde A_\mu=A_\mu+M\partial_\mu\partial_\nu A_\nu$, then the short distance (ζ) behavior of products involving $\tilde A$ will be worse than those involving A by a factor of ζ^{-2}. This leading singularity will then not be mass-independent, since M must have dimension $(\text{mass})^{-2}$. Furthermore, $\tilde A_\mu$ will not satisfy the usual ETCR's. See H. Rollnik, B. Stech, and E. Nunnemann, Z. Physik **159**, 482 (1960).

Equation (5.10) can be derived from (3.41) and the assumed asymptotic condition

$$\langle 0|A_\mu(x)|\gamma,k\rangle=(\delta_{\mu\nu}+M\partial_\mu\partial_\nu)\langle 0|A_\nu^{\text{in}}(x)|\gamma,k\rangle\,, \quad (5.14)$$

where $|\gamma,k\rangle$ is a single photon state with momentum k, valid in our gauge. Taking ET limits inside the spectral integrals, and assuming that the commutators are c numbers, (5.2) follows from (5.11) with

$$Z_1^{-1}=1+\int d\kappa\,\sigma(\kappa)\,, \quad (5.15)$$

and (5.4)–(5.6) follow from (5.10) and (5.12) with

$$Z_3^{-1}=1+\int da\frac{\pi(a)}{a}\,. \quad (5.16)$$

In view of our lengthy discussion in Sec. 3, we cannot, of course, immediately assume that the ET limits commute with the integrations. Let us show, therefore, that the procedure is valid in this case. We know that in perturbation theory the leading singularities of products of field operators at small distances ζ are given (within logs) by dimensional arguments,[13,14,56] so that $(x_\mu-y_\mu\sim\zeta\sim 0)$

$$\begin{aligned}&\bar\psi(x)\psi(y)\sim\zeta^{-3}\,,\\&A(x)A(y)\sim\zeta^{-2}\,,\\&A(x)\psi(y)\sim\zeta^{-5/2}\,,\\&\dot A(x)A(y)\sim\zeta^{-3}\,,\\&\dot A(x)\dot A(y)\sim\zeta^{-4}.\end{aligned} \quad (5.17)$$

Also, leading lower-order singularities occur as mass-independent coefficients of local-field operators.[13,14,56] Now, by locality, the ETC's, if they exist, must have the form

$$\sum_{n=0}^{N} E_n(x)\partial^n\delta(\mathbf{x}-\mathbf{y})\,.$$

Thus, since $\delta(\mathbf{x}-\mathbf{y})\sim\zeta^{-3}$ and $\partial_k\delta(\mathbf{x}-\mathbf{y})\sim\zeta^{-4}$, we see that (5.1)–(5.6) are the only possibilities and, furthermore, that the renormalization constants are at most logarithmically divergent. Also, since the leading singularities are mass-independent, the commutators must be c numbers. Indeed, the only possible q-number commutator allowed by (5.17) would be a term $A(x)\times\delta(\mathbf{x}-\mathbf{x}')$ in (5.6), but this term cannot be present by charge-conjugation covariance. We remark finally that (5.2), (5.5), and (5.6) can be derived from (5.10)–(5.12), (5.15), and (5.16) by the method of Sec. 3 using the known behavior of the spectral functions $\pi(a)$ and $\sigma(\kappa)$.

Now let us consider the fact that Z_1^{-1} and Z_3^{-1} are divergent. This implies that the ET limits (5.2), (5.5), and (5.6) do not exist in the usual sense. Thus (5.2), for example, should be replaced by

$$\{\bar\psi_\alpha(x),\psi_\beta(y)\}=\gamma_{4\beta\alpha}W(x_0-y_0)\delta(\mathbf{x}-\mathbf{y})+\Omega_{\alpha\beta}(x;y)\,, \quad (5.18)$$

where

$$W(\tau)=1+\int d\kappa\,\sigma(\kappa)\cos\kappa\tau \quad (5.19)$$

and

$$\Omega(\mathbf{x},t;\,\mathbf{y},t)=0\,, \quad (5.20)$$

in the sense of distribution theory. Equation (5.18) formally gives (5.2) for $t\to 0$ but, in a rigorous analysis, one should use

$$\lim_{\tau\to 0} W^{-1}(\tau)\{\bar\psi_\alpha(x),\psi_\beta(\mathbf{y},x_0+\tau)\}=\gamma_{4\beta\alpha}\delta(\mathbf{x}-\mathbf{y})\,. \quad (5.21)$$

Equations (5.5) and (5.6) should be replaced by similar equations. This complication does not restrict the effectiveness of our proposal (1.8). One need only take the $\tau\to 0$ limit before the ξ, $\xi'\to 0$ limits. In this and the following section, however, we shall continue to work with the somewhat formal ETCR's (5.1)–(5.6). This will simplify the calculations and enable us to work more familiar quantities. This simplification, furthermore, will not essentially change our results. The reader interested in a completely rigorous analysis can assume we are working only to fourth order in e. There, only the free-field commutators ($Z_1=Z_3=1$) are relevant, but our essential results are nevertheless displayed. In the Appendix, we shall discuss the effects of using (5.21).

B. Current Operators

We shall next exhibit the current operators defined by Eqs. (5.8) and (5.9). They have been derived and discussed in I. The electric current can be written

$$j_\mu(x)=\lim_{\xi\to 0} j_\mu(x;\,\xi)\,,\quad \xi_0=0\,, \quad (5.22)$$

where

$$\begin{aligned}j_\mu(x;\,\xi)=&\tfrac{1}{2}ie[\bar\psi(x)\gamma_\mu\psi(x+\xi)-\gamma^\mu\psi(x)\bar\psi(x+\xi)]\\&+e^2J_\mu(\xi)[\xi\cdot A(x)+\tfrac{1}{6}(\xi\cdot\partial)^2\xi\cdot A(x)-\tfrac{1}{6}e^2{:}(\xi\cdot A(x))^3{:}]\\&-eC(\xi^2)\partial_\nu F_{\mu\nu}(x)-eC'(\xi^2)(\xi\cdot\partial)\xi_\nu F_{\mu\nu}(x)-K_1(\xi^2)j_\mu(x)\\&-K_2(\xi^2)\xi_\mu\xi\cdot j(x)\,.\end{aligned} \quad (5.23)$$

Here

$$J_\mu(\xi)=\langle 0|\bar\psi(x)\gamma_\mu\psi(x+\xi)|0\rangle=\text{tr}[\gamma_\mu G(\xi)]\,, \quad (5.24)$$

and C, C', K_1, and K_2 are functions with logarithmic singularities for $\xi^2\to 0$. They are defined in I but need not be specified here. $G(\xi)$ can be taken as the Green's function

$$G(\xi)=\langle 0|T\psi(x+\xi)\bar\psi(x)|0\rangle\,, \quad (5.25)$$

and we can write

$$J_\mu(\xi)=J(\xi^2)\xi_\mu\,. \quad (5.26)$$

Finally,

$$:A_\nu(x)A_\kappa(x)A_\lambda(x):=\lim_{\eta\to 0}\mathcal{A}_{\nu\kappa\lambda}(x;\,\eta)\,,\quad \eta_0=0 \quad (5.27)$$

where

$$\mathcal{A}_{\nu\kappa\lambda}(x;\eta)=A_\nu(x+\eta)A_\kappa(x)A_\lambda(x-\eta)-\bar{B}_{2\nu\kappa\lambda}{}^{\alpha}(\eta)A^{\alpha}(x)-B_{3\nu\kappa\lambda}{}^{\alpha\beta}(\eta)A^{\alpha,\beta}(x)-B_{4\nu\kappa\lambda}{}^{\alpha\beta\gamma}(\eta)A^{\alpha,\beta\gamma}(x)-B_{6\nu\kappa\lambda}{}^{\alpha}(\eta)j^{\alpha}(x)\,, \quad (5.28)$$

with $B_6(\eta)$ having a logarithmic singularity for $\eta\to 0$, and

$$B_3=\bar{B}_3+\tilde{B}_3\,,\quad B_4=\bar{B}_4+\tilde{B}_4\,,$$

$$\bar{B}_{2\nu\kappa\lambda}{}^{\alpha}(\eta)=D_{\nu\kappa}(\eta)\delta_\lambda{}^{\alpha}+D_{\kappa\lambda}(\eta)\delta_\nu{}^{\alpha}+D_{\nu\lambda}(2\eta)\delta_\kappa{}^{\alpha}\,,$$

$$\bar{B}_{3\nu\kappa\lambda}{}^{\alpha\beta}(\eta)=D_{\nu\kappa}(\eta)\delta_\lambda{}^{\alpha}\eta^{\beta}+D_{\kappa\lambda}(\eta)\delta_\nu{}^{\alpha}\eta^{\beta}\,,$$

$$\bar{B}_{4\nu\kappa\lambda}{}^{\alpha\beta\gamma}(\eta)=\tfrac{1}{2}D_{\nu\kappa}(\eta)\delta_\lambda{}^{\alpha}\eta^{\beta}\eta^{\gamma}+\tfrac{1}{2}D_{\kappa\lambda}(\eta)\delta_\nu{}^{\alpha}\eta^{\beta}\eta^{\gamma}. \quad (5.29)$$

Here $\tilde{B}_3$ and $\tilde{B}_4$ satisfy

$$\tilde{B}_3{}^{\alpha\beta}A^{\alpha,\beta}=\tfrac{1}{2}\tilde{B}_3{}^{\alpha\beta}F^{\beta\alpha}\,,\quad \tilde{B}_4{}^{\alpha\beta\gamma}A^{\alpha,\beta\gamma}=\tfrac{1}{2}\tilde{B}_4{}^{\alpha\beta\gamma}F^{\beta\alpha,\gamma}\,, \quad (5.30)$$

respectively, and $D(\eta)$ is the Green's function

$$D_{\nu\kappa}(\eta)=\langle 0|TA_\nu(x+\eta)A_\kappa(x)|0\rangle. \quad (5.31)$$

The weak limits (5.22) and (5.27) exist and yield mathematically well-defined (finite) operator-valued distributions in each order of perturbation theory. The subtractions needed to define the currents as the finite parts of local field products correspond to the usual renormalizations[70] required to make their matrix elements finite.

We now let $j_\mu(x;\xi,\eta)$ represent (5.23) before the $\eta\to 0$ limit (5.27) is taken. Thus

$$j_\mu(x)=\lim_{\xi\to 0}\lim_{\eta\to 0} j_\mu(x;\xi,\eta). \quad (5.32)$$

In order to simplify our expressions, we shall require that the limits $\boldsymbol{\xi}\to 0$, $\boldsymbol{\eta}\to 0$ be taken in a spatially symmetric way. Then we can write

$$j_4(x;\xi)=\tfrac{1}{2}ie[\bar{\psi}(x)\gamma_4\psi(x+\xi)-\gamma_4\psi(x)\bar{\psi}(x+\xi)]-eR_1(\xi^2)\partial_\nu F_{4\nu}-K_1(\xi^2)j_4\,, \quad (5.33)$$

and

$$j_k(x;\xi,\eta)=\tfrac{1}{2}ie[\bar{\psi}(x)\gamma_k\psi(x+\xi)-\gamma_k\psi(x)\bar{\psi}(x+\xi)]-eC(\xi^2)\partial_\nu F_{k\nu}-R_2(\xi,\eta)\partial_l F_{kl}-R_3(\xi,\eta)j_k+e^2J_k(\xi)\times\{\xi\cdot A+\tfrac{1}{6}(\xi\cdot\partial)^2\xi\cdot A-\tfrac{1}{6}e^2[\xi\cdot A(x+\eta)\xi\cdot A(x)\xi\cdot A(x-\eta)-\xi_i\xi_jE_{ij}(\eta)\xi\cdot A-\xi_i\xi_jD_{ij}(\eta)(\eta\cdot\partial)^2\xi\cdot A]\}. \quad (5.34)$$

Here we have defined

$$E(\eta)=2D(\eta)+D(2\eta) \quad (5.35)$$

and have introduced new scalar functions R_1, R_2, R_3 which are combinations of previous ones. They each have logarithmic singularities at zero.

Let us compare our rigorous expressions (5.32)–(5.34) with the current given by Källén[34]:

$$j_\mu(x)=\tfrac{1}{2}ie[\bar{\psi}(x),\gamma_\mu\psi(x)]+Z_1{}^{-1}(1-Z_3)\partial_\nu F_{\mu\nu}(x)+(1-Z_1{}^{-1})j_\mu(x). \quad (5.36)$$

The main formal difference between the currents is the presence of the term $J_k(\xi)\{\cdots\}$ in (5.34). This term insures that the current is gauge invariant.[71] It must be present in order for the limit (5.32) to exist and define an operator. It is not present in (5.36) because gauge invariance is usually enforced by explicitly discarding any non-gauge-invariant terms which arise in calculations. It is, however, an important part of the complete current operator in perturbation theory. We emphasize this matter because it is this term which gives rise to the q-number structure of the ETC $[j_k,j_4]$.

Apart from this term, the currents (5.32)–(5.34) and (5.36) are formally quite similar. Indeed, it is not hard to show from expressions given in I that formally one has

$$K_1(0)=-(1-Z_1{}^{-1})\,,\quad eC(0)=-Z_1{}^{-1}(1-Z_3). \quad (5.37)$$

At this point, it would be a simple matter to use (5.32)–(5.34) together with (5.1)–(5.6) to calculate the ETC $[j_k,j_4]$ by (1.8). Essentially, the same result for the truncated commutator would be obtained, however, if we first make the formal substitution

$$eR_1=eC=-Z_1{}^{-1}(1-Z_3)\,,\quad R_2=0\,,\quad K_1=R_3=-(1-Z_1{}^{-1})\,, \quad (5.38)$$

in (5.33) and (5.34). This will shorten the calculation and again enable us to work with familiar objects. The correct truncated ETC will differ from that obtained by using (5.38) by at most a multiplicative constant. Again the reader interested in more rigor can assume we are working only to fourth order, in which case it is just as easy to use (5.33) and (5.34), since only the spinor parts will contribute to the truncated ETC. We emphasize that the simplification (5.38) is introduced for convenience only. The rigorous expressions (5.33) and (5.34) will be briefly considered in the Appendix.

Thus we write

$$j_4(x;\xi)=\tfrac{1}{2}ieZ_1(\xi^2)[\bar{\psi}(x)\gamma_4\psi(x+\xi)-\gamma_4\psi(x)\bar{\psi}(x+\xi)]+[1-Z_3(\xi^2)]\partial_\nu F_{4\nu}(x) \quad (5.39)$$

and

$$j_k(x;\xi,\eta)=\tfrac{1}{2}ieZ_1(\xi^2)[\bar{\psi}(x)\gamma_k\psi(x+\xi)-\gamma_k\psi(x)\bar{\psi}(x+\xi)]+[1-Z_3(\xi^2)]\partial_\nu F_{k\nu}(x)+e^2Z_1(\xi^2)J_k(\xi)\{\cdots\}\,, \quad (5.40)$$

where

$$Z_1(0)=Z_1\,,\quad Z_3(0)=Z_3.$$

These can be considered as the minimal extension of (5.36) required to make it consistent with gauge invariance.

The above expressions for the electric current are manifestly negative under charge conjugation ($\mathcal{C}A\mathcal{C}^{-1}=-A$ in our gauge). It was shown in I that $j_\mu(x)$ could also be represented in a form which is not *manifestly* $\mathcal{C}$-covariant but which is useful for other

[70] K. Hepp, Commun. Math. Phys. **2**, 301 (1966).

[71] It is explicitly shown in I that (5.22) is invariant under c-number gauge transformation of the second kind. The spinor term in (5.23) alone is not so invariant, because of the divergence of $\bar{\psi}(x)\psi(x+\xi)$ for $\xi\to 0$.

considerations. We have

$$j_\mu(x)=\lim_{\xi\to 0} j_\mu'(x;\xi)\,,\quad \xi_0=0\,, \tag{5.41}$$

where, employing (5.38),

$$j_\mu'(x;\xi)=ieZ_1(\xi^2)\bar\psi(x)\gamma_\mu\psi(x+\xi)+[1-Z_3(\xi^2)]\times\partial_\nu F_{\mu\nu}(x)-ieZ_1(\xi^2)J_\mu(\xi)\mathcal{E}(x;\xi)\,, \tag{5.42}$$

with[72]

$$\mathcal{E}(x;\xi)=1+ie\xi\cdot A+\tfrac{1}{2}ie(\xi\cdot\partial)\xi\cdot A+\tfrac{1}{6}ie(\xi\cdot\partial)^2\xi\cdot A -\tfrac{1}{2}e^2{:}(\xi\cdot A)^2{:}-\tfrac{1}{4}e^2\xi\cdot\partial{:}(\xi\cdot A)^2{:}-\tfrac{1}{6}ie^3{:}(\xi\cdot A)^3{:}'. \tag{5.43}$$

Here

$$\begin{aligned}{:}A_\mu(x)A_\nu(x){:}&=\lim_{\eta\to 0}[A_\mu(x)A_\nu(x-\eta)-D_{\mu\nu}(\eta)]\\ &=\lim_{\eta\to 0}[A_\mu(x+\eta)A_\nu(x)-D_{\mu\nu}(\eta)]\\ &=\lim_{\eta\to 0}[A_\mu(x+\eta)A_\nu(x-\eta)-D_{\mu\nu}(2\eta)].\end{aligned} \tag{5.44}$$

Using the fact that

$$\lim_{\xi\to 0} Z_1^{-1}(\xi^2)\xi=\lim_{\xi\to 0} Z_3^{-1}(\xi^2)\xi=\lim_{\xi\to 0} G(\xi)\xi^4=0\,, \tag{5.45}$$

it is easy to see that

$$\begin{aligned}j_\mu(x;\xi)&=\tfrac{1}{2}[j_\mu'(x;\xi)+j_\mu'(x+\xi;-\xi)]\\ &=\tfrac{1}{2}[j_\mu'(x;\xi)-\mathcal{C}j_\mu'(x;\xi)\mathcal{C}^{-1}]\end{aligned} \tag{5.46}$$

apart from terms which vanish for $\xi\to 0$. For use below, we also note here the relation

$$\begin{aligned}0&=\lim_{\xi\to 0}[j_k'(x;\xi)-j_k'(x+\xi;-\xi)]\\ &=\lim_{\xi\to 0}\{ieZ_1Z_3^{-1}[\bar\psi(x)\gamma_k\psi(x+\xi)-\bar\psi(x+\xi)\gamma_k\psi(x)]\\ &\quad -ieZ_1Z_3^{-1}J_k[2-e^2{:}(\xi\cdot A)^2{:}-\tfrac{1}{2}e^2\xi\cdot\partial{:}(\xi\cdot A)^2{:}]\}.\end{aligned} \tag{5.47}$$

This completes our discussion of the electric-current operator. We next describe the current operator $f(x)$ defined by Eq. (5.9). It is shown in I that we can write

$$f(x)=\lim_{\xi\to 0} f(x;\xi)\,,\quad \xi_0=0 \tag{5.48}$$

where

$$f(x;\xi)=e[\gamma\cdot A(x+\xi)\psi(x)-H_1(\xi)\psi(x)-H_{2\mu}(\xi)\partial_\mu\psi(x)-H_3(\xi)f(x)]. \tag{5.49}$$

The only properties of the H_i we shall need are the behaviors

$$H_1(\xi)\sim\xi^{-1},\quad H_2(\xi)\sim\ln(\xi^2)\,,\quad H_3(\xi)\sim\ln(\xi^2) \tag{5.50}$$

for $\xi\sim 0$ and the fact that H_1 and H_2 are of order e, and H_3 is of order e^2. We can also write

$$f(x;\xi)=e[1+eH_3(\xi)]^{-1}[\gamma\cdot A(x+\xi)\psi(x)-H_1(\xi)\psi(x)-H_{2\mu}(\xi)\partial_\mu\psi(x)]. \tag{5.51}$$

C. Commutation Relations

We can now use our proposal in order to calculate some ETCR's. We begin by using (5.51) to derive a useful field ETCR. Commuting Eq. (5.9) with $A_\mu(x')$, assuming the commutator can be taken inside the limit (5.48), and using the ETCR's (5.3) and (5.4), we find

$$\gamma_4[\partial_4\psi(x),A_\mu(x')] =-\lim_{\xi\to 0} e[1+eH_3(\xi)]^{-1}H_{24}(\xi)[\partial_4\psi(x),A_\mu(x')].$$

This equation is only consistent provided that

$$[\partial_4\psi(x),A_\mu(x')]=0. \tag{5.52}$$

Next we use (5.3) to obtain[65]

$$0=\partial_4[\psi(x),A_\mu(x')]=[\partial_4\psi(x),A_\mu(x')]+[\psi(x),\partial_4A_\mu(x')]\,, \tag{5.53}$$

which, with (5.52), gives

$$[\psi(x),\partial_4A_\mu(x')]=0. \tag{5.54}$$

The ETCR (5.54), which we have derived, is one of the usual canonical ones. An analysis similar to those presented earlier in this section shows that the ETC must have the form $K\delta(\mathbf{x}-\mathbf{x}')$, with K a c number. However, $\langle 0|\psi(x)A_\mu(y)|0\rangle=0$, so that K must vanish, in agreement with (5.54).

Let us next calculate the ETC of j_4 and ψ. Using (5.41) and (5.42) [or (5.22) and (5.39)] and

$$\partial_\nu F_{4\nu}=\partial_4\nabla\cdot\mathbf{A}-\Delta A_4\,, \tag{5.55}$$

together with (5.1)–(5.3) and (5.54), we obtain

$$\begin{aligned}[j_4(x),\psi(x')]&=\lim_{\xi\to 0}[j_4'(x;\xi),\psi(x')]\\ &=-ie\lim_{\xi\to 0} Z_1(\xi^2)Z_1^{-1}\psi(x+\xi)\delta(\mathbf{x}-\mathbf{x}')\\ &=-ie\psi(x)\delta(\mathbf{x}-\mathbf{x}').\end{aligned} \tag{5.56}$$

We thus find the usually assumed value for this commutator.

A difficulty now arises when we attempt to use our method to calculate the ETC of j_k and $A_{l,4}$. Using (5.42), we obtain

$$[j_k'(x;\xi),A_{l,4}(x')]_T=ie^3Z_1Z_3^{-2}J_k(\xi)[\xi_l\xi\cdot A+\tfrac{1}{2}\xi_l(\xi\cdot\partial)\times\xi\cdot A+\tfrac{1}{2}ie\xi_l{:}(\xi\cdot A)^2{:}]\delta(\mathbf{x}-\mathbf{x}'). \tag{5.57}$$

[72] Here ${:}A^3{:}'$ does not contain the $\tilde B_4$ term in (5.28) which has, by use of (5.30), been incorporated into the F term in (5.42). Likewise, B_6 has been incorporated into Z_1.

Formally defining

$$J_{kl}=\lim_{\xi\to 0} J_k(\xi)\xi_l\,,$$

$$J_{klm}=\lim_{\xi\to 0} J_k(\xi)\xi_l\xi_m\,,$$

$$J_{klmn}=\lim_{\xi\to 0} J_k(\xi)\xi_l\xi_m\xi_n\,, \qquad (5.58)$$

this becomes

$$[j_k(x),A_{l,4}(x')]_T'=ie^3Z_1Z_3^{-2}[J_{klm}A_m+\tfrac{1}{2}J_{klmn}\partial_mA_n +\tfrac{1}{2}ieJ_{klmn}{:}A_mA_n{:}]\delta(\mathbf{x}-\mathbf{x}')\,, \quad (5.59)$$

the prime denoting that $j_k'(x;\xi)$ was used. Using (5.40), on the other hand, we obtain

$$[j_k(x;\xi),A_{l,4}(x')]_T=-\tfrac{1}{2}e^4Z_1Z_3^{-2}J_k(\xi)\xi_l{:}(\xi\cdot A)^2{:} \times\delta(\mathbf{x}-\mathbf{x}')\,, \quad (5.60)$$

which becomes

$$[j_k(x),A_{l,4}(x')]_T =-\tfrac{1}{2}e^4Z_1Z_3^{-2}J_{klmn}{:}A_mA_n{:}\delta(\mathbf{x}-\mathbf{x}')\,. \quad (5.61)$$

Since (5.59) and (5.61) differ, we see that our definition of ETC depends on which sequence, (5.42) or (5.40), is used to represent $j_\mu(x)$. This difficulty, however, is easily resolved. Indeed, (5.59) cannot be correct, since it is not covariant under charge conjugation.[73] This arises from the fact that the sequence (5.42) is not manifestly negative under charge conjugation. This can be put more sharply by considering the sequence $\{j_k'(x;\xi)\xi_m\}$ which converges to the null operator 0. Our definition would imply that

$$[0,A_{l,4}(x')]_T'=\lim_{\xi\to 0}[j_k'(x;\xi)\xi_m,A_{l,4}(x')]_T =ie^3Z_1Z_3^{-2}J_{klmn}A_n\delta(\mathbf{x}-\mathbf{x}')\,,$$

an absurd result, since any definition of ETC should give $[0,B]=0$ for all B. No such contradictions for truncated commutators arise if the manifestly charge-conjugation negative sequence (5.40) is used. We conclude that commutators cannot, in general, be taken inside the limits $j(x)=\lim_{\xi\to 0} j'(x;\xi)$ or $0=\lim_{\xi\to 0} j'(x;\xi)\xi$.[74] The same is true for the limit expressed in Eq. (5.47). We shall therefore use the sequence $j(x;\xi)$ in connection with our definition of ETC in the remainder of this paper. In particular, we assume that (5.60) is correct. (J_{klmn} will be evaluated below.) This will be explicitly verified in fourth order in Sec. 6.

[73] We are here demanding that any ETC should transform as an ordinary product with respect to internal symmetry groups.

[74] It is in general dangerous to take ET limits inside of sequences of the form $j'(x;\xi)\xi$. We shall never assume that this can be done. Likewise, it is in general incorrect to take limits inside of representations of field operators in terms of each other, obtained from a current operator, such as

$$A_k(x)=\frac{3i}{2e}\lim_{\xi\to 0}[\bar\psi(x)\gamma_k\psi(x+\xi)-\gamma_k\psi(x)\bar\psi(x+\xi)][\xi^2J(\xi^2)]^{-1}.$$

See R. A. Brandt, J. Sucher, and C. H. Woo, Phys. Rev. Letters **19**, 801 (1967).

In a similar way, we easily derive

$$[j_k(x),A_{4,4}(x')]_T=0 \qquad (5.62)$$

and

$$[j_4(x),A_{\mu,\nu}(x')]_T=0\,. \qquad (5.63)$$

Since $\xi_4=0$, we can combine (5.61)–(5.63) into

$$[j_\kappa(x),A_{\mu,\nu}(x')]_T =-\tfrac{1}{2}e^4Z_1Z_3^{-2}\delta_{\nu 4}J_{\kappa\mu\alpha\beta}{:}A_\alpha A_\beta{:}\delta(\mathbf{x}-\mathbf{x}')\,. \quad (5.64)$$

We now come to our major concern, the calculation of the ETC of j_k and j_4. Here we shall keep both c numbers and q numbers, since a well-defined $\partial\Delta\delta(\mathbf{x}-\mathbf{x}')$ c-number term will arise. It follows from (5.1)–(5.6), (5.54), (5.39), and (5.40), with $\{\cdots\}$ given in (5.34), that

$$\begin{aligned}\lim_{\xi'\to 0}&[j_k(x;\xi,\eta),j_4(x';\xi')]\\ &=-\tfrac{1}{2}e^2Z_1Z_3^{-1}[\bar\psi(x)\gamma_k\psi(x+\xi)-\bar\psi(x+\xi)\gamma_k\psi(x)]\\ &\times[\xi\cdot\partial+\tfrac{1}{2}(\xi\cdot\partial)^2+\tfrac{1}{6}(\xi\cdot\partial)^3]\delta(\mathbf{x}-\mathbf{x}')-e^2Z_1(1-Z_3)\\ &\times Z_3^{-2}J_k\{[1+\tfrac{1}{6}e^2(\xi\xi)\cdot E]\xi-\tfrac{1}{6}e^2\\ &\times[(\xi\xi)\cdot(A(x)A(x-\eta)+A(x+\eta)A(x-\eta)\\ &+A(x+\eta)A(x))]\xi\}\partial\delta(\mathbf{x}-\mathbf{x}')-\tfrac{1}{6}e^4Z_1(1-Z_3)Z_3^{-2}J_k\\ &\times\{(\xi\xi)\cdot[A(x+\eta)A(x)-A(x)A(x-\eta)]\xi\eta\}\\ &\times\partial\partial\delta(\mathbf{x}-\mathbf{x}')-\tfrac{1}{6}e^2Z_1(1-Z_3)Z_3^{-2}J_k\{\xi\xi\xi+e^2(\xi\xi)\\ &\times D\eta\eta\xi-\tfrac{1}{2}e^2(\xi\xi)\cdot[A(x+\eta)A(x)+A(x)A(x-\eta)]\\ &\times\xi\eta\eta\}\partial\partial\partial\delta(\mathbf{x}-\mathbf{x}')\,,\end{aligned} \qquad (5.65)$$

apart from terms which, by virtue of

$$\lim_{\xi\to 0}G(\xi)\xi^4=\lim_{\eta\to 0}D(\eta)\eta^3=0\,, \qquad (5.66)$$

vanish for $\xi,\eta\to 0$. The first term in (5.65) arises from the spinors in j, as in Eq. (4.44). The remaining terms arise from use of (5.4)–(5.6).

We now use (5.47) in the first term of (5.65), and (5.44) in the remaining terms to obtain

$$\begin{aligned}\lim_{\eta\to 0}\lim_{\xi'\to 0}&[j_k(x;\xi,\eta),j_4(x';\xi')]\\ &=-e^2Z_1Z_3^{-2}J_k[1-\tfrac{1}{2}e^2{:}(\xi\cdot A)^2{:}](\xi\cdot\partial)\delta(\mathbf{x}-\mathbf{x}')\\ &-\tfrac{1}{2}e^2Z_1Z_3^{-1}J_k(\xi\cdot\partial)^2\delta(\mathbf{x}-\mathbf{x}')\\ &-\tfrac{1}{6}e^2Z_1Z_3^{-2}J_k(\xi\cdot\partial)^3\delta(\mathbf{x}-\mathbf{x}'),\end{aligned} \qquad (5.67)$$

apart from terms which vanish as $\xi \to 0$. Thus we find

$$[j_k(x),j_4(x')] \equiv \lim_{\xi,\eta,\xi'\to 0} [j_k(x;\xi,\eta),j_4(x';\xi')]$$
$$= -e^2 Z_1 Z_3^{-2} J_{kl}\partial_l \delta(\mathbf{x}-\mathbf{x}') - \tfrac{1}{2}e^2 Z_1 Z_3^{-1} J_{klm} \times \partial_l \partial_m \delta(\mathbf{x}-\mathbf{x}') - \tfrac{1}{2}e^2 Z_1 Z_3^{-2} J_{klmn} \times [\tfrac{1}{3}\partial_l\partial_m\partial_n - e^2 {:}A_l(x)A_m(x){:}\partial_n] \times \delta(\mathbf{x}-\mathbf{x}'), \quad (5.68)$$

where the J's are given by (5.58).

In order to calculate the J's, we shall use the representation (5.24), (5.25). Since the $\xi \to 0$ limit is to be taken in a spherically symmetric way, and since $G(\xi) \sim \xi^{-3}$ for $\xi \sim 0$, we see immediately that

$$J_{kl} = \infty\, \delta_{kl}, \quad J_{klm} = 0, \quad (5.69)$$

and

$$J_{klmn} = K\chi_{klmn} \quad (5.70)$$

for some constant K, where χ is given by Eq. (4.61). To determine K, we consider the spectral representation for $G(\xi)$ which, in view of (5.11), can be written

$$G(\xi) = \int d\kappa\, [\delta(\kappa - m) + \sigma(\kappa)] S_F(\xi;\kappa). \quad (5.71)$$

Thus

$$J_{klmn} = \lim_{\xi\to 0} J_k(\xi)\xi_l\xi_m\xi_n$$
$$= \lim_{\xi\to 0} \mathrm{tr}[\gamma_k G(\xi)]\xi_l\xi_m\xi_n$$
$$= \int d\kappa [\delta(\kappa-m)+\sigma(\kappa)] \times \lim_{\xi\to 0} \mathrm{tr}[\gamma_k S_F(\xi;\kappa)]\xi_l\xi_m\xi_n. \quad (5.72)$$

Using (4.53) and (5.15), this becomes

$$J_{klmn} = Z_1^{-1}\frac{2}{\pi^2}\frac{\xi_k\xi_l\xi_m\xi_n}{\xi^4}. \quad (5.73)$$

We now require that, after dividing out the trivial e^2 factor, Eq. (5.68) reproduces the free-field result (4.65) for $e^2 \to 0$. It is clear from (4.50), (4.57), (4.58), (4.62), and (4.63) that this requires us to use (4.64) in (5.73). Thus

$$J_{klmn} = Z_1^{-1}(1/6\pi^2)\chi_{klmn}, \quad (5.74)$$

and (5.68) becomes

$$[j_k(x),j_4(x')] = -e^2\infty\, \partial_k \delta(\mathbf{x}-\mathbf{x}') - e^2 Z_3^{-2}(1/12\pi^2) \times \partial_k \Delta\delta(\mathbf{x}-\mathbf{x}') + e^4 Z_3^{-2}(1/12\pi^2)[{:}A_l(x)A_l(x){:}\partial_k + 2{:}A_k(x)A_l(x){:}\partial_l]\delta(\mathbf{x}-\mathbf{x}'). \quad (5.75)$$

The truncated ETC is

$$[j_k(x),j_4(x')]_T = e^4 Z_3^{-2}(1/12\pi^2) \times [{:}A_lA_l{:}\partial_k + 2{:}A_kA_l{:}\partial_l]\delta(\mathbf{x}-\mathbf{x}'), \quad (5.76)$$

and the coefficient of $\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')$ is uniquely specified by, e.g.,

$$\int d\mathbf{x}[j_k(x),j_4(x')]\mathbf{x}^2 x_k = e^2 Z_3^{-2}\frac{30}{12\pi^2}. \quad (5.77)$$

We defer discussion of these equations until Sec. 6, where it will be explicitly shown that they are correct in fourth order, where Eq. (5.76) becomes the well-defined relation

$$[j_k(x),j_4(x')]_T{}^{(4)} = (e^4/12\pi^2)[{:}A_l{}^{(0)}A_l{}^{(0)}{:}\partial_k + 2{:}A_k{}^{(0)}A_l{}^{(0)}{:}\partial_l]\delta(\mathbf{x}-\mathbf{x}'). \quad (5.78)$$

In view of the simplifications (5.1)-(5.6) and (5.38) which we have employed, Eq. (5.75) can only be expected to be correct in all orders to within a (possibly logarithmically divergent) multiplicative constant. We note, however, that the fourth-order result (5.78) is a rigorous consequence of the exact expression (5.23) for the current and the meaningful free-field ETCR's obtained from (5.1)-(5.6) by setting $Z_1 = Z_3 = 1$. Indeed, in fourth order, only the spinor terms in (5.33) and (5.34) contribute to the truncated commutator, and these give a term of the form

$$e^2\{\bar\psi,\psi\}(\bar\psi\psi + \psi\bar\psi)[\xi\cdot\partial\delta(\mathbf{x}-\mathbf{x}') + \cdots]. \quad (5.79)$$

The fourth-order q-number part (5.78) of (5.79) follows from the rigorous relation

$$0 = \lim_{\xi\to 0}\{[\bar\psi(x)\gamma_\mu\psi(x+\xi) + \gamma_\mu\psi(x)\bar\psi(x+\xi)]\xi_l - J_\mu(\xi)[2 - e^2{:}(\xi\cdot A)^2{:}]\xi_l\} \quad (5.80)$$

and the free-field ETC of $\bar\psi$ and ψ.

As a final example, we consider the ETC of j_4 and j_4. Using (5.33), we easily obtain the expected result

$$[j_4(x),j_4(x')] \equiv \lim_{\xi,\xi'\to 0} [j_4(x;\xi),j_4(x';\xi')] = 0, \quad (5.81)$$

rigorously valid to all orders of perturbation theory. This follows essentially from the facts that $J_4(\xi) = 0$ and $[\partial_\nu F_{4\nu}(x), \partial_\mu' F_{4\mu}(x')] = 0$.

6. DISCUSSION

In this section, we shall discuss the ETCR's which were derived in Sec. 5. Our aim will be to show that these relations have the desirable properties mentioned in paragraph (i') of Sec. 1. In particular, we shall show that they are consistent with such general requirements as gauge invariance and that they are explicitly correct in fourth order. We shall also compare our results with those obtained by other means of calculation.

A. Commutator (5.56)

We begin with the very credible relation (5.56). It can be used, for example, together with current conservation, to derive the Ward-Takahashi identity[75,76] and its extensive generalizations.[77] The simplicity of these derivations compared with the direct derivation given in I (Sec. 7) strongly supports the usefulness of incorporating meaningful ETCR's into field theory.

Equation (5.56) is also of interest in that it is a convenient mathematical representation of the fact that the physical charge of the electron is e. Indeed, defining the electric charge as

$$Q=-i\int d\mathbf{x}\; j_4(x)\,, \tag{6.1}$$

so that (5.56) becomes

$$[Q,\psi(x)]=-e\psi(x)\,, \tag{6.2}$$

the desired result

$$\langle \mathbf{p}'|Q|\mathbf{p}\rangle = e(2\pi)^3\delta(\mathbf{p}-\mathbf{p}') \tag{6.3}$$

follows immediately from inserting a complete set of intermediate states in the vacuum-electron matrix element of (6.2).[78] Equation (6.3) also follows directly from the usual formalism[79]:

$$\begin{aligned}\langle \mathbf{p}'|Q|\mathbf{p}\rangle &= (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')e(m/p_0)\bar{u}(\mathbf{p})\\ &\quad\times[\Gamma_4(0)+\Pi_{4\mu}(0)D_{\mu\nu}(0)\Gamma_\nu(p,p)]u(\mathbf{p})\\ &= (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')e(m/p_0)\bar{u}(\mathbf{p})\gamma_4 u(\mathbf{p})\\ &= e(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\,.\end{aligned}$$

Equation (5.56) also shows that $j_4(x)$ is the generator of local gauge transformations on ψ. This is not quite the case for A_μ. We find

$$[j_4(x),A_\nu(x')]=-(Z_3^{-1}-1)\partial_\nu\delta(\mathbf{x}-\mathbf{x}')\,. \tag{6.4}$$

However, we still have

$$[Q,A_\nu(x)]=0\,, \tag{6.5}$$

corresponding to the photon's lack of charge.

B. Commutators (5.76) and (5.61)

In contrast to (5.56), the ETCR (5.76) is somewhat surprising. It has usually been assumed that (5.75) is a (nonvanishing) c number, so that (5.76) vanishes. We shall therefore present a detailed discussion of the consistency of (5.76) and, simultaneously, of (5.61). We shall afterwards directly establish these relations, and in fact all of (5.75), in fourth order. For convenience, let us here rewrite (5.76) and (5.61) as

$$[j_k(x),j_4(x')]_T = (e^4/12\pi^2 Z_3^2)\chi_{klmn}{:}A_lA_m{:}\partial_n\delta(\mathbf{x}-\mathbf{x}') \tag{6.6}$$

and

$$[j_k(x),A_{l,4}(x')]_T = -(e^4/12\pi^2 Z_3^2)\chi_{klmn}{:}A_mA_n{:}\delta(\mathbf{x}-\mathbf{x}')\,, \tag{6.7}$$

where we have introduced (5.74) into (5.61).

We first note that (6.6) and (6.7) are consistent with charge-conjugation covariance and with the dimensional restriction on perturbation-theoretic singularities:

$$\dim jj = 2\dim A+\dim\delta' = 6\,,$$
$$\dim j\dot{A} = 2\dim A+\dim\delta = 5.$$

On the other hand, Eq. (6.6) appears to be highly non-gauge-covariant. Indeed, the left side is manifestly gauge invariant, in sharp contrast to the right side. We recall, however, that (6.6) has been derived in the particular Gupta-Bleuler gauge and hence can only be expected to be correct in this gauge. This is the gauge for which the assumed field ETCR's (5.1)-(5.6) are valid[34] and for which the expression (5.23) for the electric current has been derived.[14] Equation (5.7) has also been assumed valid. Thus (6.6) by no means immediately contradicts gauge invariance. It simply instructs one to compute matrix elements of the gauge-invariant quantity $[j_k,j_4]$ in any gauge by evaluating the matrix element of the right side of (6.6) in the Gupta-Bleuler gauge.

However, it is not immediately clear that (6.6) in the Gupta-Bleuler gauge is consistent with the general gauge-invariance requirements of quantum electrodynamics. Let us therefore attempt to show that this is the case. We first note that the gauge invariance of S-matrix elements is always guaranteed whatever are the values of (6.6) and (6.7). In fact, no on-mass-shell quantity can depend on them. Indeed, the gauge invariance of electrodynamics can be rigorosuly derived[14] with no assumptions whatever about any ETC's. Nevertheless, gauge invariance, together with the assumption that certain often-used formal procedures are valid, does impose consistency conditions on ETCR's.

To see how this comes about, let us consider the photon-photon scattering amplitude. It can be written, apart from kinematical factors, as[80]

$$T_{\mu\nu}{}^{\lambda_1\lambda_2}(k_1k_2k_3k_4)=\int dxdy\; e^{-ik_3\cdot x-ik_4\cdot y}\Box_x\Box_y \times\langle k_1\lambda_1|TA_\mu(x)A_\nu(y)|k_2\lambda_2\rangle\,, \tag{6.8}$$

[75] Y. Takahashi, Nuovo Cimento **6**, 370 (1957).

[76] We disagree here with the criticism of Takahashi's derivation advanced by K. Bardakci, M. B. Halpern, and C. Segrè, Phys. Rev. **158**, 1544 (1967), Appendix. These authors seem to neglect the $\partial_\mu\partial_\nu A_\nu$ term in Eq. (5.36) and hence find an extra factor of Z_3^{-1} on the right side of Eq. (5.56). Inclusion of this term restores the validity of Takahashi's derivation. (I thank Dr. M. Halpern for a confirmation of this point.) If, on the other hand, one could consistently assume $\partial_\nu A_\nu=0$ as an operator relation, then the alternative derivation of Bardacki *et al.* could also be correct. We feel that this is unlikely, however, since this assumption leads to the conclusion that a certain product of renormalized functions is divergent and also that $\int d\mathbf{x}j_0(x)$ is e/Z_3, rather than e, between one-electron states, as can be seen from Eqs. (A4) and (A17) of Bardacki *et al.*

[77] K. Nishijima, Phys. Rev. **119**, 485 (1960); N. P. Chang and H. S. Mani, *ibid.* **134**, B896 (1964); R. J. Rivers, J. Math. Phys. **7**, 385 (1966); N. M. Kroll, Nuovo Cimento **45A**, 65 (1966).

[78] See, e.g., S. Fubini and G. Furlan, Physics **1**, 229 (1965).

[79] This was first shown by G. Källén, Helv. Acta **26**, 755 (1953).

[80] This representation is valid on the mass-shell in our gauge even though (5.14) is valid. See R. S. Willey, Ann. Phys. (N. Y.) **45**, 167 (1967).

where $|k_i\lambda_i\rangle \equiv |i\rangle$ is a one-photon state with momentum k_i and polarization λ_i, $i=1, 2$. We assume that $|1\rangle \neq |2\rangle$ in order to avoid c-number contributions. Now (6.8) can formally be written as

$$T_{\mu\nu}{}^{\lambda_1\lambda_2}(k_1k_2k_3k_4)=\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\langle 1|Tj_\mu(x)j_\nu(y)|2\rangle$$
$$-i\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\delta(x_0-y_0)\times\langle 1|[j_\mu(x),A_{\nu,4}(y)]|2\rangle. \quad (6.9)$$

Here we have assumed that the ETC's $[A,A]$, $[\dot{A},A]$, and $[j,A]$ are c numbers. This follows for $[A,A]$ and $[\dot{A},A]$ from the field ETCR's (5.4) and (5.5), and for $[j,A]$ from our proposal. Since we want to check our proposal, we should show independently that $[j,A]$ is a c number. This is easy. Dimensionality gives

$$[j(x),A(x')]=C_1A\,\delta(\mathbf{x}-\mathbf{x}')+C_2\delta'(\mathbf{x}-\mathbf{x}'),$$

and charge-conjugation covariance gives $C_1=0$.

Now, although (6.8) is well defined and (on the mass shell) independent of the values of (6.6) and (6.7), the individual terms in (6.9) need not be well defined, and they certainly depend on the ETC's. Let us assume, however, that the individual terms in (6.9) *are* meaningful. If this were not the case, then the ETCR (6.6) would be of little practical value. Then, although (6.8) is independent of ETC's involving the current, the required equality of (6.8) and (6.9) imposes consistency conditions on these ETC's. Since we want our ETCR's to be suitable for use in expressions of the form (6.9), we shall require that they satisfy such consistency conditions.

Let us therefore determine the constraints imposed by the requirement that (6.9) be gauge invariant. We find

$$k_{4\nu}T_{\mu\nu}{}^{\lambda_1\lambda_2}(k_1k_2k_3k_4)=\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\delta(x_0-y_0)$$
$$\times\langle 1|[j_\mu(x),j_4(y)]|2\rangle-\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\frac{\partial}{\partial y_\nu}$$
$$\times\delta(x_0-y_0)\langle 1|[j_\mu(x),A_{\nu,4}(y)]|2\rangle. \quad (6.10)$$

Again Eq. (6.10) is only formally valid, but, in the above spirit, we shall assume that it too is meaningful. Then we see that gauge invariance, which requires (6.10) to vanish, implies that

$$\delta(x_0-y_0)\langle 1|[j_\mu(x),j_4(y)]|2\rangle$$
$$=\frac{\partial}{\partial y_\nu}\delta(x_0-y_0)\langle 1|[j_\mu(x),A_{\nu,4}(y)]|2\rangle. \quad (6.11)$$

The same relation [contracted with the polarization tensor $\epsilon_\mu{}^{\lambda_3}(k_3)$] must be valid for *any* states $|1\rangle$ and $|2\rangle$.

We can easily see that our ETC's do indeed satisfy (6.11). For $\mu=4$, the left side vanishes by (5.81), and, by (5.63), so does the right side. For $\mu=k$, by virtue of (5.62), (6.11) becomes

$$\delta(x_0-y_0)\langle 1|[j_k(x),j_4(y)]|2\rangle$$
$$=\delta(x_0-y_0)\langle 1|[j_k(x),A_{l,4l}(y)]|2\rangle, \quad (6.12)$$

which is clearly satisfied by (6.6) and (6.7) (since χ is totally symmetric). Furthermore, (5.62) is itself a *consequence* of (6.11) since the left-hand side of (6.11) contains no $\partial_0\delta(x_0-y_0)$ term.

We conclude that our ETCR's are consistent with at least the above requirement of gauge invariance. We note, moreover, that since the electron-electron matrix elements of (6.6) and (6.7) vanish in fourth order, the above argument is sufficient to establish the consistency of our ETCR's with fourth-order quantum electrodynamics. We shall see below, in fact, that in fourth order, gauge invariance essentially implies (6.6) and (6.7). Since fourth-order perturbation theory is a "model" satisfying both gauge invariance and our ETCR's, we can conclude that an ETCR of the form (6.6) is *in general* consistent with gauge invariance.

For the same reason, our ETCR's are consistent with the field equations. To see an explicit example of this, consider the field ETCR's (5.4) and (5.5). They imply (respectively) that

$$0=\Delta[A_\mu(x),A_\nu(x')]=[\Delta A_\mu(x),A_\nu(x')], \quad (6.13)$$

and[65]

$$0=\partial_4[A_{\mu,4}(x),A_\nu(x')]=[A_{\mu,44}(x),A_\nu(x')]+[A_{\mu,4}(x),A_{\nu,4}(x')]. \quad (6.14)$$

Adding (6.13) and (6.14), and using the field equation (5.8), we find

$$[j_\mu(x),A_\nu(x')]=[A_{\mu,4}(x),A_{\nu,4}(x')]. \quad (6.15)$$

It is easily checked that (6.15) is satisfied by our commutators.

In addition to requirements such as the above imposed on ETCR's by properties of the objects being computed, there are other properties which ordinary commutators must possess by virtue of their algebraic meaning. Abstractly, a commutator is required to satisfy the conditions

$$[\alpha,\alpha]=0 \quad (6.16)$$

and

$$[[\alpha,\beta],\gamma]+[[\beta,\gamma],\alpha]+[[\gamma,\alpha],\beta]=0, \quad (6.17)$$

the Jacobi identity, for all α, β, γ in the commutator algebra. Now (6.16) is, by the definition of our ETC's, immediately satisfied. Equation (6.17), however, should not be expected to hold for the following reason. In deriving, for example, (5.68), we used (5.47). This latter equation is correct, but, as we have seen, ET commutation does not commute with this limit $\xi\to 0$. Thus one should not be able to correctly calculate, say, the double commutator $[[j_4,j_k],\dot{A}_l]$ by commuting $\dot{A}$ with the right side of (5.68). Indeed, it follows from (6.6) and (5.5) that $(x_0=x_0'=x_0'')$

$$[[j_4(x'),j_k(x)],A_{l,4}(x'')]=-(e^4/6\pi^2)Z_3^{-3}\chi_{klmn}A_m \times\delta(\mathbf{x}-\mathbf{x}'')\partial_n\delta(\mathbf{x}-\mathbf{x}'). \quad (6.18)$$

Likewise, from (5.39), (5.55), and (5.3)–(5.5), we obtain

$$\lim_{\eta\to 0}[A_i(x)A_j(x+\eta),j_4(x')] =(1-Z_3)Z_3^{-1}[A_i\partial_j'\delta(\mathbf{x}-\mathbf{x}')+A_j\partial_i'\delta(\mathbf{x}-\mathbf{x}')], \quad (6.19)$$

so that, by (6.7).

$$[[j_k(x),A_{l,4}(x'')],j_4(x')]=(e^4/6\pi^2)(1-Z_3^{-3}\chi_{klmn}A_m \times\delta(\mathbf{x}-\mathbf{x}'')\partial_n\delta(\mathbf{x}-\mathbf{x}'). \quad (6.20)$$

Finally, (5.63) implies that

$$[[A_{l,4}(x''),j_4(x')],j_k(x)]=0. \quad (6.21)$$

We thus see that the Jacobi identity (6.18)+(6.20) +(6.21)=0 is not satisfied. To correctly calculate the double commutator, one must return to Eq. (5.65) and use it without introducing limits such as (5.47) which do not commute with ET commutation.

C. Explicit Fourth-Order Calculation

Having satisfied ourselves of the consistency of our results, we turn next to the problem of explicitly verifying them in fourth order. For the reasons mentioned in paragraph (ii) of Sec. 1, we cannot evaluate $[j_\mu(x),j_\nu(y)]$ and take the ET limit. An indirect method is, however, available in fourth order. We consider again Eq. (6.9) in fourth order, which we write as

$$\epsilon_\alpha(1)\epsilon_\beta(2)T_{\alpha\beta kl}{}^{(4)}(\tilde{k})=\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y} \times\langle 1|Tj_k(x)j_l(y)|2\rangle^{(4)}-i\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y} \times\delta(x_0-y_0)\langle 1|[j_k(x),A_{l,4}(y)]|2\rangle^{(4)}. \quad (6.22)$$

Here $|1\rangle$ is a photon state k_1, λ_1, $|2\rangle$ is a photon state $-k_2,\lambda_2$, $|1\rangle\neq|2\rangle$, $\epsilon_\alpha(1)=\epsilon_\alpha{}^{\lambda_1}(k_1)$, $\epsilon_\alpha(2)=\epsilon_\alpha{}^{\lambda_2}(-k_2)$, and $\tilde{k}$ represents the quartuplet (k_1,k_2,k_3,k_4). We shall also write $\delta(\tilde{k})=\delta(k_1+k_2+k_3+k_4)$ and $\tilde{0}=(0,0,0,0)$.

As we mentioned above, the individual terms in (6.22) are somewhat ambiguous, whereas the sum is well defined. It was shown in I that one could write

$$\langle 1|Tj_k(x)j_l(y)|2\rangle^{(4)} =\lim_{\xi\to 0}\langle 1|Tj_k(x;\xi)j_l(y)|2\rangle^{(4)}, \quad (6.23)$$

when $j_l(y)$ was any product of fields A, ψ at different points. Since $j_l(y)$ involves a limiting process, however, the right side of (6.23) can be ambiguous. Also, although $\lim_{\xi\to 0}\langle 1|[j_k(x;\xi),A_{l,4}(y)]|2\rangle$ is well defined, it is not immediately clear whether the second term in (6.22) should be taken as $\lim\delta_0\langle\ \ \rangle$ or $\delta_0\lim\langle\ \ \rangle$. The former possibility corresponds to our definition of ETC which we have seen to be *a priori* slightly ambiguous. Our purpose now will be to relate these ambiguities. We shall see that they are all resolved by the requirement that (6.22) (with additional terms corresponding to c-number contributions) also holds when $|1\rangle$ and $|2\rangle$ are the same state.

We first note that (6.23) only requires the expressions for the individual currents to third order. By (5.40), we can write

$$j_k(x;\xi)=Z_1Z_3^{-1}\tilde{j}_k(x;\xi)+Z_3^{-1}(1-Z_3)\partial_kA_{\mu,\mu}(x) +e^2Z_1Z_3^{-1}J_k(\xi)[\xi\cdot A+\tfrac{1}{6}(\xi\cdot\partial)^2\xi\cdot A]+O(e^4), \quad (6.24)$$

where

$$\tilde{j}_k(x;\xi)=\tfrac{1}{2}ie[\bar{\psi}(x)\gamma_k\psi(x+\xi)-\gamma_k\psi(x)\bar{\psi}(x+\xi)]. \quad (6.25)$$

It follows that

$$\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\langle 1|T\tilde{j}_k(x;\xi)\tilde{j}_l(y)|2\rangle =\epsilon_\alpha(1)\epsilon_\beta(2)U_{\alpha\beta kl}{}^{(4)}(\tilde{k};\xi)\delta(\tilde{k})+O(e^5), \quad (6.26)$$

where $U_{\alpha\beta kl}{}^{(4)}(\tilde{k};0)$ is the usual (slightly ambiguous) unrenormalized fourth-order photon-photon box diagram:

$$\epsilon_\alpha(1)\epsilon_\beta(2)U_{\alpha\beta kl}{}^{(4)}(\tilde{k};0)=\frac{e^4}{4!}\int dxdydzdz'e^{-ik_3\cdot x-ik_4\cdot y} \times\langle 1|T\bar{\psi}_0\gamma_k\psi_0(x)\bar{\psi}_0\gamma_l\psi_0(y)\bar{\psi}_0\gamma \times A_0\psi_0(z)\bar{\psi}_0\gamma\cdot A_0\psi_0(z')|2\rangle, \quad (6.27)$$

where a 0-subscript indicates a free field. Now, one easily sees that the remaining terms in (6.24) do not contribute to (6.23) for $|1\rangle\neq|2\rangle$. To fourth order, they only renormalize the vacuum expectation value.

Some insight into the meaning of (6.26) from our point of view can be obtained by considering the (heuristic) equality

$$\epsilon_\alpha(1)\epsilon_\beta(2)U_{\alpha\beta kl}{}^{(4)}(\tilde{k})=\int dydx\, e^{-ik_3\cdot x-ik_4\cdot y}\Box_x\Box_y \times\langle 1|T\tilde{A}_k(x)\tilde{A}_l(y)|2\rangle^{(4)}, \quad (6.28)$$

where $\tilde{A}_k(x)$ represents the unrenormalized vector potential, so that

$$\Box\tilde{A}_k(x)=\tfrac{1}{2}ie_0Z_1[\bar{\psi},\gamma_k\psi]=\tfrac{1}{2}ieZ_1Z_3^{-1/2}[\bar{\psi},\gamma_k\psi] =Z_1Z_3^{-1/2}\tilde{j}_k. \quad (6.29)$$

Since

$$\delta(x_0-y_0)[\tilde{j}_k(x),\tilde{A}_{l,4}(y)]=0, \quad (6.30)$$

(6.28) becomes

$$\epsilon_\alpha(1)\epsilon_\beta(2)U_{\alpha\beta kl}{}^{(4)}(\tilde{k}) =\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\langle 1|T\tilde{j}_k(x)\tilde{j}_l(y)|2\rangle^{(4)}, \quad (6.31)$$

which is (6.26),

It follows from (6.24) and (6.26) that we can write

$$\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\langle 1|Tj_k(x+\xi)j_l(y)|2\rangle^{(4)} = \epsilon_\alpha(1)\epsilon_\beta(2)U_{\alpha\beta kl}^{(4)}(\tilde{k};\xi)\delta(\tilde{k}). \quad (6.32)$$

In particular, the limits of each side of (6.32) will be the same for any approach to $\xi=0$. It is well known, however, that (by gauge invariance) the fourth-order renormalized box amplitude is given by

$$T_{\alpha\beta kl}^{(4)}(\tilde{k})=U_{\alpha\beta kl}^{(4)}(\tilde{k})\delta(\tilde{k})-U_{\alpha\beta kl}^{(4)}(\tilde{k}=0)\delta(\tilde{k}). \quad (6.33)$$

Again the individual terms in (6.33) are not well defined They depend on how the loop integration is performed or, if a regularization is employed, on how it is removed. If, in particular, the regularization corresponding to $\xi^2>0$ is used, they depend on how $\xi\rightarrow 0$. No approach to $\xi=0$, however, will give $U_{\alpha\beta kl}^{(4)}(\tilde{k}=0)=0$.

Comparison of (6.22), (6.32), and (6.33) now gives

$$i\int dxdy\, e^{-ik_3\cdot x-ik_4\cdot y}\delta(x_0-y_0)\langle 1|[j_k(x),A_{l,4}(y)]|2\rangle^{(4)} = \epsilon_\alpha(1)\epsilon_\beta(2)U_{\alpha\beta kl}^{(4)}(\tilde{k}=0)\delta(\tilde{k}). \quad (6.34)$$

It immediately follows that $[j_k(x),A_{l,4}(x')]$ is a q number, and, by (6.12), so is $[j_k(x),j_4(x')]$. We can, however, be more explicit. It follows from Ward's identity that[81]

$$\begin{aligned} U_{\alpha\beta kl}^{(4)}(\tilde{k}=0) &= -\lim_{\xi\to 0} e^4\int dp\, e^{ip\cdot\xi}\partial_\beta\partial_k\partial_l J_\alpha^{(0)}(p) \\ &= -(2\pi)^4 ie^4 \lim_{\xi\to 0} J_\alpha^{(0)}(\xi)\xi_\beta\xi_k\xi_l \\ &\equiv -(2\pi)^4 ie^4 J_{\alpha\beta kl}^{(0)}. \end{aligned} \quad (6.35)$$

Now $J_{\alpha\beta kl}^{(0)}$ and hence $[j_k,A_{l,4}]$ depend on how the $\xi\rightarrow 0$ limit is taken, corresponding to the ambiguities in the individual terms in (6.33) and in (6.22). We saw previously that our definition of $[j_k,A_{l,4}]$ [see Eq. (5.60)] depended on how $\xi\rightarrow 0$ was taken, because the individual terms in $j(x;\xi)$, particularly $\bar{\psi}(x)\gamma\psi(x+\xi)$ and $J(\xi)(\xi\cdot A)^3$, had such a dependence, although their sum did not. We see from (6.34) that these ambiguities are precisely the same. This also follows from the fact that in fourth order the purpose of the $J(\xi)(\xi\cdot A)^3$ term in $j(x;\xi)$ is to renormalize $U^{(4)}(\tilde{k})$ by removing $U^{(4)}(\tilde{k}=0)$ from it. This will be discussed in more detail below [Eq. (6.42)].

Now, (6.35) cannot vanish for any approach to $\xi=0$. In fact, (6.35) is given a definite finite value by the requirement that in the analog of (6.22) for the vacuum expectation value, the ETC expression, in connection with (6.12), reproduces the well-defined orthodox $\Delta\partial\delta(\mathbf{x}-\mathbf{x}')$ term. That is, all ambiguities are resolved by the requirement that (6.22) [with suitable additional terms corresponding to the c-number ETC's $[\dot{A}_k,A_l]$, etc.] holds for *all* states $|1\rangle$ and $|2\rangle$. This follows from the results of Sec. 5 in order e^2 [which is actually sufficient to define (6.35)] and will be established below in order e^4. Thus we must take

$$\begin{aligned} J_{\alpha\beta kl}^{(0)} &= (1/6\pi^2)\chi_{klmn}, \quad \alpha=m, \quad \beta=n \\ &=0, \qquad \alpha=4 \text{ or } \beta=4 \end{aligned} \quad (6.36)$$

the second line corresponding to $\xi_4=0$.

We next observe that the most general form for the ETC allowed by dimensionality and charge-conjugation covariance is

$$[j_k(x),A_{l,4}(x')]_T=eM_{\alpha\beta kl}:A_\alpha A_\beta:\delta(\mathbf{x}-\mathbf{x}'). \quad (6.37)$$

On substituting (6.35) and (6.37) into (6.34), we find

$$2ie\epsilon_\alpha(1)\epsilon_\beta(2)M_{\alpha\beta kl}^{(3)}\int dxdy\, e^{-i(k_1+k_2+k_3+k_4)\cdot x}\delta(x-y) = -(2\pi)^4 ie^4\epsilon_\alpha(1)\epsilon_\beta(2)J_{\alpha\beta kl}^{(0)}\delta(\tilde{k}). \quad (6.38)$$

Hence

$$M_{\alpha\beta kl}^{(3)}=-\tfrac{1}{2}e^3 J_{\alpha\beta kl}^{(0)}, \quad (6.39)$$

and, from (6.36),

$$[j_k(x),A_{l,4}(x')]_T^{(4)} = -(e^4/12\pi^2)\chi_{klmn}:A_m^{(0)}A_n^{(0)}:\delta(\mathbf{x}-\mathbf{x}'), \quad (6.40)$$

in exact agreement with (6.7) in fourth order. Equation (6.6) then follows from the general requirement (6.21) of gauge invariance. Thus we have succeeded in verifying that Eq. (6.6), which we previously derived by interchanging the equal-time and $\xi\rightarrow 0$ limits, is correct in fourth order.

We mentioned above that in fourth order the purpose of the $J(\xi)(\xi\cdot A)^3$ term in $j(x)$ is to renormalize the photon-photon box diagram.[82] In order to see this in the present context, we write the box diagram in the form

$$\epsilon_\alpha(1)\epsilon_\beta(2)\epsilon_\gamma(3)T_{\alpha\beta\gamma l}^{(4)}(\tilde{k}) = -\int dy\, e^{-ik_4\cdot y}\langle 1;3|j_l(y)|2\rangle^{(4)}. \quad (6.41)$$

To evaluate (6.41), we need the expression (5.40) for j to fourth order. Taking $|1\rangle\neq|2\rangle\neq|3\rangle\neq|1\rangle$, only the $\bar{j}_l(x;\xi)$ term (6.25) and the $\frac{1}{6}e^4J_k(\xi)(\xi\cdot A)^3$ term [in (5.34)] will contribute. Then (6.41) becomes

$$\begin{aligned} \epsilon_\alpha(1)\epsilon_\beta(2)\epsilon_\gamma(3)T_{\alpha\beta\gamma l}^{(4)}(\tilde{k}) &= -\int dy\, e^{-ik_4\cdot y}\langle 1;3|\bar{j}_l(y)|2\rangle^{(4)}-\tfrac{1}{6}e^4 J_{\alpha\beta\gamma l}^{(0)}\int dy \\ &\quad\times e^{-ik_4\cdot y}\langle 1;3|A_\alpha(y)A_\beta(y)A_\gamma(y)|2\rangle^{(0)} \\ &= \epsilon_\alpha(1)\epsilon_\beta(2)\epsilon_\gamma(3)U_{\alpha\beta\gamma l}^{(4)}(\tilde{k})\delta(\tilde{k})+(2\pi)^4 ie^4\epsilon_\alpha(1) \\ &\quad\times\epsilon_\beta(2)\epsilon_\gamma(3)J_{\alpha\beta\gamma l}^{(0)}\delta(\tilde{k}), \end{aligned} \quad (6.42)$$

---

[81] See I, Eqs. (5.44), (6.49), and (7.74).

[82] In higher orders, it renormalizes all diagrams containing the box diagram.

which, by virute of (6.35), is the same as (6.33). The A^3 subtraction in j is thus seen to provide the $U(\tilde{k}=0)$ subtraction in $T(\tilde{k})$.

Having explicitly verified (5.76) in fourth order, we next turn to the problem of verifying the entire commutator (5.75). If we had considered the generalization of (6.2) which allows $|1\rangle=|2\rangle$, the c-number terms in (5.75) would have contributed. In particular, using (6.12), the $\partial\Delta\delta(\mathbf{x}-\mathbf{x}')$ term would have been present, with a coefficient involving the same $\xi\to 0$ ambiguity as in (6.35), exactly as in (5.68). This ambiguity was resolved by requiring the coefficient to be such that the orthodox ETC is reproduced. In second order, we saw that this required us to use (4.64), and this was also used above. For consistency, we must show that (4.64) is also correct in fourth order, that is, we must establish that the coefficient of $\partial\Delta\delta(\mathbf{x}-\mathbf{x}')$ given in (5.75) is correct in fourth order. This will simultaneously show that our definition (1.8) of ETC gives the correct result for the term in this order.

We have already computed the orthodox result for the coefficient of $\partial_k\Delta\delta(\mathbf{x}-\mathbf{x}')$ in $[j_k(x),j_4(x')]^{(4)}$. By (3.54) and (3.75), it is

$$-\frac{e^4}{72\pi^4}\lim_{N\to\infty}\ln\frac{N^2}{\alpha}. \tag{6.43}$$

Our result for this coefficient, given in (5.75), is

$$-(e^2/12\pi^2)(Z_3^{-2})^{(2)}. \tag{6.44}$$

We shall use the well-known expression

$$Z_3^{-1}=1+\frac{e^2}{12\pi^2}\lim_{N\to\infty}\ln\frac{N^2}{\alpha}+O(e^4). \tag{6.45}$$

This can be derived, for example, from (5.16) using (3.66)–(3.70). Thus (6.44) becomes

$$-\frac{e^2}{12\pi^2}\frac{2e^2}{12\pi^2}\lim_{N\to\infty}\ln\frac{N^2}{\alpha},$$

in perfect agreement with (6.43). We see that our method has unambiguously reproduced the correct ETC in fourth order.

D. Comparison with Other Methods

We should mention that conclusions opposite to ours have appeared in the literature,[83,84] namely, that (5.75) is only a c number, so that (6.6) vanishes. Let us comment on the source of these differences. In Ref. 83, full quantum electrodynamics is presented as a set of Feynman rules and is not based on field equations with explicit current operators. Formal manipulations with spectral representations are performed so that the $\partial\Delta\delta(\mathbf{x})$ term, which we have shown in Sec. 3 to be present in all orders of perturbation theory, is missed. It is clear from (5.68), however, that removing this term will simultaneously remove the q-number structure of the ETC.

In Ref. 84, the expression (6.22) for the renormalized box diagram is considered and written as

$$T(\tilde{k})=T^T(\tilde{k})+T^P(\tilde{k}), \tag{6.46}$$

with T^P denoting the ETC term. A corresponding decomposition for the unrenormalized box diagram [Eq. (6.28)] is written as

$$U(\tilde{k})=U^T(\tilde{k})+U^P(\tilde{k}), \tag{6.47}$$

and the relation (6.33) is noted[85]:

$$T(\tilde{k})=U(\tilde{k})-U(\tilde{k}=0). \tag{6.48}$$

It is then *assumed* that

$$U^T(\tilde{k}),\ T^T(\tilde{k})\xrightarrow[k_{40}\to\infty]{}0 \tag{6.49}$$

and that $U^P(\tilde{k})$ and $T^P(\tilde{k})$ are the nonvanishing parts of $U(\tilde{k})$ and $T(\tilde{k})$ in the limit $k_{40}\to\infty$ with $\mathbf{k}_4$ fixed. Thus, since

$$U(0,0,k_3,k_4)=U(0,0,0,0)\neq 0 \tag{6.50}$$

it is concluded from (6.48) that

$$T^P(0,0,k_3,k_4)=U^P(0,0,k_3,k_4)-U(\tilde{k})=0.$$

We have seen above [Eqs. (6.28), (6.32), (6.34)], however, that

$$U^T(\tilde{k})=T^T(\tilde{k})=U(\tilde{k}) \tag{6.51}$$

so that, in view of (6.50), (6.49) is not valid. Let us comment on the reason for this. Equation (6.49) would follow, more or less, if the Fourier transform of a general time-ordered product $\tau(t)=\theta(t)F(t)$ could be written as

$$\hat{\tau}(\omega)=\int d\omega'\frac{\sigma(\omega')}{\omega-\omega'}. \tag{6.52}$$

However, if the distribution $F(t)$ behaves like t^{-n} for $t\sim 0$, then only an n-times subtracted version of (6.52) is valid.[12] That such representations of U^T and T^T need subtractions follows from (6.50) and (6.51).

Let us next attempt to compute the ETC by using our current (5.40) in connection with the functional differentiation formalism (2.22). Assuming that the external electromagnetic field enters into (2.22) in the same way as does the quantized field, and that the differentiation can be commuted with the $\xi\to 0$ limit,[86]

[83] D. G. Boulware, Phys. Rev. **151**, 1024 (1966). I wish to thank Dr. Boulware for an interesting discussion on this matter.

[84] T. Nagylaki, Phys. Rev. **158**, 1534 (1967). I thank Dr. Nagylaki for a clarifying correspondence concerning his approach.

[85] Although not stated in Ref. 84, this relation is only valid in fourth order. However, since U was not explicitly defined in Ref. 84, this criticism may not be relevant. In any case, the situation is clear in fourth order.

[86] We are also assuming that it is the renormalized fields which are relevant in (2.22).

we find for $[j_k(x),j_4(x')]$ the expression

$$-e^2Z_1Z_3^{-1}J_{kl}\partial_l\delta(\mathbf{x}-\mathbf{x}')-(Z_3^{-1}-1)\partial_l\Delta\delta(\mathbf{x}-\mathbf{x}') -\tfrac{1}{6}e^2Z_1Z_3^{-1}J_{klmn}\partial_l\partial_m\partial_n\delta(\mathbf{x}-\mathbf{x}') +\tfrac{1}{2}e^4Z_1Z_3^{-1}J_{klmn}{:}A_lA_m{:}\partial_n\delta(\mathbf{x}-\mathbf{x}'). \quad (6.53)$$

In second order, this gives a divergent coefficient of $\partial_l\Delta\delta(\mathbf{x}-\mathbf{x}')$, contrary to the orthodox result. Furthermore, if one ignores the $(Z_3^{-1}-1)$ term and takes the limit defining J_{klmn} to be such that the orthodox results for the VEV are reproduced, one finds a contradiction between the second- and fourth-order results. Indeed, second order requires (4.65), which we have seen is also correct in fourth order when Z_3^{-2} is present, as in (5.68). With Z_3^{-1} present as in (6.53), however, the fourth-order term will differ from the orthodox one by a factor of 2. Nevertheless, the form of Eq. (6.53) is the same as that of our result (5.68).

We mention finally that if ETC's are defined by Eq. (2.19), results different from ours will be obtained. This can be seen, for example, from consideration of the Fourier transform of Eq. (6.9), which reads

$$\langle 1|Tj_\mu(x)j_\nu(y)|2\rangle=\frac{1}{(2\pi)^8}\int dk_3dk_4\, e^{ik_3\cdot x+ik_4\cdot y}T_{\mu\nu}{}^{\lambda_1\lambda_2} \times(k_1k_2k_3k_4)+i\langle 1|[j_\mu(x),A_{\nu,4}(y)]|2\rangle\delta(x_0-y_0). \quad (6.54)$$

Since $T_{\mu\nu}{}^{\lambda_1\lambda_2}(k_1k_2k_3k_4)$ vanishes for, say, $k_1=0$, the commutator $\langle 1|[j_\mu(x),j_\nu(y)]|2\rangle$ vanishes for $k_1=0$ and $x_0\neq y_0$, so that (2.19) gives $\langle 1|[j_k(x),j_4(x')]|2\rangle|_{k_1=0}=0$. From (6.6), we find, on the other hand,

$$\langle 1|[j_k(x),j_4(x')]|2\rangle^{(4)}=(e^4/6\pi^2)\chi_{klmn}\epsilon_l{}^{\lambda_1}(k_1)\epsilon_m{}^{\lambda_2} \times(k_2)e^{-ik_1\cdot x+ik_2\cdot x'}\partial_n\delta(\mathbf{x}-\mathbf{x}'), \quad (6.55)$$

which does not vanish for $k_1=0$. The source of this difference is, of course, that the second term in (6.54) represents a discrete ET singularity which Eq. (2.19) overlooks. We see that our definition (1.8) amounts to a regularization of this singularity. We take the consistency of our results and their verification in fourth order as evidence that this regularization is a reasonable one.

Let us conclude by reemphasizing that beyond fourth order, the exact coefficient of A^2 in, say, Eq. (6.6) should not be taken too seriously. For purposes of abstracting ETCR's from perturbation theory, one need only assume that the general form of (6.6) is correct. Similar results are expected in any renormalizable field theory.

ACKNOWLEDGMENTS

I am greatly indebted to Professor Joseph Sucher for his continuous encouragement, cogent criticisms, and useful suggestions. For helpful discussions, I am grateful also to Professor H. Araki, Professor O. W. Greenberg, and Professor C. H. Woo.

APPENDIX

Because of the fact that the renormalization constants Z_1^{-1} and Z_3^{-1} are infinite beyond first order, the equations of Secs. 5 and 6 are somewhat heuristic beyond fourth order. In this appendix, therefore, we shall briefly indicate how a more rigorous analysis should proceed. Thus we assume only CR's of the form (5.21) and that the $\tau\rightarrow 0$ limit commutes with the $\xi\rightarrow 0$ limit in the correct current operator (5.32)–(5.34).

We define

$$V(\xi^2)=[1+K_1(\xi^2)]^{-1}, \quad (A1)$$

so that (5.32) and (5.33) become

$$j_4(x)=\lim_{\xi\to 0} j_4(x;\xi), \quad (A2)$$

with

$$j_4(x;\xi)=\tfrac{1}{2}ieV(\xi^2)[\bar{\psi}(x)\gamma_4\psi(x+\xi)-\gamma_4\psi(x)\bar{\psi}(x+\xi)] -eV(\xi^2)R_1(\xi^2)\partial_\nu F_{4\nu}(x). \quad (A3)$$

We shall use the rigorous ETCR's

$$\lim_{\tau\to 0} W^{-1}(\tau)\{\bar{\psi}_\alpha(x),\psi_\beta(\mathbf{y},x_0+\tau)\}=\gamma_{4\beta\alpha}\delta(\mathbf{x}-\mathbf{y}), \quad (A4)$$

$$\begin{aligned}\lim_{\tau\to 0}\{\psi_\alpha(x),\psi_\beta(\mathbf{y},x_0+\tau)\}&=\lim_{\tau\to 0}\{\bar{\psi}_\alpha(x),\bar{\psi}_\beta(\mathbf{y},x_0+\tau)\}\\ &=\lim_{\tau\to 0}\{A_\mu(x),\psi_\alpha(\mathbf{y},x_0+\tau)\}\\ &=\lim_{\tau\to 0}\{A_{\mu,4}(x),\psi_\alpha(\mathbf{y},x_0+\tau)\}\\ &=0. \quad (A5)\end{aligned}$$

We thus have

$$\lim_{\xi\to 0}\lim_{\tau\to 0} V^{-1}(\xi)W^{-1}(\tau)[j_4(x;\xi),\psi(\mathbf{y},x_0+\tau)] =-ie\psi(x)\delta(\mathbf{x}-\mathbf{y}). \quad (A6)$$

We can now *define* an ETC by writing (A6) as

$$Y[j_4(x),\psi(x')]=-ie\psi(x)\delta(\mathbf{x}-\mathbf{x}'), \quad (A7)$$

with Y a constant which depends on how the $\tau=0$ and $\xi=0$ limits are taken. Finally, (6.1) and (6 2) could be used to specify that $Y=1$, so that (5.56) is again obtained.

The other ETCR's considered in Sec. 5 can be discussed from a similar point of view. The analysis in Secs. 5 and 6 should, however, be sufficient for purposes of abstracting ETCR's from perturbation theory.