PHYSICAL REVIEW VOLUME 166, NUMBER 5 25 FEBRUARY 1968

Algebraic Solution in the V-θ Sector of the Lee Model*

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(Received 31 July 1967)

A set of equations for the V-θ sector of the Lee model is solved by purely algebraic techniques. It is shown that the integral equations involved are like those that arise in scattering by a separable potential, in that their solution requires no integral-equation techniques.

INTRODUCTION

THE V-θ sector of the Lee model was first solved by Amado[1]; other methods of solution have subsequently appeared.[2,3] All of them involve solving singular integral equations by techniques of analytic continuation. In this paper, the basic integral equations are chosen to be a redundant set; the four equations could be reduced to two. The redundant set of equations is shown to be soluble by purely algebraic techniques, while it can be demonstrated that the reduced set of two equations is not. Because of the algebraic nature of the solution, it is simple to give a complete description of the V-θ sector.

NOTATION AND N-θ SECTOR

The Hamiltonian for the Lee model is

$$H=E_V V^\dagger V+E_N N^\dagger N+\int \omega(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})d\mathbf{k}$$
$$+\int \mu(\mathbf{k})a^\dagger(\mathbf{k})d\mathbf{k}N^\dagger V+V^\dagger N\int \mu(\mathbf{k})a(\mathbf{k})d\mathbf{k}, \quad (1)$$

where E_V and E_N are the energies of the source in its V and N states, respectively, $\omega(\mathbf{k})$ is the θ meson energy, and $\mu(\mathbf{k})$ is the source form factor. We let the θ field be a Bose field. In the sectors of interest it does not matter whether V and N are both Bose fields or both Fermi fields; we take them to be Bose fields. Then the nonvanishing equal-time commutators are

$$[N(t),N^\dagger(t)]=[V(t),V^\dagger(t)]=1, \quad (2)$$

$$[a(\mathbf{k},t),a^\dagger(\mathbf{k}',t)]=\delta(\mathbf{k}-\mathbf{k}'). \quad (3)$$

We work in the Heisenberg picture with time variables usually suppressed. The equations of motion are

$$i\partial_t N=E_N N+\int \mu(\mathbf{k})a^\dagger(\mathbf{k})d\mathbf{k}V, \quad (4)$$

$$i\partial_t V=E_V V+N\int \mu^*(\mathbf{k})a(\mathbf{k})d\mathbf{k}, \quad (5)$$

$$i\partial_t a(\mathbf{k})=\omega(\mathbf{k})a(\mathbf{k})+\mu(\mathbf{k})N^\dagger V. \quad (6)$$

The vacuum state $|0\rangle$ satisfies

$$\langle 0|a^\dagger(\mathbf{k})=\langle 0|V^\dagger=\langle 0|N^\dagger=\langle 0|H=0. \quad (7)$$

We define the notation $A\doteq B$ by

$$A\doteq B\rightleftarrows\langle 0|A=\langle 0|B; \quad (8)$$

then (7) can be written

$$a^\dagger(\mathbf{k})\doteq V^\dagger\doteq N^\dagger\doteq H\doteq 0. \quad (9)$$

It follows from (4) and (9) that

$$i\partial_t N\doteq E_N N, \quad (10)$$

so that $\langle 0|N$ is an eigenstate of H with eigenvalue E_N; it is the N-particle state $\langle N|$. (The notation is backward, but convenient in that it avoids daggers and gives equations for annihilation operators.) Similarly,

$$i\partial_t a(\mathbf{k})\doteq\omega(\mathbf{k})a(\mathbf{k}), \quad (11)$$

so that the θ-particle state

$$\langle\mathbf{k}|=\langle 0|a(\mathbf{k}) \quad (12)$$

is an eigenstate of H with eigenvalue $\omega(\mathbf{k})$.

In order to illustrate the procedure to be used in the V-θ sector, we solve the N-θ sector in some detail. The coupled equations in the N-θ sector are

$$i\partial_t V\doteq E_V V+N\int \mu^*(\mathbf{k})a(\mathbf{k})d\mathbf{k},$$
$$i\partial_t Na(\mathbf{k})\doteq[E_N+\omega(\mathbf{k})]Na(\mathbf{k})+\mu(\mathbf{k})V. \quad (13)$$

Equation (13) will be solved by looking for operators $Y_\alpha(t)$ that have a single frequency, so that $Y_\alpha(t)$ can be written $Y_\alpha e^{-i\omega_\alpha t}$. Then the operator Y_α will be the annihilation operator for a quantum of energy ω_α. We

* Work performed under the auspices of the U. S. Atomic Energy Commission.

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introduce the Fourier transforms

$$V(t)=\int e^{-i(E_T+\lambda)t}W(\lambda)d\lambda\,,\qquad Na(\mathbf{k})=\int e^{-i(E_T+\lambda)t}W(\mathbf{k},\lambda)d\lambda\,,\tag{14}$$

where E_T is the N-θ threshold energy;

$$E_T=E_N+\omega(\mathbf{0})\,.\tag{15}$$

Then the equations become

$$(\lambda-\Delta)W(\lambda)\doteq\int\mu^*(\mathbf{k})W(\mathbf{k},\lambda)d\mathbf{k}\,,\qquad [\lambda-\xi(\mathbf{k})]W(\mathbf{k},\lambda)\doteq\mu(\mathbf{k})W(\lambda)\,,\tag{16}$$

where Δ and $\xi(\mathbf{k})$ are defined by

$$\Delta=E_V-E_T\,,\qquad \xi(\mathbf{k})=\omega(\mathbf{k})-\omega(\mathbf{0})\,.\tag{17}$$

Now the Fourier transforms of eigenoperators $Y_\alpha(t)$ take the form $Y_\alpha\delta(\lambda-(\omega_\alpha-E_T))$. The second of Eqs. (16) has the general solution

$$W(\mathbf{k},\lambda)\doteq n(\mathbf{k})Y(\mathbf{k})\delta(\lambda-\xi(\mathbf{k}))+\frac{\mu(\mathbf{k})}{\lambda-\xi(\mathbf{k})+i0}W(\lambda)\,,\tag{18}$$

where $Y(\mathbf{k})$ is the eigenoperator for wave number $\mathbf{k}$, and $n(\mathbf{k})$ is a function that will be chosen to normalize the commutation relations of the eigenoperators. Then the first of Eqs. (16) becomes

$$D^+(\lambda)W(\lambda)\doteq\int\mu^*(\mathbf{k})n(\mathbf{k})Y(\mathbf{k})\delta(\lambda-\xi(\mathbf{k}))d\mathbf{k}\,,\tag{19}$$

where the notation $D^+(\lambda)$ is defined by

$$D^+(\lambda)=D(\lambda+i0)\tag{20}$$

and D is a function that is analytic in the cut z plane

$$D(z)=z-\Delta-I(z)\,,\tag{21}$$

$$I(z)=\int\frac{|\mu(\mathbf{k})|^2}{z-\xi(\mathbf{k})}d\mathbf{k}=\int_0^\infty\frac{\rho(\epsilon)}{z-\epsilon}d\epsilon\,,\tag{22}$$

$$\rho(\epsilon)=\int|\mu(\mathbf{k})|^2\delta(\epsilon-\xi(\mathbf{k}))d\mathbf{k}=\frac{1}{2\pi i}(D^+(\epsilon)-D^-(\epsilon)).\tag{23}$$

The function $D(z)$ is real and monotonically increasing on the negative real axis. We are interested in the case that there is a discrete state in the N-θ sector; this means that $D(z)$ has a zero at $z=-B$, where B is the binding energy of the state with respect to E_T:

$$D(-B)=0\,.\tag{24}$$

Again, the general solution of (19) involves a δ function at the zero of $D^+(\lambda)$, so that another eigenoperator Y_B is required:

$$\begin{aligned}W(\lambda)&\doteq n_BY_B\delta(\lambda+B)+\frac{1}{D^+(\lambda)}\int\mu^*(\mathbf{k})n(\mathbf{k})Y(\mathbf{k})\delta(\lambda-\xi(\mathbf{k}))d\mathbf{k}\\&\doteq n_BY_B\delta(\lambda+B)+\int\frac{\mu^*(\mathbf{k})}{D^+(\xi(\mathbf{k}))}n(\mathbf{k})Y(\mathbf{k})\delta(\lambda-\xi(\mathbf{k}))d\mathbf{k}\doteq e_{B,V}Y_B\delta(\lambda+B)+\int e_{\mathbf{q},V}Y(\mathbf{q})\delta(\lambda-\xi(\mathbf{q}))d\mathbf{q}\,,\end{aligned}\tag{25}$$

and hence

$$\begin{aligned}W(\mathbf{k},\lambda)&\doteq\int\left[\delta(\mathbf{k}-\mathbf{q})+\frac{\mu(\mathbf{k})\mu^*(\mathbf{q})}{[\xi(\mathbf{q})-\xi(\mathbf{k})+i0]D^+(\xi(\mathbf{q}))}\right]\\&\quad\times n(\mathbf{q})Y(\mathbf{q})\delta(\lambda-\xi(\mathbf{q}))d\mathbf{q}-\frac{\mu(\mathbf{k})n_B}{B+\xi(\mathbf{k})}Y_B\delta(\lambda+B)\\&\doteq\int e_{\mathbf{q}}(\mathbf{k})Y(\mathbf{q})\delta(\lambda-\xi(\mathbf{q}))d\mathbf{q}\\&\qquad+e_B(\mathbf{k})Y_B\delta(\lambda+B)\,.\end{aligned}\tag{26}$$

The Fourier transforms give

$$V(t)\doteq e_{B,V}Y_Be^{-iE_Bt}+\int e_{\mathbf{q},V}Y(\mathbf{q})e^{-iE(\mathbf{q})t}d\mathbf{q}\,,\qquad N(t)a(\mathbf{k},t)\doteq e_B(\mathbf{k})Y_Be^{-iE_Bt}+\int e_{\mathbf{q}}(\mathbf{k})Y(\mathbf{q})e^{-iE(\mathbf{q})t}d\mathbf{q}\,.\tag{27}$$

It follows that if the n's are chosen properly, then

$$\langle B|=\langle 0|Y_B\,,\qquad \langle N\mathbf{q}^{\text{in}}|=\langle 0|Y(\mathbf{q})\tag{28}$$

are normalized eigenstates of H with eigenvalues

$$E_B=E_T-B\,,\qquad E(\mathbf{q})=E_T+\xi(\mathbf{q})\,,\tag{29}$$

respectively.

The normalization factors are chosen by first noting that

$$\langle 0|V(0)|\alpha\rangle=e_{\alpha,V}\,,\qquad \langle 0|N(0)a(\mathbf{k},0)|\alpha\rangle=e_\alpha(\mathbf{k})\,,\tag{30}$$

and if α and β are in the N-θ sector

$$\begin{aligned}\langle\beta|V^\dagger V+\int a^\dagger(\mathbf{k})a(\mathbf{k})d\mathbf{k}|\alpha\rangle&=\delta_{\alpha\beta}\\&=\langle\beta|V^\dagger V+\int a^\dagger(\mathbf{k})N^\dagger Na(\mathbf{k})d\mathbf{k}|\alpha\rangle\\&=e_{\beta V}{}^*e_{\alpha V}+\int e_\beta{}^*(\mathbf{k})e_\alpha(\mathbf{k})d\mathbf{k}\,,\end{aligned}\tag{31}$$

It is not difficult to verify that (31) is satisfied if

$$n(\mathbf{q})=1,\quad n_B=Z^{1/2},\quad Z^{-1}\equiv D'(-B). \tag{32}$$

Then (27) can be inverted to give

$$\begin{aligned} Y_B(t)&=e^{-iE_Bt}Y_B\doteq e_{B,V}{}^*V(t)+\int e_B{}^*(\mathbf{k})N(t)a(\mathbf{k},t)d\mathbf{k},\\ Y(\mathbf{q},t)&=e^{-iE(\mathbf{q})t}Y(\mathbf{q})\doteq e_{\mathbf{q},V}{}^*V(t)+\int e_{\mathbf{q}}{}^*(\mathbf{k})N(t)a(\mathbf{k},t)d\mathbf{k}.\end{aligned} \tag{33}$$

That $Y^\dagger(\mathbf{q})|0\rangle$ is actually the state with an incident θ particle with momentum $\mathbf{q}$ follows from investigating the amplitude

$$\langle 0|Na(\mathbf{k})|N\mathbf{q}^{\text{in}}\rangle=e_{\mathbf{q}}(\mathbf{k}), \tag{34}$$

and noting that in $\mathbf{k}$ it has a plane wave and outgoing spherical waves. Finally, we list the matrix $e_{\alpha,a}$

$\alpha \backslash a$	V	$Na(\mathbf{k})$
B	$Z^{1/2}$	$-Z^{1/2}\dfrac{\mu(\mathbf{k})}{B+\xi(\mathbf{k})}$
q	$\dfrac{\mu^*(\mathbf{q})}{D^+(\xi(\mathbf{q}))}$	$\delta(\mathbf{q}-\mathbf{k})+\dfrac{\mu^*(\mathbf{q})\mu(\mathbf{k})}{D^+(\xi(\mathbf{q}))[\xi(\mathbf{q})-\xi(\mathbf{k})+i0]}$.

(35)

The elements in (35) really correspond to $e_{\mathbf{q}}{}^{\text{in}}$; there is a similar matrix if we use the states $|N\mathbf{q}^{\text{out}}\rangle$ instead of $|N\mathbf{q}^{\text{in}}\rangle$. The functions $e_{\mathbf{q}}{}^{\text{out}}$ are obtained from $e_{\mathbf{q}}{}^{\text{in}}$ by changing $+i0$ to $-i0$.

V-θ SECTOR

The coupled equations in the V-θ sector can be written

$$\begin{aligned} i\partial_tY_\alpha a(\mathbf{k})&\doteq[\omega(\mathbf{k})+E_\alpha]Y_\alpha a(\mathbf{k})+\mu(\mathbf{k})Y_\alpha N^\dagger V,\\ i\partial_tY_\alpha N^\dagger V&\doteq(E_\alpha+E_V-E_N)Y_\alpha N^\dagger V+Y_\alpha(N^\dagger N-V^\dagger V)\int\mu^*(k)a(\mathbf{k})d\mathbf{k},\end{aligned} \tag{36}$$

where α stands for B or $\mathbf{q}$, and we have used the equations

$$i\partial_tY_\alpha\doteq E_\alpha Y_\alpha \tag{37}$$

that follow from (33). The last term is transformed as follows:

$$\begin{aligned} &Y_\alpha(N^\dagger N-V^\dagger V)\\ &=\left(e_{\alpha,V}{}^*V+\int e_\alpha{}^*(\mathbf{k})Na(\mathbf{k})d\mathbf{k}\right)(N^\dagger N-V^\dagger V)\\ &\doteq -e_{\alpha,V}{}^*V+\int e_\alpha{}^*(\mathbf{k})Na(\mathbf{k})d\mathbf{k}\\ &\doteq Y_\alpha-2e_{\alpha,V}{}^*V\\ &\doteq Y_\alpha-2e_{\alpha,V}{}^*\left[e_{B,V}Y_B+\int e_{\mathbf{q},V}Y(\mathbf{q})d\mathbf{q}\right],\end{aligned} \tag{38}$$

so that the coupled equations become

$$\begin{aligned} i\partial_tY_\alpha a(\mathbf{k})&\doteq[\omega(\mathbf{k})+E_\alpha]Y_\alpha a(\mathbf{k})+\mu(\mathbf{k})Y_\alpha N^\dagger V,\\ i\partial_tY_\alpha N^\dagger V&\doteq(E_\alpha+E_V-E_N)Y_\alpha N^\dagger V+Y_\alpha\int\mu^*(\mathbf{k})a(\mathbf{k})d\mathbf{k}\\ &\quad-2e_{\alpha,V}{}^*\left[e_{B,V}Y_B\int\mu^*(\mathbf{k})a(\mathbf{k})d\mathbf{k}+\int\mu^*(\mathbf{k})e_{\mathbf{q},V}Y(\mathbf{q})a(\mathbf{k})d\mathbf{k}d\mathbf{q}\right].\end{aligned} \tag{39}$$

Equations (39) are redundant because $Y_\alpha N^\dagger V$ can be expressed in terms of $Y_\alpha a(\mathbf{p})$ by using (33) and (27). However, it is apparently just this redundancy that makes it possible to solve (39) by using algebraic techniques.

As in (14), we introduce the Fourier transforms

$$\begin{aligned} Y_B(t)N^\dagger(t)V(t)&=\int e^{-i(E_2+\lambda)t}U(\lambda)d\lambda,\\ Y(\mathbf{q},t)N^\dagger(t)V(t)&=\int e^{-i(E_2+\lambda)t}U(\mathbf{q},\lambda)d\lambda,\\ Y_B(t)a(\mathbf{k},t)&=\int e^{-i(E_2+\lambda)t}T(\mathbf{k},\lambda)d\lambda,\\ Y(\mathbf{q},t)a(\mathbf{k},t)&=\int e^{-i(E_2+\lambda)t}T(\mathbf{k},\mathbf{q},\lambda)d\lambda,\end{aligned} \tag{40}$$

with E_2 the $N-2\theta$ threshold:

$$E_2=E_T+\omega(\mathbf{0})=E_N+2\omega(\mathbf{0}), \tag{41}$$

and substitute the explicit forms of the $e_{\alpha,a}$ to obtain the coupled equations

$$[\lambda+B-\xi(\mathbf{k})]T(\mathbf{k},\lambda)=\mu(\mathbf{k})U(\lambda)\,,$$
$$[\lambda-\xi(\mathbf{k})-\xi(\mathbf{q})]T(\mathbf{k},\mathbf{q},\lambda)=\mu(\mathbf{k})U(\mathbf{q},\lambda)\,,$$
$$(\lambda+B-\Delta)U(\lambda)=(1-2Z)\int\mu^*(\mathbf{k})T(\mathbf{k},\lambda)d\mathbf{k}-2Z^{1/2}\int\mu^*(\mathbf{k})\beta^*(\mathbf{p})T(\mathbf{k},\mathbf{p},\lambda)d\mathbf{k}d\mathbf{p}\,,\tag{42}$$
$$[\lambda-\xi(\mathbf{q})-\Delta]U(\mathbf{q},\lambda)=\int\mu^*(\mathbf{k})T(\mathbf{k},\mathbf{q},\lambda)d\mathbf{k}-2Z^{1/2}\beta(\mathbf{q})\int\mu^*(\mathbf{k})T(\mathbf{k},\lambda)d\mathbf{k}-2\beta(\mathbf{q})\int\mu^*(\mathbf{k})\beta^*(\mathbf{p})T(\mathbf{k},\mathbf{p},\lambda)d\mathbf{k}d\mathbf{p}\,,$$

where

$$\beta(\mathbf{q})=e_{\mathbf{q},V}^*.\tag{43}$$

Before actually solving the equations, we need to know what sort of eigenoperators to expect in this sector. If there is a $B-\theta$ or $N-2\theta$ bound state, then we will have an operator X_C with eigenvalue E_2-C, $C>B$. We can also have scattering states $|B\mathbf{q}^{\text{in}}\rangle$ with corresponding $X(\mathbf{q})$ and, finally, states $|N\mathbf{q}_1\mathbf{q}_2{}^{\text{in}}\rangle$ with operators $X(\mathbf{q}_1,\mathbf{q}_2)$. As in (25) and (26), we seek the functions f, g that give

$$U(\lambda)=f_{C,B}X_C(\lambda)+\int f_{\mathbf{p},B}X(\mathbf{p},\lambda)+\tfrac{1}{2}\int f_{\mathbf{p}_1\mathbf{p}_2,B}X(\mathbf{p}_1\mathbf{p}_2,\lambda)\,,$$
$$U(\mathbf{q},\lambda)=f_C(\mathbf{q})X_C(\lambda)+\int f_{\mathbf{p}}(\mathbf{q})X(\mathbf{p},\lambda)+\tfrac{1}{2}\int f_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{q})X(\mathbf{p}_1\mathbf{p}_2,\lambda)\,,$$
$$T(\mathbf{k},\lambda)=g_C(\mathbf{k})X_C(\lambda)+\int g_{\mathbf{p}}(\mathbf{k})X(\mathbf{p},\lambda)+\tfrac{1}{2}\int g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k})X(\mathbf{p}_1\mathbf{p}_2,\lambda)\,,\tag{44}$$
$$T(\mathbf{k},\mathbf{q},\lambda)=g_C(\mathbf{k},\mathbf{q})X_C(\lambda)+\int g_{\mathbf{p}}(\mathbf{k},\mathbf{q})X(\mathbf{p},\lambda)+\tfrac{1}{2}\int g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k},\mathbf{q})X(\mathbf{p}_1\mathbf{p}_2,\lambda)\,,$$

$$X_C(\lambda)=X_C\delta(\lambda+C)\,,$$
$$X(\mathbf{p},\lambda)=X(\mathbf{p})\delta(\lambda-\xi(\mathbf{p})+B)\,,\tag{45}$$
$$X(\mathbf{p}_1\mathbf{p}_2,\lambda)=X(\mathbf{p}_1\mathbf{p}_2)\delta(\lambda-\xi(\mathbf{p}_1)-\xi(\mathbf{p}_2))\,.$$

In order to check δ-function parts, we note that the bound state normalization is a separate operation. As for the others, first

$$\langle 0|Y_B(0)N^\dagger(0)V(0)|\alpha\rangle=f_{\alpha,B}\,,$$
$$\langle 0|Y(\mathbf{q},0)N^\dagger(0)V(0)|\alpha\rangle=f_\alpha(\mathbf{q})\,,\tag{46}$$
$$\langle 0|Y_B(0)a(\mathbf{k},0)|\alpha\rangle=g_\alpha(\mathbf{k})\,,$$
$$\langle 0|Y(\mathbf{q},0)a(\mathbf{k},0)|\alpha\rangle=g_\alpha(\mathbf{k},\mathbf{q})\,,$$

where α is C or $B\mathbf{p}^{\text{in}}$ or $N\mathbf{p}_1\mathbf{p}_2{}^{\text{in}}$. Clearly $f_{\alpha,B}$ can have only the bound state delta. If α is $B\mathbf{p}^{\text{in}}$, then the δ part of $f_{\mathbf{p}}(\mathbf{q})$ is

$$Z^{1/2}\langle 0|Y(\mathbf{q},0)|N\mathbf{p}^{\text{in}}\rangle=Z^{1/2}\delta(\mathbf{p}-\mathbf{q})\,.\tag{47}$$

Similarly, the δ part of $g_{\mathbf{p}}(\mathbf{k})$ is

$$\langle 0|Y_B(0)|B\rangle\delta(\mathbf{p}-\mathbf{k})=\delta(\mathbf{p}-\mathbf{k})\,.\tag{48}$$

The functions $g_{\mathbf{p}}(\mathbf{k},\mathbf{q})$, $f_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{q})$, and $g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k})$ have no pure δ part. The δ part of $g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k},\mathbf{q})$ is

$$\delta(\mathbf{k}-\mathbf{p}_1)\delta(\mathbf{q}-\mathbf{p}_2)+\delta(\mathbf{k}-\mathbf{p}_2)\delta(\mathbf{q}-\mathbf{p}_1)\,.\tag{49}$$

Now we can solve the first two of Eqs. (42):

$$T(\mathbf{k},\mathbf{q},\lambda)=\tfrac{1}{2}[X(\mathbf{k},\mathbf{q})+X(\mathbf{q},\mathbf{k})]\delta(\lambda-\xi(\mathbf{k})-\xi(\mathbf{q}))+\frac{\mu(\mathbf{k})}{\lambda-\xi(\mathbf{q})-\xi(\mathbf{k})+i0}U(\mathbf{q},\lambda)\,,$$
$$T(\mathbf{k},\lambda)=X(\mathbf{k})\delta(\lambda-\xi(\mathbf{k})+B)+\frac{\mu(\mathbf{k})}{\lambda+B-\xi(\mathbf{k})+i0}U(\lambda)\,.\tag{50}$$

Substitution into the equation for $U(\mathbf{q},\lambda)$ gives

$$(\lambda-\xi(\mathbf{q})-\Delta-I^+[\lambda-\xi(\mathbf{q})])U(\mathbf{q},\lambda)=b(\mathbf{q},\lambda)-2\beta(\mathbf{q})[a(\lambda)+Z^{1/2}c(\lambda)+A(\lambda)+Z^{1/2}I^+(\lambda+B)U(\lambda)],\tag{51}$$

$$a(\lambda)=\tfrac{1}{2}\int\mu^*(\mathbf{k})\beta^*(\mathbf{p})[X(\mathbf{k},\mathbf{p})+X(\mathbf{p},\mathbf{k})]\times\delta(\lambda-\xi(\mathbf{k})-\xi(\mathbf{p}))d\mathbf{k}d\mathbf{p}\,,$$
$$b(\mathbf{q},\lambda)=\tfrac{1}{2}\int\mu^*(\mathbf{k})[X(\mathbf{k},\mathbf{q})+X(\mathbf{q},\mathbf{k})]\times\delta(\lambda-\xi(\mathbf{k})-\xi(\mathbf{q}))d\mathbf{k}\,,\tag{52}$$
$$c(\lambda)=\int\mu^*(\mathbf{k})X(\mathbf{k})\delta(\lambda+B-\xi(\mathbf{k}))d\mathbf{k}\,,$$
$$A(\lambda)=\int\beta^*(\mathbf{p})I^+(\lambda-\xi(\mathbf{p}))U(\mathbf{p},\lambda)d\mathbf{p}\,.$$

The function multiplying U on the left side is $D^+(\lambda-\xi(\mathbf{q}))$, and it has a zero at $\lambda=\xi(\mathbf{q})-\mathrm{B}$. Therefore, the solution of (50) is [using (47)]

$$U(\mathbf{q},\lambda)=Z^{1/2}X(\mathbf{q})\delta(\lambda+B-\xi(\mathbf{q}))+\frac{1}{D^{+}(\lambda-\xi(\mathbf{q}))}\{b(\mathbf{q},\lambda)-2\beta(\mathbf{q})[a(\lambda)+Z^{1/2}c(\lambda)+A(\lambda)+Z^{1/2}I^{+}(\lambda+B)U(\lambda)]\}. \tag{53}$$

Substitution into $A(\lambda)$ gives

$$[1-J^{+}(\lambda)]A(\lambda)=d(\lambda)+Z^{1/2}f(\lambda)+J^{+}(\lambda)\times[a(\lambda)+Z^{1/2}c(\lambda)+Z^{1/2}I^{+}(\lambda+B)U(\lambda)], \tag{54}$$

where

$$d(\lambda)=\int\frac{\beta^{*}(\mathbf{p})I^{+}(\lambda-\xi(\mathbf{p}))}{D^{+}(\lambda-\xi(\mathbf{p}))}b(\mathbf{p},\lambda)d\mathbf{p},$$

$$f(\lambda)=\int\beta^{*}(\mathbf{p})I^{+}(\lambda-\xi(\mathbf{p}))X(\mathbf{p})\delta(\lambda+B-\xi(\mathbf{p}))d\mathbf{p}=I(-B)\int\beta^{*}(\mathbf{p})X(\mathbf{p})\delta(\lambda+B-\xi(\mathbf{p}))d\mathbf{p}, \tag{55}$$

$$J(z)=-2\int\frac{|\beta(\mathbf{p})|^{2}I(z-\xi(\mathbf{p}))}{D(z-\xi(\mathbf{p}))}d\mathbf{p}.$$

Now the equation for $U(\lambda)$ becomes

$$\{[1-J^{+}(\lambda)]D^{+}(\lambda+B)+2ZI^{+}(\lambda+B)\}U(\lambda)=[1-2Z-J^{+}(\lambda)]c(\lambda)-2Z^{1/2}[a(\lambda)+d(\lambda)+Z^{1/2}f(\lambda)]. \tag{56}$$

First we consider $J(z)$, which can be written

$$J(z)=-2\int\frac{\rho(\epsilon)I(z-\epsilon)}{D^{+}(\epsilon)D^{-}(\epsilon)D(z-\epsilon)}d\epsilon=\frac{1}{\pi i}\int_{C}\frac{I(z-\epsilon)}{D(\epsilon)D(z-\epsilon)}d\epsilon=\frac{1}{\pi i}\int_{C}\frac{z-\epsilon-\Delta}{D(\epsilon)D(z-\epsilon)}d\epsilon-\frac{1}{\pi i}\int_{C}\frac{d\epsilon}{D(\epsilon)}, \tag{57}$$

where C is the contour in Fig. 1. The second integral is easily seen to give

$$\int_{C}\frac{d\epsilon}{D(\epsilon)}=\frac{2\pi i}{D'(-B)}-\int_{C_{\infty}}\frac{d\epsilon}{\epsilon}=2\pi i(Z-1), \tag{58}$$

where C_{∞} is the circle at infinity. On the other hand, we have

$$\int_{C}\frac{\epsilon d\epsilon}{D(\epsilon)D(z-\epsilon)}=\left[\int_{C_{-B}}+\int_{C_{z+B}}-\int_{C_{\infty}}-\int_{C'}\right]\times\frac{\epsilon d\epsilon}{D(\epsilon)D(z-\epsilon)}, \tag{59}$$

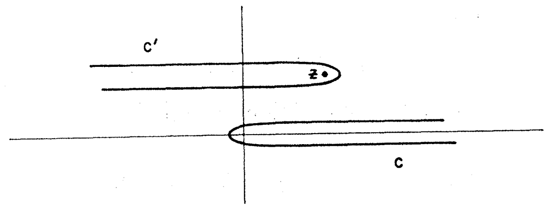

FIG. 1. Contours used in Eqs. (57)–(62).

where C_{-B} and C_{z+B} are infinitesimal counterclockwise contours around $\epsilon=-B$ and $\epsilon=z+B$, respectively, and C' is shown in Fig. 1. Then

$$\begin{aligned}\int_{C_{-B}}\frac{\epsilon d\epsilon}{D(\epsilon)D(z-\epsilon)}&=-\frac{2\pi iBZ}{D(z+B)},\\ \int_{C_{z+B}}\frac{\epsilon d\epsilon}{D(\epsilon)D(z-\epsilon)}&=-\frac{2\pi iZ(z+B)}{D(z+B)},\\ \int_{C_{\infty}}\frac{\epsilon d\epsilon}{D(\epsilon)D(z-\epsilon)}&=-\int_{C_{\infty}}\frac{d\epsilon}{\epsilon}=-2\pi i,\\ \int_{C'}\frac{\epsilon d\epsilon}{D(\epsilon)D(z-\epsilon)}&=-\int_{C}\frac{(z-\nu)d\nu}{D(z-\nu)D(\nu)}.\end{aligned} \tag{60}$$

Therefore,

$$\int_{C}\frac{\epsilon d\epsilon}{D(\epsilon)D(z-\epsilon)}=\pi i\left[1-\frac{Z(z+2B)}{D(z+B)}-\frac{z}{2}K(z)\right], \tag{61}$$

where

$$K(z)=-\frac{1}{2\pi i}\int_{C}\frac{\epsilon d\epsilon}{D(z-\epsilon)D(\epsilon)}=\int_{0}^{\infty}\frac{\rho(\epsilon)d\epsilon}{D^{+}(\epsilon)D^{-}(\epsilon)D(z-\epsilon)}=\int\frac{|\beta(\mathbf{q})|^{2}}{D(z-\xi(\mathbf{q}))}d\mathbf{q}. \tag{62}$$

Then

$$J(z)=1-2Z+Z(z+2B)/D(z+B)-(z-2\Delta)K(z) \tag{63}$$

and

$$[1-J^{+}(\lambda)]D^{+}(\lambda+B)+2ZI^{+}(\lambda+B)=(\lambda-2\Delta)[Z+K^{+}(\lambda)D^{+}(\lambda+B)]. \tag{64}$$

It is now simple algebra to see that

$$\begin{aligned}a(\lambda)+d(\lambda)&=(\lambda-2\Delta)\tfrac{1}{2}\int\beta^{*}(\mathbf{k})\beta^{*}(\mathbf{p})X(\mathbf{k},\mathbf{p})\times\delta(\lambda-\xi(\mathbf{k})-\xi(\mathbf{p}))d\mathbf{k}d\mathbf{p},\\ [1-2Z-J^{+}(\lambda)]c(\lambda)-2Zf(\lambda)&=(\lambda-2\Delta)\int\beta^{*}(\mathbf{k})\times[K^{+}(\lambda)D^{+}(\lambda+B)-Z]X(\mathbf{k})\delta(\lambda+B-\xi(\mathbf{k}))d\mathbf{k},\end{aligned} \tag{65}$$

Since the point $\lambda=2\Delta$ has nothing to do with anything, we can write (56) as

$$F^+(\lambda)U(\lambda)=\int\beta^*(\mathbf{k})[F^+(\xi(\mathbf{k})-B)-2Z]X(\mathbf{k})\times\delta(\lambda+B-\xi(\mathbf{k}))d\mathbf{k}-Z^{1/2}\int\beta^*(\mathbf{k})\beta^*(\mathbf{p})X(\mathbf{k},\mathbf{p})\times\delta(\lambda-\xi(\mathbf{k})-\xi(\mathbf{p}))d\mathbf{k}d\mathbf{p}\,,\quad(66)$$

$$F(z)=K(z)D(z+B)+Z\,.\quad(67)$$

The condition for a discrete state in the V-θ sector is that $F(z)$ have a zero for z real and less than $-B$. As $|z|\rightarrow\infty$, $K(z)\rightarrow(1-Z)/z$, so that $F(z)\rightarrow 1$. Since $K(z)$ is negative for z real and less than $-B$ and $D(z+B)$ is negative for z real and less than $-2B$, it follows that C is less than $2B$. At $z=-B$, $F(z)=Z+D(0)K(-B)$. It is easy to see that $K(-B)$ is negative, so the condition for a bound state is

$$|K(-B)|>Z/D(0)\,.\quad(68)$$

We will assume that there is a bound state at $z=-C$, $2B>C>B$. Then we see that

$$f_{C,B}=n_C\,,$$
$$f_{\mathbf{p},B}=\beta^*(\mathbf{p})\frac{F^+(\xi(\mathbf{p})-B)-2Z}{F^+(\xi(\mathbf{p})-B)}\,,\quad(69)$$
$$f_{\mathbf{p}_1\mathbf{p}_2,B}=-2Z^{1/2}\frac{\beta^*(\mathbf{p}_1)\beta^*(\mathbf{p}_2)}{F^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2))}\,,$$

where n_C is a normalization factor to be determined later.

Next we can solve for $U(\mathbf{q},\lambda)$, with the results

$$f_C(\mathbf{q})=\frac{\beta(\mathbf{q})D(B-C)}{Z^{1/2}D(-C-\xi(\mathbf{q}))}n_C\,,$$

$$f_{\mathbf{p}}(\mathbf{q})=Z^{1/2}\delta(\mathbf{p}-\mathbf{q})-\frac{2\beta(\mathbf{q})}{D^+(\xi(\mathbf{p})-B-\xi(\mathbf{q}))}\frac{Z^{1/2}\mu^*(\mathbf{p})}{F^+(\xi(\mathbf{p})-B)}\,,\quad(70)$$

$$f_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{q})=\beta^*(\mathbf{p}_1)\delta(\mathbf{p}_2-\mathbf{q})+\beta^*(\mathbf{p}_2)\delta(\mathbf{p}_1-\mathbf{q})-2\frac{D^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)+B)}{D^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)-\xi(\mathbf{q}))}\frac{\beta(\mathbf{q})\beta^*(\mathbf{p}_1)\beta^*(\mathbf{p}_2)}{F^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2))}\,.$$

Finally,

$$g_C(\mathbf{k})=\frac{\mu(\mathbf{k})}{B-C-\xi(\mathbf{k})}n_C\,,$$

$$g_{\mathbf{p}}(\mathbf{k})=\delta(\mathbf{p}-\mathbf{k})+\frac{\mu(\mathbf{k})\beta^*(\mathbf{p})}{\xi(\mathbf{p})-\xi(\mathbf{k})+i0}\frac{F^+(\xi(\mathbf{p})-B)-2Z}{F^+(\xi(\mathbf{p})-B)}\,,$$

$$g_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{k})=-2Z^{1/2}\frac{\mu(\mathbf{k})\beta^*(\mathbf{p}_1)\beta^*(\mathbf{p}_2)}{[\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)+B-\xi(\mathbf{k})+i0]F^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2))}\,,$$

$$g_C(\mathbf{k},\mathbf{q})=-\frac{\mu(\mathbf{k})\beta(\mathbf{q})D(B-C)}{Z^{1/2}[C+\xi(\mathbf{k})+\xi(\mathbf{q})]D(-C-\xi(\mathbf{q}))}n_C\,,\quad(71)$$

$$g_{\mathbf{p}}(\mathbf{k},\mathbf{q})=-\frac{Z^{1/2}\mu(\mathbf{k})\delta(\mathbf{p}-\mathbf{q})}{\xi(\mathbf{k})+B}-\frac{2Z^{1/2}\mu(\mathbf{k})\mu^*(\mathbf{p})\beta(\mathbf{q})}{[\xi(\mathbf{p})-\xi(\mathbf{k})-\xi(\mathbf{q})-B+i0]D^+(\xi(\mathbf{p})-B-\xi(\mathbf{q}))F^+(\xi(\mathbf{p})-B)}\,,$$

$$g_{\mathbf{p}_1\mathbf{p}_2}(\mathbf{k},\mathbf{q})=\delta(\mathbf{p}_1-\mathbf{k})\delta(\mathbf{p}_2-\mathbf{q})+\delta(\mathbf{p}_1-\mathbf{q})\delta(\mathbf{p}_2-\mathbf{k})+\frac{\mu(\mathbf{k})\beta^*(\mathbf{p}_1)}{\xi(\mathbf{p}_1)-\xi(\mathbf{k})+i0}\delta(\mathbf{p}_2-\mathbf{q})$$
$$+\frac{\mu(\mathbf{k})\beta^*(\mathbf{p}_2)}{\xi(\mathbf{p}_2)-\xi(\mathbf{k})+i0}\delta(\mathbf{p}_1-\mathbf{q})-2\frac{\mu(\mathbf{k})\beta(\mathbf{q})\beta^*(\mathbf{p}_1)\beta^*(\mathbf{p}_2)D^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)+B)}{[\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)-\xi(\mathbf{k})-\xi(\mathbf{q})+i0]D^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)-\xi(\mathbf{q}))F^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2))}\,.$$

In the V-θ sector, the normalization condition takes the form

$$\langle\beta|[V^\dagger V+\int a^\dagger(\mathbf{k})a(\mathbf{k})d\mathbf{k}]|\alpha\rangle=2\delta_{\alpha\beta}=\langle\beta|[V^\dagger NN^\dagger V+\int a^\dagger(\mathbf{k})a(\mathbf{k})d\mathbf{k}]|\alpha\rangle$$

$$=\langle\beta|\left[V^\dagger N\left(Y_B^\dagger Y_B+\int Y^\dagger(\mathbf{q})Y(\mathbf{q})d\mathbf{q}\right)N^\dagger V+\int a^\dagger(\mathbf{k})\left(Y_B^\dagger Y_B+\int Y^\dagger(\mathbf{q})Y(\mathbf{q})d\mathbf{q}\right)a(\mathbf{k})d\mathbf{k}\right]|\alpha\rangle$$

$$=f_{\beta,B}{}^*f_{\alpha,B}+\int f_\beta{}^*(\mathbf{q})f_\alpha(\mathbf{q})d\mathbf{q}+\int g_\beta{}^*(\mathbf{k})g_\alpha(\mathbf{k})d\mathbf{k}+\int g_\beta{}^*(\mathbf{k},\mathbf{q})g_\alpha(\mathbf{k},\mathbf{q})d\mathbf{k}d\mathbf{q}. \tag{72}$$

We have chosen

$$\delta_{\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_1,\mathbf{p}_2}=\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2')+\delta(\mathbf{p}_1-\mathbf{p}_2')\delta(\mathbf{p}_1'-\mathbf{p}_2). \tag{73}$$

Substitution gives

$$|n_C|^2=2K(-C)/F'(-C). \tag{74}$$

The other orthonormality relations hold with the f and g functions given above.

Now Eqs. (44) can be inverted to give

$$X_\alpha(t)\doteq e^{-iE_\alpha t}X_\alpha\doteq\tfrac{1}{2}\Big[f_{\alpha,B}{}^*Y_B(t)N^\dagger(t)V(t)+\int f_\alpha{}^*(\mathbf{q})Y(\mathbf{q},t)N^\dagger(t)V(t)d\mathbf{q}$$
$$+\int g_\alpha{}^*(\mathbf{k})Y_B(t)a(\mathbf{k},t)d\mathbf{k}+\int g_\alpha{}^*(\mathbf{k},\mathbf{q})Y(\mathbf{q},t)a(\mathbf{k},t)d\mathbf{k}d\mathbf{q}\Big]. \tag{75}$$

SCATTERING MATRIX ELEMENTS

We obtain S-matrix elements by using the equation for $a(\mathbf{k},t)$ that follows from (6):

$$a(\mathbf{k},t)=a^{\mathrm{in}}(\mathbf{k},t)-i\mu(\mathbf{k})\int_{-\infty}^{\infty}e^{-i\omega(\mathbf{k})(t-t')}\theta(t-t')N^\dagger(t')V(t')dt'. \tag{76}$$

Then

$$\langle B\mathbf{p}^{\mathrm{out}}|\alpha^{\mathrm{in}}\rangle=\langle B|a^{\mathrm{out}}(\mathbf{p},0)|\alpha^{\mathrm{in}}\rangle=\lim_{t\to\infty}e^{i\omega(\mathbf{p})t}\langle B|a(\mathbf{p},t)|\alpha^{\mathrm{in}}\rangle$$
$$=\langle B|a^{\mathrm{in}}(\mathbf{p},0)|\alpha^{\mathrm{in}}\rangle-i\mu(\mathbf{k})\int e^{i\omega(\mathbf{p})t'}\langle B|N^\dagger(t')V(t')|\alpha^{\mathrm{in}}\rangle dt'$$
$$=\langle B|a^{\mathrm{in}}(\mathbf{p},0)|A^{\mathrm{in}}\rangle-2\pi i\delta(E_B+\omega(\mathbf{p})-E_\alpha)\mu(\mathbf{p})\langle B|N^\dagger(0)V(0)|\alpha^{\mathrm{in}}\rangle$$
$$=\langle B|a^{\mathrm{in}}(\mathbf{p},0)|\alpha^{\mathrm{in}}\rangle-2\pi i\delta(E_B+\omega(\mathbf{p})-E_\alpha)\mu(\mathbf{p})f_{\alpha,B}, \tag{77}$$

$$\langle N\mathbf{p}_1\mathbf{p}_2{}^{\mathrm{out}}|\alpha^{\mathrm{in}}\rangle=\langle N\mathbf{p}_2{}^{\mathrm{out}}|a^{\mathrm{out}}(\mathbf{p}_1,0)|\alpha^{\mathrm{in}}\rangle$$
$$=\langle N\mathbf{p}_2{}^{\mathrm{out}}|a^{\mathrm{in}}(\mathbf{p}_1,0)|\alpha^{\mathrm{in}}\rangle-2\pi i\delta(E_N+\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)-E_\alpha)\mu(\mathbf{p}_1)\langle N\mathbf{p}_2{}^{\mathrm{out}}|N^\dagger(0)V(0)|\alpha^{\mathrm{in}}\rangle$$
$$=\langle N\mathbf{p}_2{}^{\mathrm{out}}|a^{\mathrm{in}}(\mathbf{p}_1,0)|\alpha^{\mathrm{in}}\rangle-2\pi i\delta(E_N+\omega(\mathbf{p}_1)+\omega(\mathbf{p}_2)-E_\alpha)\mu(\mathbf{p}_1)\int d\mathbf{k}\langle N\mathbf{p}_2{}^{\mathrm{out}}|N\mathbf{k}^{\mathrm{in}}\rangle f_\alpha(\mathbf{k}). \tag{78}$$

These give

$$\langle B\mathbf{p}^{\mathrm{out}}|B\mathbf{q}^{\mathrm{in}}\rangle=\delta(\mathbf{p}-\mathbf{q})-2\pi i\delta(\xi(\mathbf{p})-\xi(\mathbf{q}))\mu(\mathbf{p})f_{\mathbf{q},B},$$
$$\langle B\mathbf{p}^{\mathrm{out}}|N\mathbf{q}_1\mathbf{q}_1{}^{\mathrm{in}}\rangle=-2\pi i\delta(\xi(\mathbf{p})-B-\xi(\mathbf{q}_1)-\xi(\mathbf{q}_2))\mu(\mathbf{p})f_{\mathbf{q}_1\mathbf{q}_2,B},$$
$$\langle N\mathbf{p}_1\mathbf{p}_2{}^{\mathrm{out}}|B\mathbf{q}^{\mathrm{in}}\rangle=-2\pi i\delta(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)+B-\xi(\mathbf{q}))\mu(\mathbf{p}_1)\int d\mathbf{k}\langle N\mathbf{p}_2{}^{\mathrm{out}}|N\mathbf{k}^{\mathrm{in}}\rangle f_{\mathbf{q}}(\mathbf{k}), \tag{79}$$

$$\langle N\mathbf{p}_1\mathbf{p}_2{}^{\mathrm{out}}|N\mathbf{q}_1\mathbf{q}_2{}^{\mathrm{in}}\rangle=\langle N\mathbf{p}_2{}^{\mathrm{out}}|N\mathbf{q}_1{}^{\mathrm{in}}\rangle\delta(\mathbf{p}_1-\mathbf{q}_2)+\langle N\mathbf{p}_2{}^{\mathrm{out}}|N\mathbf{q}_2{}^{\mathrm{in}}\rangle\delta(\mathbf{p}_1-\mathbf{q}_1)$$
$$-2\pi i\delta(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)-\xi(\mathbf{q}_1)-\xi(\mathbf{q}_2))\mu(\mathbf{p}_1)\int d\mathbf{k}\langle N\mathbf{p}_2{}^{\mathrm{out}}|N\mathbf{k}^{\mathrm{in}}\rangle f_{\mathbf{q}_1\mathbf{q}_2}(\mathbf{k}).$$

If we write

$$\langle\alpha^{\text{out}}|\beta^{\text{in}}\rangle=\langle\alpha^{\text{in}}|\beta^{\text{in}}\rangle-2\pi i\delta(E_\alpha-E_\beta)\langle\alpha|T|\beta\rangle, \tag{80}$$

then we have

$$\langle B\mathbf{p}|T|B\mathbf{q}\rangle=\mu(\mathbf{p})\beta^*(\mathbf{q})\left(1-\frac{2Z}{F^+(\xi(\mathbf{p})-B)}\right)=\frac{\mu(\mathbf{p})\mu^*(\mathbf{q})}{D^+(\xi(\mathbf{q}))}\left(1-\frac{2Z}{F^+(\xi(\mathbf{p})-B)}\right),$$

$$\langle B\mathbf{p}|T|N\mathbf{q}_1\mathbf{q}_2\rangle=-2Z^{1/2}\frac{\mu(\mathbf{p})\mu^*(\mathbf{q}_1)\mu^*(\mathbf{q}_2)}{D^+(\xi(\mathbf{q}_1))D^+(\xi(\mathbf{q}_2))F^+(\xi(\mathbf{q}_1)+\xi(\mathbf{q}_2))},$$

$$\langle N\mathbf{p}_1\mathbf{p}_2|T|B\mathbf{q}\rangle=-2Z^{1/2}\frac{\mu(\mathbf{p}_1)\mu(\mathbf{p}_2)\mu^*(\mathbf{q})}{D^+(\xi(\mathbf{p}_1))D^+(\xi(\mathbf{p}_2))F^+(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2))}, \tag{81}$$

$$\langle N\mathbf{p}_1\mathbf{p}_2{}^{\text{out}}|N\mathbf{q}_1\mathbf{q}_2{}^{\text{in}}\rangle=\langle N\mathbf{p}_1{}^{\text{out}}|N\mathbf{q}_1{}^{\text{in}}\rangle\langle N\mathbf{p}_2{}^{\text{out}}|N\mathbf{q}_2{}^{\text{in}}\rangle+\langle N\mathbf{p}_1{}^{\text{out}}|N\mathbf{q}_2{}^{\text{in}}\rangle\langle N\mathbf{p}_2{}^{\text{out}}|N\mathbf{q}_1{}^{\text{in}}\rangle$$
$$-2\pi i\delta(\xi(\mathbf{p}_1)+\xi(\mathbf{p}_2)-\xi(\mathbf{q}_1)-\xi(\mathbf{q}_2))\langle N\mathbf{p}_1\mathbf{p}_2|U|N\mathbf{q}_1\mathbf{q}_2\rangle,$$

$$\langle N\mathbf{p}_1\mathbf{p}_2|U|N\mathbf{q}_1\mathbf{q}_2\rangle=-2\frac{\mu(\mathbf{p}_1)\mu(\mathbf{p}_2)\mu^*(\mathbf{q}_1)\mu^*(\mathbf{q}_2)}{D^+(\xi(\mathbf{p}_1))D^+(\xi(\mathbf{p}_2))D^+(\xi(\mathbf{q}_1))D^+(\xi(\mathbf{q}_2))}\frac{D^+(\xi(\mathbf{q}_1)+\xi(\mathbf{q}_2)+B)}{F^+(\xi(\mathbf{q}_1)+\xi(\mathbf{q}_2))}.$$

These are like the S-matrix elements given by Maxon[3] when the correspondences between Maxon's A and G functions and the function K and D of the present work are noted:

$$A(z)=-K(z-B+\omega(0))/Z^2,\qquad G(z)=ZD(z-B+\omega(0)), \tag{82}$$

although here we have not assumed anything about the function $\mu(\mathbf{k})$.

By considering the analytic properties of $1/D(z)$ it is easy to see that

$$\frac{1}{D(z)}=\frac{Z}{z+B}+\int\frac{|\beta(\mathbf{q})|^2}{z-\xi(\mathbf{q})}d\mathbf{q}, \tag{83}$$

so that $K(z)$ can be written

$$K(z)=Z\int\frac{|\beta(\mathbf{q})|^2}{z+B-\xi(\mathbf{q})}d\mathbf{q}+\int\frac{|\beta(\mathbf{q})|^2|\beta(\mathbf{p})|^2}{z-\xi(\mathbf{q})-\xi(\mathbf{p})}d\mathbf{q}d\mathbf{p}. \tag{84}$$

Therefore, the function $K(z)-Z/D(z+B)$ has a pole at $z=-2B$ and a cut from $z=0$ to $z=\infty$; it follows that it is real from $z=-B$ to $z=0$, that is, in the region in which only the $B-\theta$ channel is open. Moreover, if we write

$$F(z)=D(z+B)[K(z)-Z/D(z+B)]+2Z, \tag{85}$$

then it is clear that the discontinuity of F across its cut for $-B<\omega<0$ is

$$F^+(\omega)-F^-(\omega)=[K(z)-Z/D(z+B)]\times[D^+(\omega+B)-D^-(\omega+B)]=2\pi i[K(z)-Z/D(z+B)]\rho(\omega+B). \tag{86}$$

Since we can write

$$\langle B\mathbf{p}|T|B\mathbf{q}\rangle=\mu(\mathbf{p})\mu^*(\mathbf{q})[K(\xi(\mathbf{p})-B)-Z/D(\xi(\mathbf{p}))]/F^+(\xi(\mathbf{p})-B), \tag{87}$$

it follows in the usual way[4] that if μ is spherically symmetric, then the S-wave phase shift below the inelastic threshold is just the negative of the phase of F^+.

RENORMALIZATION AND OTHER SECTORS

For any of the usual choices of $\mu(k)$ and $\omega(k)$, only the function $D(z)$ requires renormalization. If, on the other hand, the renormalized coupling constant is such as to produce ghosts, this will be reflected in the V-θ sector also.

It seems at least possible that a technique similar to the one presented here could lead to a solution in the V-2θ sector. The idea would be to work with the operators $X_\alpha a(\mathbf{k})$ and $X_\alpha N^\dagger V$ and use the equations corresponding to (39). Further investigations along these lines are being undertaken.

[4] See, e.g., M. Bolsterli and J. MacKenzie, Physics **2**, 141 (1965).