Construction of Solutions to Superconvergence Relations*

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Superconvergence relations for $t \leqq 0$ are studied in the one-tower approximation, and a simple method for their solution is introduced. In general, we find that, given a mass spectrum and one solution for the couplings, an infinite number of other solutions can be constructed. In simple cases, we also construct the first solution.

THE saturation of nonforward superconvergence relations[1] with an infinite set of single-particle states[2,3] has recently become a popular pursuit. The problem divides itself naturally into two parts: the saturation with (a) a finite number of external particles, and (b) the whole tower as external particles. The latter problem is clearly much more difficult than the former, and we shall have only some indirect comments to make about it. Our purpose in this paper is to note a rather simple procedure for construction of solutions with a finite number of external particles. In general, we learn that, given a mass spectrum and one solution for the couplings, one can construct an infinite number of other solutions.[4] In simple cases, we can also construct the first solution. Unless some further physical principle can be invoked to distinguish between the solutions,[5] it appears that the nonforward superconvergence relations are dangerously close to being empty.[6,7]

Our order of presentation is as follows. First we study an example of a superconvergence relation derived from large isospin and/or strangeness in the t channel. Here, given any mass spectrum μ_j^2 bounded by j^2 for large j, we can construct an infinite number of absolutely convergent solutions for the couplings. Then we go on to consider the helicity-flip superconvergence relation recently discussed in some detail by Klein.[3] Here things are more difficult, but we can show that if one solution exists, then one can construct an infinity of other, different, solutions. At least this weakened result appears true in general for helicity-flip relations. Finally, we mention the application of our method to the saturation of an infinite number of superconvergence relations for form factors, and to the saturation of current algebra sum rules.

Our first case is the superconvergence relation in the K-$\bar{K}$ channel, due to $I=0$, strangeness 2 in the cross (K-K) channel. By charge-conjugation invariance, only isospin-1 (0) resonances can appear in the even (odd) partial waves. Moreover, from the isospin crossing matrix, the $I=0$ resonances appear with an extra minus sign. Putting one (stable) particle in each partial wave, and using crossing, the superconvergence relation takes the form[8]

$$\sum_{j=0}^{\infty} P_j\left(1+\frac{2t}{\mu_j^2-4}\right)g_j=0\,, \tag{1}$$

where the couplings g_j are constrained to be positive (negative) for j even (odd), and μ_j^2 is the mass spectrum. By expanding the left side of this equation in powers of t, and setting the coefficient of each power to zero, we obtain the upper triangular array

$$\sum_{j=n}^{\infty} a_{n,j}g_j=0, \quad n=0,1\cdots, \tag{2}$$

where

$$a_{n,j}=\frac{1}{(\mu_j^2-4)^n}\frac{(j+n)!}{(j-n)!}, \quad n\leqq j \tag{3}$$

is positive definite. The question is whether solutions of this system exist.

We could proceed with a naive construction as follows. We define a sequence of stages. The zeroth stage is to pick g_0, say, equal to unity. The first stage is defined by

$$a_{0,0}+a_{0,1}g_1=0\,, \tag{4}$$

thus determining g_1. In the next stage, we add the $n=1$

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[1] S. Fubini, Nuovo Cimento **43**, 475 (1966); S. Fubini and G. Segrè, *ibid.* **65**, 641 (1966); V. De Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966).

[2] S. Fubini, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967* (W. H. Freeman and Co., San Francisco, 1967).

[3] S. Klein, Phys. Rev. Letters **18**, 1074 (1967).

[4] The basic reason for this is that the resulting infinite array of equations for an infinite number of unknowns can be viewed without harm as a rectangular matrix equation, with many more unknowns than equations.

[5] It has recently been noted by I. T. Grodsky [Phys. Letters **25**, 149 (1967)] that a single-tower saturation of a superconvergence relation can easily lead to anomalous singularities in t. Probably any one-tower solution will have this difficulty, as one is not attempting to maintain crossing. Part of this can be fixed up by adding (lower) daughter and/or conspirator trajectories to the game; e.g., see K. Bardakci and G. Segrè, Berkeley Report (unpublished).

[6] In particular, one seems to lose the information gained from the $t=0$ superconvergence relation with the usual techniques. In fact any subset of couplings that one chooses can be specified arbitrarily, as long as there remains an infinite subset to be determined.

[7] In a way this may be good, because without this circumstance, one could not hope to push on toward solving the harder problem of the whole tower as external particles. That is to say, one may hope that by putting a larger infinity of constraints on the couplings and masses, the multiplicity of solutions may be reduced.

[8] Note that, in the degenerate-mass case, these equations have only the trivial solution.

equation, and, say, two more couplings g_2, g_3, while demanding that the solution of the first stage is not altered:

$$a_{0,2}g_2+a_{0,3}g_3=0\,,\qquad a_{1,2}g_2+a_{1,3}g_3=-a_{1,1}g_1\,. \tag{5}$$

Having two equations in two unknowns, we can determine g_2 and g_3. In a similar way, the third stage (with three equations) may be satisfied leaving $g_0\cdots g_3$ unchanged by introducing couplings up to g_6, and so on in each stage. Clearly, at each stage, we could add many more couplings, most of which could be specified arbitrarily; the degree of arbitrariness in such a procedure is vast.

There are two difficulties inherent in the above scheme. The first is that the various infinite series in j may not converge. Secondly, the above construction does not yet guarantee the correct signs of the couplings.

The first problem has been solved by Pólya,[9] who found a sufficient condition that the infiite sums be absolutely convergent: There are an infinite number of (linearly independent) absolutely convergent solutions to infinite matrix equations of the form of Eq. (2), if

$$\lim_{j\to\infty}\frac{|a_{0,j}|+|a_{1,j}|+\cdots+|a_{n-1,j}|}{|a_{n,j}|}=0,\quad n=1,2\cdots. \tag{6}$$

With a mass spectrum going asymptotically as $\mu_j^2\sim j^\alpha$, the conditions of Pólya's theorem are met for $\alpha<2$.[10] Both Refs. 3 and 11 point out that these mass spectra are probably the only onces that can saturate these relations, so we will limit our discussion to these cases.[12]

We will illustrate Pólya's theorem by showing, e.g., how the third stage is satisfied, assuming $g_0\cdots g_3$ have already been determined. Instead of introducing three new g_j's ($j=4,5,6$), we introduce four, which we call g_{j_1}, g_{j_2}, g_{j_3}, and g_{j_4}, where $j_1<j_2<j_3<j_4$ and where all four numbers may in fact be large. All other g_j's ($3<j<j_4$) are set to zero. Then, taking account of the previous stages, and choosing

$$g_{j_4}=(-a_{2,2}g_2+a_{2,3}g_3)/a_{2,j_4}\,, \tag{7}$$

we obtain the equations of the third stage

$$\begin{aligned}
a_{0,j_1}g_{j_1}+a_{0,j_2}g_{j_2}+a_{0,j_3}g_{j_3}&=(a_{0,j_4}/a_{2,j_4})(a_{2,2}g_2+a_{2,3}g_3)\,,\\
a_{1,j_1}g_{j_1}+a_{1,j_2}g_{j_2}+a_{1,j_3}g_{j_3}&=(a_{1,j_4}/a_{2,j_4})(a_{2,2}g_2+a_{2,3}g_3)\,,\\
a_{2,j_1}g_{j_1}+a_{2,j_2}g_{j_2}+a_{2,j_3}g_{j_3}&=0\,.
\end{aligned} \tag{8}$$

According to the condition (6), we can now pick j_4 so large that the right-hand sides of Eq. (8) are as small as we wish. It follows then that the solutions $g_{j_1}g_{j_2}g_{j_3}$ are arbitrarily small,[13] and in particular

$$|a_{n,j_1}g_{j_1}|+|a_{n,j_2}g_{j_2}|+|a_{n,j_3}g_{j_3}|+|a_{n,j_4}g_{j_4}|$$

can be made arbitrarily small for $n=0$, 1 [since g_{j_4} is also arbitrarily small for large j_4—see Eq. (7)]. By an inductive procedure, Pólya shows that the sum of the moduli of all the terms added in all stages to the first and second equations can be made arbitrarily small—that is, the series converges absolutely. Of course, $a_{2,j_4}g_{j_4}$ is not small in general, but the additions to the third equation in the next stage will be small, and so on; in the end, all the sums will be absolutely convergent.[14]

We now turn to the second problem; namely, that of guaranteeing the correct signs of the g's. Continuing to illustrate with the third stage, we can guarantee the correct sign of g_{j_4} by choosing j_4 even (odd) if the right side of Eq. (7) is positive (negative); beyond that, j_4 is not specified. Suppose that j_1 is already much larger than $n=2$, so that the asymptotic form of the kernel ($j^{n(2-\alpha)}e^{-2n}$) is applicable. Then the solution of the system (8) may be written

$$g_{j_1}=(-g_{j_4})\begin{vmatrix}1&1&1\\ j_4^{2-\alpha}&j_2^{2-\alpha}&j_3^{2-\alpha}\\ 0&j_2^{2(2-\alpha)}&j_3^{2(2-\alpha)}\end{vmatrix}\Bigg/\begin{vmatrix}1&1&1\\ j_1^{2-\alpha}&j_2^{2-\alpha}&j_3^{2-\alpha}\\ j_1^{2(2-\alpha)}&j_2^{2(2-\alpha)}&j_3^{2(2-\alpha)}\end{vmatrix} \tag{9}$$

and similarly for g_{j_2}, g_{j_3}. The important point is that, if we choose $j_1\ll j_2\ll j_3\ll j_4$, then the sign of the denominator determinant is that of the product of the diagonal elements (positive). This may be seen by a columnwise expansion of the determinant, starting with $j_3^{2(2-\alpha)}$, its largest element. The same technique may be applied to the numerator determinant, where it is clear that the sign is negative, as the determinant is dominated by $(-j_4^{2-\alpha})$ multiplied by its (positive) minor. Hence g_{j_1} has the same sign as g_{j_4}, and we need only choose j_1 to be some even (odd) integer if j_4 was even (odd).[15] These considerations can be generalized to arbitrarily large-order determinants, as should be obvious. Beyond the various evenness and oddness

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[9] G. Pólya, Commentarii Math. Helvitici **11**, 234 (1938–9); R. G. Cooke, *Infinite Matrices and Sequence Spaces* (MacMillan and Co., Ltd., London, 1950).

[10] That is, if one imagines the saturating particles to lie on a Regge trajectory, then, asymptotically $\alpha(s)$ must increase more rapidly than $s^{1/2}$.

[11] I. T. Grodsky, M. Martinis, and M. Święcki, Phys. Rev. Letters **19**, 332 (1967).

[12] It would be very interesting to establish the conditions on the mass spectrum such that (a) no solution existed to the superconvergence relation, or (b) a finite number existed.

[13] Pólya shows that j_1, j_2, j_3 can always be picked so that Eq. (8) is not singular, if an infinite number of elements a_{0j}, $j=0,1\cdots$ are nonzero, as is our case.

[14] The "tails," where everything is smaller than some ϵ, will start higher in j for higher n.

[15] For g_{j_2}, the j_4 column is in the center, making the numerator determinant positive, so that we simply choose j_2 of opposite parity to j_4. j_3 will have the parity of j_4 again.

requirements, there clearly remains a great deal of arbitrariness in the construction.[16]

Now we turn our attention to superconvergence relations arising from helicity flip in the cross channel, e.g., the popular one in the π-ρ system with $I=1$ and helicity flip 2 in the t channel. Klein[3] has recently written down the relation, attempting to saturate with ω and A_2 towers and the pion, obtaining the system of equations

$$\sum_{j=n}^{\infty} \frac{a_j}{(\mu_j^2-4)^n} G_n(j) \frac{(j+n)!}{(j-n)!} \tag{10}$$

in which all the a_j's (couplings) are constrained to be positive. There is a chance of solution because $G_n(j)$ begins negative but has an n-dependent zero and then goes positive. The form of this equation is very similar to our Eq. (2), and indeed, for this equation, the conditions of Pólya's theorem are still satisfied for $\mu_j^2<j^2$. On the other hand, the unitarity constraints are quite different as, for any n, there are not an infinite number of negative terms. Thus our basic technique, which involves setting large blocks of each equation (far to the right) to zero, cannot help in a direct manner.[17] What we shall show, however, is that if one solution of Klein's equation exists, then there are infinitely many others. For this discussion, we redefine Klein's a's and kernel as our g's and a's.

Suppose we have a solution to Eq. (10) that we can examine, and, in particular, learn the behavior of g_j as $j\rightarrow\infty$. We can now construct an infinite number of solutions g_j', à la Pólya, in which $g_j'=0$ for all values of j for which the given solution vanishes, and for which[18] $\lim_{j\to\infty} g_j'/g_j=0$. For each of these it is possible to find a nonzero number κ such that $|\kappa g_j'|\leqq g_j$ for all j. Then since g_j and $\kappa g_j'$ satisfy the linear equations (10), it follows that $(g_j+\kappa g_j')$ are new solutions that satisfy the positivity requirement.

One might ask, in the general case, how many solutions exist with the physical requirement that all the couplings are nonzero (no gaps in the trajectory). By the reasoning of the previous paragraph, our answer is that, if there is one, there are an infinite number. Moreover, from a cursory examination of higher helicity-flip relations, it appears that at least this weakened multiplicity result holds in general for helicity-flip relations. On the basis of the cases discussed above, we feel it worth conjecturing that, in general, superconvergence relations with a finite number of external particles[19] have (given a mass spectrum) an infinite number of solutions for the couplings.

To conclude, we mention some other possible applications of our method. The first is to form factors. If electromagnetic form factors fall off faster than any power of t (the momentum transfer) one might hope that they satisfy an infinite set of higher superconvergence relations

$$\int_0^\infty dt'(t')^n \,\mathrm{Im}F(t')=0 \quad n=0,1,\cdots. \tag{11}$$

In the approximation of an infinite number of (say, spin 1) stable resonances, the equations can be written

$$\sum_{i=0}^{\infty}(\mu_i^2)^n g_i=0 \qquad n=0,1,\cdots \tag{12}$$

where there are no sign restrictions on the couplings g_i, and μ_i^2 is the mass spectrum. The conditions of Pólya's therem are satisfied in this case for *any* rising mass spectrum, so, given a mass spectrum, we can construct an infinite number of absolutely convergent solutions[20] to Eq. (12). The method is also applicable in principle to the tower-saturation of individual current algebra sum rules[11]—given the (form factor) inhomogeneities. Again because of the spin complications, we need one solution in order to construct an infinite number of solutions to these equations.[20a]

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[16] There are various other superconvergence relations, derived from large isospin in the cross channel, to which the method may be applied directly. For example, in π-π scattering with $I=2$ in the cross channel, a first moment superconvergence relation may be written

$$\int_0^\infty \nu' \,\mathrm{Im}A(\nu',t)d\nu'=0$$

The method works as well for these higher moment relations. In this case, one plays $I=0$, 1 resonances (with different signs from the crossing matrix) against one another.

[17] The reason for this difference is that, by putting in just these particles, Klein has used only the diagonal (positive) entries in the helicity crossing matrix. If one were to put in as well the whole pion tower, there would be, in particular, an additional infinity of negative terms, so that one might hope to extend our method to this case. However there will be certain obvious Schwarz-like inequalities, due to factorization, which would be difficult to incorporate.

[18] This is possible if the Regge trajectory does not increase more than linearly in s. Details will be given by D. Atkinson, University of Rome (unpublished).

[19] Our results can be extended to any finite number of external particles: In such a case, one has a finite number of other equations of the form of Eq. (2), each to be satisfied by the same set of couplings. This array can be written as a super-matrix equation of the same form as Eq. (2). The question of what happens when the entire tower is on the outside remains entirely open.

[20] It is not likely that any of the absolutely convergent solutions attain the Martin bound [A. Martin, Nuovo Cimento 37, 671 (1965)] though it is possible that some conditionally convergent ones do.

[20a] *Note added in proof.* The argument in the paragraph containing Eq. (11) is incorrect. It should be replaced by the following. Suppose we have a solution to Eq. (10) that we can examine, and, in particular, learn the behavior of g_j as $j\rightarrow\infty$. Now we can construct an infinite number of solutions g_j', à la Pólya, in which $g_j'=0$ for all values of j for which the given solution g_j vanishes, and for which $\lim_{j\to\infty} g_j'/g_j=0$. This turns out to be possible if the Regge trajectory does not increase faster than linearly; details will be given in D. Atkinson: [University of Rome Report (unpublished)]. For each of these it is possible to find a nonzero number κ such that $|\kappa g_j'|\leqq g_j$ for all j. Since g_j and $\kappa g_j'$ satisfy the linear equations (10), it follows that $g_j+\kappa g_j'$ are new solutions that satisfy the positivity requirement. In the general case, one might ask how many solutions exist with the physical requirement that all the couplings are nonzero (no gaps in the trajectory). By the same reasoning, our answer is that if there is one, there are an infinite number.