Hagen and Macfarlane,[15] $\tilde{S}_n$ can be given in terms of homogeneous product sums "h" appropriate to $SU(6)$:

$$\tilde{S}_n = h_{n+3}h_n^* - h_{n+2}h_{n-1}^*,$$

where[15]

$$h_n = \sum_{k=0}^{I(\frac{1}{2}n)} \tfrac{1}{2}(n-2k)(n-2k,\, k)$$

and

$$h_n^* = \sum_{k=0}^{I(\frac{1}{2}n)} \tfrac{1}{2}(n-2k)(k,\, n-2k),$$

$$I(x) = \text{integral part of } x.$$

Thus

$$\tilde{S}_n = \sum_{k=0}^{I(\frac{1}{2}(n+3))} \tfrac{1}{2}(n+3-2k)(n+3-2k,\, k)$$

$$\otimes \sum_{l=0}^{I(\frac{1}{2}n)} \tfrac{1}{2}(n-2l)(l,\, n-2l) - \sum_{k=0}^{I(\frac{1}{2}(n+2))} \tfrac{1}{2}(n+2-2k)$$

$$\times (n+2-2k,\, k) \otimes \sum_{l=0}^{I(\frac{1}{2}(n-1))} \tfrac{1}{2}(n-1-2l)(l,\, n-1-2l).$$

[15] C. R. Hagen and A. J. Macfarlane, J. Math. Phys. **6**, 1355 (1965).

The reduction of the $SU(2)$ part, e.g., $\frac{1}{2}(n+3-2k) \otimes \frac{1}{2}(n-2)$ causes no problem. The reduction of the $SU(3)$ part can be achieved by using the elegant formula of Coleman[16] which we quote below.

$$(n,m) \otimes (n',m') = \sum_{i=0}^{\min(n,m')} \sum_{j=0}^{\min(m,n')} (n-i, n'-j; m-j, m'-i),$$

where

$$(n,n';m,m') = (n+n',\, m+m') \oplus \sum_{i=1}^{\min(n,n')} (n+n'-2i,\, m+m'+i) + \sum_{j=1}^{\min(m,m')} (n+n'+i,\, m+m'-2i).$$

Proceeding in this fashion, one obtains $\tilde{\sigma}_n$ and $\tilde{\Sigma}_n$. After a straightforward but very lengthy and laborious calculation, one then obtains the expression $\tilde{R}_n$ as quoted.

[16] S. Coleman, in *Proceedings of the Seminar in High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965).

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Regge-Pole Exchange and Direct-Channel Resonances in Models for High-Energy Scattering Amplitudes*

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The behavior of the forward and backward πN scattering amplitudes for momenta of 1–5 BeV/c has been analyzed recently using models in which Breit-Wigner amplitudes describing direct-channel resonances are added to a background amplitude given by the Regge-pole-exchange model. Although remarkably successful in practice, the model has severe theoretical limitations, especially with regard to the treatment of the tails of the resonant terms, double counting of the background contributions, and the restriction to the Breit-Wigner approximation for sets of isolated resonances. The theory of Regge-pole–plus–resonance (RPR) models is examined in detail for both single-channel potential scattering and the many-channel relativistic case. A modified RPR model is developed in which (i) the double-counting problems are eliminated, and (ii) direct-channel resonances are described in terms of their Regge-trajectory functions. There is no difficulty with the tails of the resonant amplitudes in this formulation of the RPR model. Moreover, the contributions of the entire set of resonances on a given Regge trajectory can be included in the scattering amplitude. The relevance of these modifications of the RPR model to past analyses of πN scattering is discussed briefly.

I. INTRODUCTION

IT has become clear in the past year that a remarkably successful description of πN scattering for laboratory momenta of 1–5 BeV/c can be obtained by adding appropriate direct-channel resonance terms to the Regge-pole-exchange amplitudes deduced from fits to high-energy scattering cross sections. The rationale for such models is simple: the Regge-pole-exchange amplitude is used to represent the smooth average behavior of the complete amplitudes, while the resonant terms take into account the large deviations of a few partial-wave amplitudes from that average behavior. Models

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of this type have been used to fit the resonance fluctuations in backward $\pi^{\pm}p$ scattering cross sections,[1] the difference between π^+p and π^-p total cross sections,[2] and the forward[2] and nonforward[3,4] $\pi^-p \to \pi^0 n$ charge-exchange cross section. The interference of resonance contributions with the Reggeized ρ-meson exchange amplitude has also been advanced as a possible explanation of the polarization observed in the charge-exchange reaction at 5.9 and 11.2 BeV/c.[5-10] More recently, the model has been used to predict the polarization to be expected in the charge-exchange reaction at lower energies,[4,11] and in a tentative analysis of the polarization observed in elastic $\pi^{\pm}p$ scattering in the range of 1–3 BeV/c.[12]

Although notably successful from a phenomenological point of view, the Regge-pole–plus–resonance (RPR) model is not entirely satisfactory theoretically. First, the resonant amplitudes have generally been described by simple Breit-Wigner terms with constant widths and elasticities.[13] Such a description is valid only in the immediate neighborhood of the resonance energy. At lower energies, the resonant contribution may be sharply suppressed by angular momentum barrier factors, while at higher energies, the increasing competition from opening channels may lead to a rapid decrease in the elasticity parameter.[14] The smooth residual contribution of the resonant partial wave far from resonance is presumably given by the background amplitude, that is, the exchange amplitude in the RPR model. The tails of the Breit-Wigner resonance amplitudes may lead to a significant overestimate of the background amplitude. Difficulties of this type were noted by Barger and Cline[1] in their study of backward π^+p scattering, and by the author[10] in connection with the early fits of the π^-p charge-exchange polarization.[5-7]

A second objection to the RPR model as generally used concerns the treatment of the πN resonances as isolated or uncorrelated. The work of Barger and Cline[1] and Barger and Olsson[2] has demonstrated rather convincingly that the higher πN resonances are Regge recurrences of the lower states.[15] It is clearly desirable from a theoretical point of view to treat the contributions of these direct-channel Regge trajectories in their entirety. As we shall see, such a treatment eliminates most of the problems noted in the preceding paragraph. The fact that the parameters of all the resonances on a given trajectory can be expressed in terms of a single trajectory function $\alpha(s)$ and the Regge residue $\beta(s)$, both of which vary smoothly with s, has further practical advantages. First, it is probable that many fewer parameters are needed in the phenonenological analyses than have actually been used.[16] Second, since the smoothness conditions on the Regge trajectory and

[1] V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966); Phys. Rev. **155**, 1792 (1967).

[2] V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966).

[3] J. Baacke and M. Yvert, Nuovo Cimento **51A**, 761 (1967).

[4] D. Reeder and K. Sarma (private communication), and University of Wisconsin Report No. COO-881-149 (unpublished).

[5] R. J. N. Phillips, Nuovo Cimento **45A**, 245 (1966).

[6] R. K. Logan and L. Sertorio, Phys. Rev. Letters **17**, 834 (1966).

[7] R. K. Logan, J. Beaupre, and L. Sertorio, Phys. Rev. Letters **18**, 259 (1967).

[8] G. Altarelli *et al.*, Nuovo Cimento **48A**, 245 (1967).

[9] B. R. Desai, D. T. Gregorich, and R. Ramachandran, Phys. Rev. Letters **18**, 565 (1967).

[10] The charge-exchange polarization was measured by P. Bonamy *et al.*, Phys. Letters **23**, 501 (1966). The polarization of the recoil nucleon in this reaction would vanish if only ρ exchange were present. There are unfortunately a number of objections to the treatment of the resonance contributions to the amplitude in the foregoing theoretical papers, and it is unlikely that these explanations would survive a more careful calculation. A possible exception is the model of Desai *et al.*, Ref. 9. Alternative explanations for the observed polarization in terms of Regge cuts or lower-lying ρ-type trajectories have also been advanced [V. M. de Lany *et al.*, Phys. Rev. Letters **18**, 148 (1967); H. Hogassen and W. Fischer, Phys. Rev. **22**, 516 (1966); R. K. Logan *et al.*, Ref. 7]. For more detailed comments, see the talk of L. Durand in the Report of the Argonne Symposium on Regge Poles [Argonne National Laboratory Report, 1966 (unpublished)].

[11] A. Yokosawa, talk given at the Colloquim on Polarized Targets, Saclay, 1966 (unpublished), and private communication.

[12] P. D. Grannis, H. M. Steiner, and L. Valentin, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, Calif., 1967).

[13] Modified resonance forms have been used in Refs. 8 and 9. However, the form used in Ref. 8 is not simply connected to the direct-channel Regge-pole amplitudes. The Khuri form used in Ref. 9 is more satisfactory, but leads to difficulties with unitarity at high energies (see Sec. II).

[14] Since the elasticity of a resonance gives the probability that the πN system scatters through the eigenchannel [in the sense of Sec. IIIB] in which it sees the resonance, rather than through the increasingly large number of inelastic channels, it is clear physically that this parameter should decrease rapidly with increasing energy. It is plausible also that at a fixed energy, the elasticities (in the foregoing sense) should decrease monotonically for decreasing j, corresponding to our expectation that close collisions should be less elastic than grazing collisions. If this is the case, we may use the results of Ref. 1 for the high spin resonances to bound those parameters for the lower partial waves, hence, to study their general behavior with increasing energy. For example, the elasticities of the $\frac{3}{2}^-$, $\frac{7}{2}^-$, $\frac{11}{2}^-$, and $\frac{15}{2}^-$ members of the N_γ sequence are given at resonance as 0.76, 0.20, 0.08, and 0.01. The foregoing argument suggests that the first three parameters should be smaller than the fourth at the position of the $\frac{15}{2}^-$ state, hence, decrease extremely rapidly with increasing s. This results in strong suppression of the high-energy tails of the lower resonances [for example, the contribution of the $\frac{3}{2}^-$ state at the position of the $\frac{15}{2}^-$ state would be $\sim 1/76$ of that obtained from the simple Breit-Wigner result], a point which can be crucial for phenomenological studies. As we shall see, these correlations are built into the Regge-pole model.

[15] Three baryon Regge trajectories seem to be well established: Δ_δ with isotopic spin $I=\frac{3}{2}$, positive parity $P=+1$, and odd signature $\alpha=(-1)^{j-1/2}=-1$; N_α with $I=\frac{1}{2}$, $P=+1$, $\tau=+1$; and N_γ with $I=\frac{1}{2}$, $P=-1$, $\tau=-1$. Recent phase-shift analyses provide some evidence for the existence of additional trajectories with different quantum numbers, and of low-lying secondary trajectories with the foregoing quantum numbers. [Cf. V. Barger and D. Cline, Phys. Rev. Letters (to be published).]

[16] The Barger-Cline and Barger-Olsson analyses of backward and forward $\pi^{\pm}p$ scattering [Refs. 1 and 2] include some 16 resonances on the N_α, N_γ, and Δ_δ trajectories. To specify the masses, widths, and elasticities of these resonances using even simple Breit-Wigner resonance amplitudes thus requires 48 parameters, more than can be fitted uniquely using current data [the author would like to thank V. Barger for a useful comment on this point]. If additional resonances or Regge trajectories must be included, the number of free parameters in an "isolated resonance" treatment becomes prohibitive. The apparent smoothness of the masses, widths, and elasticities of the resonances used in Refs. 1 and 2 suggests that rather few parameters will be necessary to specify the Regge trajectory and residue functions.

residue functions restrict rather severely the possible variation in the parameters of a given resonance, it is less likely that the contributions of very inelastic resonances, for example, those associated with low-lying Regge trajectories,[15] will be obscured than in analyses in which the parameters of the stronger resonances are allowed to vary freely. Finally, the parameters of resonances with masses higher than those so far observed can be predicted once the energy dependence of the trajectory and residue functions is known approximately. As noted by Altarelli *et al.*[8] and Desai *et al.*,[9] such higher resonances are of particular importance for the RPR model of the high-energy $\pi^- p \to \pi^0 n$ charge-exchange polarization.

A final point which has caused some confusion with respect to the RPR model concerns the possibility that in adding a Regge-pole-exchange amplitude and a sum of resonant amplitudes, one is counting the same contributions to the scattering amplitude twice. It could be argued, for example, that the complete scattering amplitude is given by the sum of all the s-channel Regge-pole amplitudes, whence the t-channel exchange amplitude is superfluous. Although possibly correct in principle, this argument is deceptive in that only the highest few trajectories may lead to resonances. The important, but smooth contributions of the s-channel poles in the left half of the angular momentum plane are contained in the Regge background integral. It is presumably these contributions which can be described by the t-channel exchange amplitude.[17] The remaining double-counting and background problems associated with the treatment of the tails of the resonant amplitudes are eliminated in the modified RPR model discussed in the following sections.

In the present paper, we wish to consider in some detail the theoretical basis of the RPR model. For clarity, we will consider initially the case of single-particle potential scattering, for which we can obtain relatively simple and rigorous results. The generalizations necessary for the realistic case of multichannel scattering unfortunately introduce some practical problems. The physical ideas basic to the proper description of resonance phenomena in the RPR model nevertheless remain clear. The main uncertainties in a proper treatment of the model are fortunately confined to the lower partial waves, and as a consequence are probably not of major importance for high-energy scattering.

II. SINGLE-CHANNEL POTENTIAL SCATTERING

The scattering amplitude for a superposition of Yukawa potentials

$$V(r)=-\frac{1}{r}\int_\mu^\infty \sigma(\mu')e^{-\mu' r}d\mu' \qquad (1)$$

can be expressed using the Sommerfeld-Watson transform in the familiar form

$$A(s,t)=-\pi\sum_n \frac{2\alpha_n+1}{\sin\pi\alpha_n}\beta_n(s)P_{\alpha_n}(-x) + \tfrac{1}{2}i\int_{-1/2-i\infty}^{-1/2+i\infty} dj\frac{2j+1}{\sin\pi j}A(j,s)P_j(-x). \qquad (2)$$

Here s is the square of the momentum, $s=p^2(2m=1)$, x is the cosine of the scattering angle, and $t=2p^2(x-1)$. The sum runs over all the Regge poles in the right half j plane, $\mathrm{Re}\alpha_n>-\frac{1}{2}$. We have ignored the minor complications associated with exchange potentials and the signature of the trajectories. The pole terms lead to resonances in the scattering amplitude whenever one of the trajectory functions $\alpha_n(s)$ approaches a positive integer:

$$-\pi\frac{2\alpha_n+1}{\sin\pi\alpha_n}\beta_n(s)P_{\alpha_n}(-x) \underset{\alpha_n\to j}{\longrightarrow} \frac{2j+1}{s_j-s+i\Gamma/2}\frac{\beta_n(s_j)}{\mathrm{Re}\alpha_n'(s_j)}P_j(x), \qquad (3)$$

for $\mathrm{Re}\alpha_n(s_j)=j$, $\pi\,\mathrm{Im}\alpha_n(s_j)\ll 1$. Here s_j is the value of s for which $\mathrm{Re}\alpha_n=j$, and $\Gamma/2=\mathrm{Im}\alpha_n(s_j)/\mathrm{Re}\alpha_n'(s_j)$. At sufficiently high energies, all the Regge poles retreat into the left half j plane, $\mathrm{Re}\alpha_n<-\frac{1}{2}$, all n, and the entire scattering amplitude is given by the background integral in Eq. (2). In contrast to the pole terms, this function is expected to vary smoothly with s, approaching the Born approximation to the scattering amplitude at high energies. It is plausible that one can obtain a useful representation for the scattering amplitude at somewhat lower energies by approximating the background integral by the Born amplitude [the equivalent for potential scattering of the t-channel exchange amplitude of the relativistic theory], and retaining appropriate Regge-pole terms to represent any resonant contributions to the full amplitude. In particular, if the pole terms are approximated by a finite sum of Breit-Wigner amplitudes as in (3), one obtains the analog for potential scattering of the simple RPR model used in in Refs. 1–7.

The difficulties with the simple RPR model noted in the Introduction are also present in the potential scattering model. It is clear, for example, that there is some double counting of the exchange contributions, since each Regge-pole term contributes asymptotically to the Born amplitude. Secondly, the approximation of a Regge-pole term by a sum of Breit-Wigner amplitudes is clearly not valid. The Breit-Wigner approximation [Eq. (3)] is adequate only in the immediate vicinity of a resonance [$\pi|\alpha_n(s)-j|\ll 1$] and fails with respect to

[17] This is at least the case for nonrelativistic potential scattering (Sec. II).

a given resonance as $\alpha_n(s)$ moves away from the value of j in question. Furthermore, representation of the pole term by a sum of resonant amplitudes will evidently lead to an extraneous background amplitude: Only the single resonant term is present for $\alpha_n(s)$ near an integer [Eq. (3)]; the tails of distant resonances do not contribute. On the other hand, one cannot use the Regge-pole amplitudes as they appear in Eq. (2) for scattering near the forward direction, as $P_\alpha(-x)$ is logarithmically divergent for $x\to +1$ for noninteger values of α. The same problem arises for backward scattering if the trajectories of even and odd signature are distinct.[18] As noted by Khuri,[19] the divergent parts of the Regge-pole amplitudes are cancelled by nonresonant contributions from the exact background integral. The Born amplitude unfortunately does not contain the relevant terms.

It is evident that to obtain a satisfactory version of the RPR model for potential scattering, we must use a representation for the scattering amplitude with the following properties: (i) The resonant parts of the Regge-pole amplitudes should be treated in their entirety; the Breit-Wigner approximation should be avoided. (ii) The Regge-pole amplitudes should have the correct analyticity properties for $x\sim\pm 1$. (iii) The asymptotic parts of the Regge-pole amplitudes (those parts which are contained in the Born amplitude) should be deleted. Two such representations have been proposed: the Khuri representation[19] as modified by Ahmadzadeh,[20] and the Cheng representation[21] as modified by Abbe *et al.*[22] The Khuri representation is not entirely satisfactory, as will be seen later. We shall therefore use the Cheng representation. In its simple (unmodified) form, the Cheng representation expresses the S matrix as a product of factors each of which corresponds to a single Regge pole:

$$S(j,s)=\prod_n S_n(j,s)\,, \tag{4}$$

$$S_n(j,s)=\exp\left\{\int_{-\infty}^{\alpha_n^*(s)}\frac{e^{(\lambda-j)\xi(s)}}{\lambda-j}d\lambda-\int_{-\infty}^{\alpha_n(s)}\frac{e^{(\lambda-j)\xi(s)}}{\lambda-j}d\lambda\right\}. \tag{5}$$

The parameter ξ is defined in terms of the minimum value of μ in Eq. (1) as

$$\xi=\cosh^{-1}(1+\mu^2/2s)\to\mu/\sqrt{s}\,,\quad \mu^2/s\ll 1\,. \tag{6}$$

It is readily verified that $|S_n(j,s)|=1$ for real j and s, $s>0$. The Cheng representation thus has the important property that the unitarity of the S matrix is preserved even if the product in Eq. (4) is restricted to a finite number of Regge poles. [This feature is not shared by the simpler Khuri representation.]

The behavior of $S_n(j,s)$ in various limits is easily deduced. We will confine our attention to the case $\xi\,\mathrm{Im}\alpha_n(s)<1$ which appears to be of the most interest physically.[23] For this case, $\ln S_n(j,s)$ is well approximated for $|j-\mathrm{Re}\alpha_n|\,\xi$ not too large as

$$\ln S_n(j,s)\approx\ln\frac{j-\alpha_n^*}{j-\alpha_n}+2i\,\mathrm{Im}\alpha_n\frac{e^{(\mathrm{Re}\alpha_n-j)\xi}-1}{j-\mathrm{Re}\alpha_n}\,, \tag{7}$$

where the argument of $(j-\alpha_n^*)/(j-\alpha_n)$ is defined so as to approach $0+$ for $j\to\infty$. For $j\sim\mathrm{Re}\alpha_n$, this expression shows the expected resonant structure:

$$S_n(j,s)\xrightarrow[j\sim\mathrm{Re}\alpha_n]{}\frac{j-\alpha_n^*}{j-\alpha_n}e^{-2i\xi\,\mathrm{Im}\alpha_n}; \tag{8}$$

the background phase $2\xi\,\mathrm{Im}\alpha_n$ is generally small for the cases of interest. On the other hand, for j sufficiently large, $j-\mathrm{Re}\alpha_n\gg\mathrm{Im}\alpha_n$, the phase shift is exponentially damped as would be expected for a potential of finite range[24]:

$$\ln S_n(j,s)\to\frac{2i\,\mathrm{Im}\alpha_n}{j-\mathrm{Re}\alpha_n}e^{-(j-\mathrm{Re}\alpha_n)\xi}\,, \tag{9}$$

and the partial-wave scattering amplitude in a single-Regge-pole approximation may be expanded as

$$A(j,s)=\frac{(S-1)}{2i\sqrt{s}}\to\frac{1}{\sqrt{s}}\frac{\mathrm{Im}\alpha}{j-\mathrm{Re}\alpha}e^{-(j-\mathrm{Re}\alpha)\xi}. \tag{10}$$

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[18] The origin of this difficulty is easily seen by considering the partial-wave series for a single Regge-pole term with definite signature,

$$-\tfrac{1}{2}\pi\frac{2\alpha+1}{\sin\pi\alpha}[P_\alpha(-x)\pm P_\alpha(x)]=\sum_j(2j+1)A^{\mathrm{pole}}(j,s)\tfrac{1}{2}[1\pm(-1)^j]P_j(x)\,,$$

$$A^{\mathrm{pole}}(j,s)=(2\alpha+1)/[(j-\alpha)(j+\alpha+1)].$$

For large j and $x=\pm 1$, the terms in the series decrease only as j^{-1}, and the series consequently diverges logarithmically. [The series converges absolutely for $|x|<1$ because of the $j^{-1/2}$ decrease of the Legendre functions for large j, $|x|<1$.] In contrast, the complete partial-wave amplitudes for the Yukawa potential of Eq. (1) decrease for large j as $e^{-j\xi}/\sqrt{j}$, where ξ is defined in Eq. (6). The exponential decrease of the amplitude for $j\to\infty$ is associated with the angular momentum barrier penetration factor for a finite range potential. This factor appears in both the Cheng and Khuri forms for the Regge-pole amplitudes, Eqs. (9) and (12). There is consequently no convergence problem for $x=\pm 1$ when those representations are used.

[19] N. N. Khuri, Phys. Rev. **130**, 429 (1963).

[20] A. Ahmadzadeh, Phys. Rev. **133**, B1074 (1964).

[21] H. Cheng, Phys. Rev. **144**, 1237 (1966). This representation holds for continuous superpositions of Yukawa potentials, Eq. (1), for which the weight function $\sigma(\mu')$ decreases exponentially for $\mu'\to\infty$. We will restrict our attention to these potentials.

[22] W. J. Abbe *et al.*, Phys. Rev. **140**, B1595 (1965); **141**, 1513 (1966).

[23] Even for the rather large value $\mu=1$ BeV, $\xi\,\mathrm{Im}\alpha<0.4$ for the baryon Regge trajectories discussed in Ref. 1 if $\mathrm{Im}\alpha$ is estimated using the Breit-Wigner approximation [Eq. (3)] for the resonances at laboratory momenta below 5 BeV/c.

[24] This estimate actually gives an upper bound on the partial phase shift for $j>\mathrm{Re}\alpha_n$,

$$|\ln S_n(j,s)|\le\frac{2\,\mathrm{Im}\alpha_n}{j-\mathrm{Re}\alpha_n}e^{-(j-\mathrm{Re}\alpha_n)\xi}.$$

For large j, the limit in Eq. (9) should be used rather than the full expression in Eq. (7). The two expressions differ by terms of order $[\mathrm{Im}\alpha_n/(j-\mathrm{Re}\alpha_n)]^3$. The approximation in Eq. (7) is most accurate near the resonance.

A similar result for $\ln S_n$ is obtained for $j<\mathrm{Re}\alpha_n$; $\mathrm{Re}\alpha_n - j \gg \mathrm{Im}\alpha_n$,

$$\ln S_n(j,s) \to \frac{2i\,\mathrm{Im}\alpha_n}{j-\mathrm{Re}\alpha_n} e^{(\mathrm{Re}\alpha_n - j)\xi} + 2\pi i. \quad (11)$$

However, the exponential factor in Eq. (11) may be large enough in this limit that $|\ln S_n| \gtrsim 1$. It is not possible in that case to approximate the scattering amplitude as in Eq. (10).

It is interesting to compare the foregoing results with those of Khuri,[19,25] who obtains for each Regge pole in the right half plane a contribution to the partial-wave scattering amplitude of the form

$$A_n{}^K(j,s) = \frac{1}{\sqrt{s}} \frac{\beta_n(s)}{j-\alpha_n} e^{-(j-\alpha_n)\xi}. \quad (12)$$

In a single-pole model, the residue function $\beta_n(s)$ is equal to $\mathrm{Im}\alpha_n$, and Khuri amplitude coincides with the large-j form of the Cheng amplitude, Eq. (9).[26] It is evident, however, that the individual Khuri amplitudes may exceed the unitarity limit for large $\mathrm{Re}\alpha_n$ and small j, as the exponential factor may become quite large. The complete partial-wave amplitude is unitary only if the background contributions are included. As a result, use of the simple form of the Khuri representation in an RPR model, with the resonant contributions described by a sum of partial-wave amplitudes of the form given in Eq. (12), may result in large spurious contributions from the low partial waves. [For $\alpha\xi \gg 1$, these are of order $e^{\alpha\xi}/\alpha\xi^2$, large compared to the resonant contributions.] In contrast, the Cheng representation maintains the unitarity of the S matrix, albeit at the expense of some analytical complication.

Although the Cheng representation is suitable for the treatment of resonance phenomena, the form given in Eq. (5) is not satisfactory for the discussion of high-energy limits or for use in an RPR type model, since each Regge pole contributes asymptotically to the Born amplitude. The necessary modification of the Cheng representation has been given by Abbe *et al.*,[22] who observed that the S matrix could be written as

$$S(j,s) = e^{2i\delta_B(j,s)} \prod_n \hat{S}_n(j,s), \quad (13)$$

where δ_B is the Born approximation to the phase shift,

$$\delta_B(j,s) = \frac{1}{2\sqrt{s}} \int_\mu^\infty \sigma(\mu') Q_j\left(1+\frac{\mu'^2}{2s}\right) d\mu', \quad (14)$$

[25] The assumption of Ref. 19 that the background integral can be eliminated in terms of the Regge poles in the left half plane is not correct. Thus, $A(j,s)$ cannot be written simply as a sum over the Khuri amplitudes $A_n{}^K(j,s)$. The result is nevertheless correct for the poles in the right half plane, which lead to the interesting resonance effects.

[26] More generally, if the individual phase shifts $\ln S_n$ and their sum are all small, as for $j\to\infty$, $A(j,s)$ is given for the Cheng representation by a sum of terms of the form given in Eq. (10), and the Khuri and Cheng representations coincide.

and

$$\hat{S}_n(j,s) = \exp\left\{ \int_{-\infty}^{\alpha_n^*(s)} \frac{\exp[(\lambda-j)\xi]}{\lambda-j} d\lambda - \int_{-\infty}^{\alpha_n(s)} \frac{\exp[(\lambda-j)\xi]}{\lambda-j} d\lambda - \frac{i}{\sqrt{s}} \frac{\exp[-(j+n)\bar{\xi}]}{j+n} \times \int_\mu^\infty \sigma(\mu') P_{n-1}\left(1+\frac{\mu'^2}{2s}\right) d\mu' \right\}. \quad (15)$$

Here $\sigma(\mu')$ is the weight function for the potential of Eq. (1), and the parameter $\bar{\xi}$ is defined as[27]

$$\bar{\xi} = \cosh^{-1}(1+2\mu^2/s). \quad (16)$$

It is easily checked using the asymptotic form of the normal trajectory functions

$$\alpha_n(s) \underset{s\to\infty}{\longrightarrow} -n + \frac{i}{2\sqrt{s}} \int_\mu^\infty \sigma(\mu') P_{n-1}\left(1+\frac{\mu'^2}{2s}\right) d\mu', \quad (17)$$

that $(\sqrt{s}) \ln \hat{S}_n(j,s)$ vanishes for $s\to\infty$. The functions $\ln S_n(j,s)$ in the unmodified form of the Cheng representation do not have this property, each term contributing part of the Born approximation for the phase shift as $s\to\infty$.[22]

The modified Cheng representation appears to be satisfactory for practical calculations for simple Yukawa potential scattering.[28] However, only the asymptotic contributions of the normal Regge trajectories [those which approach the negative integers for $s\to\infty$] have been removed. Abnormal trajectories which move to ∞ in the left half plane are also known to exist.[29] While their contributions to the phase shift vanish for $s\to\infty$ [but not uniformly in n], it is quite possible that important nonresonant contributions persist at finite energies, especially for potentials for

[27] Since the Born approximation to the phase shift has been extracted, the potential must act at least twice in generating $\ln \hat{S}_n$. The maximum effective range of the interaction which appears in the angular momentum barrier penetration factors is therefore one-half that characteristic of the Born approximation, and μ in the parameter ξ is replaced in $\bar{\xi}$ by 2μ. Abbe *et al.*, Ref. 22, show that this choice leads to an amplitude having the correct analyticity as a function of s and x.

[28] The smallness of the contributions to $\hat{S}$ of low-lying trajectories is evident from Eqs. (15) and (17). Numerical calculations by Abbe *et al.*, Ref. 22, show in fact that a single-pole approximation for $\hat{S}$ gives remarkably accurate results for the S matrix for an example of Yukawa potential scattering in which only a single Regge trajectory reaches the right half plane. The results are much better than those obtained using the modified Khuri fepresentation, Ref. 20. Cf. also W. J. Abbe and G. A. Gary [Phys. Rev. **160**, 1510 (1967)], for a more extensive comparison of the modified Cheng representation with exact results for strong Yukawa potentials. We are unaware of any tests of the Cheng representation for potentials which lead to many resonances and high rising trajectories. Calculations for the potential in footnote 30 would be particularly interesting.

[29] Y. I. Azimov, A. A. Ansel'm, and V. M. Shekhter, Zh. Eksperim. i Teor. Fiz. **44**, 361 (1963); **44**, 1078 (1963) [English transls.: Soviet Phys.—JETP **17**, 246 (1963); **17**, 726 (1963)]. N. F. Bali, S. Y. Chu, and R. W. Haymaker, Phys. Rev. **161**, 1450 (1967).

which $\mathrm{Re}\alpha_n$ becomes large for one or more trajectories.[30] In the latter case, the partial phase shifts for small j and a given n may increase exponentially over some range of s as $\mathrm{Re}\alpha_n$ increases [cf. Eqs. (11) and (15)]. The resulting unphysically rapid increase of the complete phase shift is not cancelled by contributions from the lower normal trajectories, nor is it eliminated at intermediate energies by subtracting the Born contribution as in Eq. (15). It is probable that the increase is prevented by contributions from the abnormal trajectories,[31] but rather little is known about these terms. We expect, in any case, that the complete phase shift will increase by $\sim\pi$, but not much more, as the energy of a resonance is passed. The exponential increase in the phase apparent in the individual terms in Eqs. (11) or (15) is clearly spurious.

The foregoing problem can be eliminated, at least for practical purposes, by working with the difference between the phases $\ln\hat{S}_n(j,s)$ and $\ln\hat{S}_n(0,s)$ given by Eq. (15), and treating the total S-wave phase shift separately. Thus, for "unusual" potentials [or in the more interesting case of relativistic scattering], we can rewrite $S(j,s)$ as

$$S(j,s)=\exp\{2i\delta_B(j,s)-2i\delta_B(0,s)+2i\delta(0,s)\}\prod_n \tilde{S}_n(j,s)\,, \tag{18}$$

where

$$\ln\tilde{S}_n(j,s)=\ln\hat{S}_n(j,s)-\ln\hat{S}_n(0,s)\,, \tag{19}$$

and the phase shift $\delta(0,s)$ is to be calculated directly, or treated phenomenologically. The unphysical increase in the low j phase shifts for increasing α_n, $\mathrm{Re}\alpha_n\gg j$, is substantially offset by the subtraction. It is possible, therefore, that for reasonable potentials, the effects of the abnormal trajectories will be important only in the calculation of the S-wave phase shift. [For example, it is readily checked that the phase difference $\ln\tilde{S}_n$ does not become large for the known πN trajectories.] The modifications in the following results introduced by this change in representation are rather minor, and will not be considered in detail.

The partial-wave scattering amplitude $A(j,s)$ can be written using the modified Cheng representation in the convenient form

$$A(j,s)=(e^{2i\delta_B}-1)/2i\sqrt{s}+e^{2i\delta_B}(\hat{S}-1)/2i\sqrt{s}\,, \tag{20}$$

where $\hat{S}=\prod_n \hat{S}_n$. For s sufficiently large, $|2\delta_B|<1$, and the first term approaches the Born approximation $A_B(j,s)=\delta_B(j,s)/\sqrt{s}$. The resonant contributions to $A(j,s)$ are contained entirely in the second term. Since this term vanishes more rapidly than the first for $s\to\infty$, $A(j,s)$ approaches $A_B(j,s)$ in this limit as would be expected. We are most interested in high, but not asymptotic, energies, for which δ_B is small, and at most a few Regge poles are in the right half plane. Provided the potential is such that a region of this type exists (this is apparently the case for the Yukawa potential[28]), we can approximate the leading term in Eq. (20) by the Born approximation. Furthermore, since the contributions of the low-lying Regge trajectories to $\ln\tilde{S}$ are generally quite small, it is reasonable to retain only the leading (resonant) trajectories in the calculation of that function.[28] The complete scattering amplitude then assumes the form

$$A(s,t)=A_B(s,x)+(2i\sqrt{s})^{-1}\times\sum_j(2j+1)e^{2i\delta_B}(\hat{S}-1)P_j(x)\,, \tag{21}$$

where

$$A_B(s,t)=\int_\mu^\infty \sigma(\mu')\frac{1}{\mu'^2-t}d\mu'\,, \tag{22}$$

with $t=2s(x-1)$. The contributions to $\hat{S}$ of the Regge poles in the right half plane can be approximated using Eqs. (7) and (15)[32]:

$$\ln\hat{S}_n(j,s)\approx\ln\frac{j-\alpha_n^*}{j-\alpha_n}+2i\,\mathrm{Im}\alpha_n\frac{\exp[(\mathrm{Re}\alpha_n-j)\xi]-1}{j-\mathrm{Re}\alpha_n}-\frac{i}{\sqrt{s}}\frac{\exp[-(j+n)\xi]}{j+n}\int_\mu^\infty\sigma(\mu')P_{n-1}\left(1+\frac{\mu'^2}{2s}\right)d\mu'\,. \tag{23}$$

The results in Eqs. (21)–(23) constitute an analog of the RPR model for potential scattering, but an analog in which the resonance contributions are treated properly. Several points should be noted in this connection. First, the use of the modified Cheng representation permits the resonant contributions to the scattering amplitude of any set of Regge poles to be included in their entirety. In particular, the treatment of the resonances is not restricted to the Breit-Wigner approximation. Second, since the smooth asymptotic form of the Regge-pole terms is explicitly extracted and appears only in the Born (or exchange) amplitude, there is no problem of double counting of these background contributions. Third, the contribution of a given resonance is large only for $j\sim\mathrm{Re}\alpha_n$, $s\sim s_j$; the resonant amplitude is sharply cut off for $j>\mathrm{Re}\alpha_n$ by the angular momentum barrier factors, especially if the potential is of short range; the residual phases $\ln\hat{S}_n$ are

[30] For example, many resonances and rather high rising trajectories would be obtained for a damped oscillator potential, $r^2e^{-\mu r}$, μ small.

[31] The author would like to thank Professor Peter Kaus for an interesting discussion of this point. The difficulties with the modified Cheng representation have been noted independently by Professor Kaus and Professor R. Blankenbecler (private communication to P. Kaus).

[32] For a simple Yukawa potential with a range parameter μ, the last term in Eq. (23) is replaced by

$$i\frac{g}{\sqrt{s}}\frac{\exp[-(j+n)\xi]}{j+n}P_{n-1}\left(1+\frac{\mu^2}{2s}\right),$$

where g is the coupling strength. It may be possible to use this simpler form much more generally as an approximation to the integral if μ is replaced by an average or effective range parameter, and g by an effective strength. The effective value of μ should also appear in ξ.

reduced above resonance by the background subtraction, and vanish for $s\to\infty$. Finally, the resonant contributions to the S matrix appear in a product form, rather than as a sum, a point which may be important in situations in which more than one trajectory must be considered. Perhaps the main practical defect of the present results is the apparent necessity of describing the Regge-pole contributions through the partial-wave series in Eq. (21); we have in any case been unable to find a simple approximation for this series. However, the exponential cutoff for large j makes the series suitable for computer evaluation.

III. RELATIVISTIC SCATTERING

A. Elastic Scattering

In the absence of an adequate dynamical model for relativistic scattering, the generalization of the results of the preceding section to physically interesting cases is somewhat speculative. Nevertheless, the physical ideas involved are clear. It is plausible, first, that the role of the Born amplitude in Eq. (21) should be assumed in the relativistic theory by the Regge-pole-exchange amplitude[33]: The Born amplitude describes the smooth high-energy behavior of the scattering amplitude in potential theory, the Regge-pole model, the corresponding behavior of the relativistic amplitude. In either case, the asymptotic amplitude gives the smooth limiting variation of the partial-wave amplitudes as functions of the angular momentum j, and the square of the total energy in the center-of-mass system s. The Regge-pole model seems, furthermore, to describe properly the average behavior of the scattering amplitude at rather low energies, well into the resonance region. We consequently expect the exchange amplitude to give an adequate description of the smooth background scattering, with resonant contributions to the amplitude confined to rather narrow bands in j or s.

It is clear that we should look for a relativistic analog of Eq. (23) for the modified partial-wave amplitudes. The problem is unfortunately complicated by the multichannel character of high-energy scattering. Furthermore, rather little is known about the j-plane properties of the complete S matrix. There are several possibilities at this point. Perhaps the most satisfactory theoretically for elastic scattering is to follow the procedure of Cheng[21] [or Abbe *et al.*[22] for the modified amplitude] and write a Cauchy representation for the logarithm of the relevant S-matrix element. Assuming that the boundary conditions for $|j|\to\infty$ are those given by Cheng, and that the (important) singularities of S are isolated poles, one obtains a representation for the diagonal S-matrix elements similar to that in Eq. (5), with branch points at the poles α_n and the zeroes $\bar{\alpha}_n$ of the matrix elements. [The representation is correct for many-channel potential scattering, with potentials of the type considered by Cheng.[21]] Thus, for a particular matrix element S_{ii},[34]

$$S_{ii}=\prod_n (S_{ii})_n\,, \tag{24}$$

with

$$(S_{ii})_n=\exp\left[\int_{-\infty}^{\bar{\alpha}_n(s)}\frac{e^{(\lambda-j)\xi(s)}}{\lambda-j}d\lambda-\int_{-\infty}^{\alpha_n(s)}\frac{e^{(\lambda-j)\xi(s)}}{\lambda-j}d\lambda\right]. \tag{25}$$

The parameter $\xi(s)$ depends as before on the location of the leading singularities of the partial-wave amplitudes.[35] Both ξ and the position of the zeros $\bar{\alpha}_n$ may depend on the channel in question. The pole positions α_n are common to all channels.

Because of the multichannel character of the scattering, the zeros $\bar{\alpha}_n(s)$ are displaced from the points $\alpha_n{}^*(s)$ characteristic of the single-channel problem, and the individual matrix elements are not unitary. As a consequence, the low-lying trajectories may change the magnitude of S_{ii} even if they do not contribute significantly to the real phase shift. Note, however, that the effects of these trajectories decrease rapidly for increasing j, in agreement with our expectation that the scattering should be more nearly elastic in grazing collisions.

It is interesting to examine the inelastic effects in detail for a single factor $(S_{ii})_n$. We will again consider only that case which appears to be of the most interest physically. If $|\alpha_n-\bar{\alpha}_n|\xi\ll1$, we can approximate $\ln(S_{ii})_n$ quite accurately as

$$\ln(S_{ii})_n\approx\ln\frac{j-\bar{\alpha}_n}{j-\alpha_n}+(\alpha_n-\bar{\alpha}_n)\frac{\exp[(\hat{\alpha}_n-j)\xi]-1}{j-\hat{\alpha}_n}\,,$$

$$\hat{\alpha}_n=\tfrac{1}{2}(\alpha_n+\bar{\alpha}_n)\,. \tag{26}$$

This approximation reduces to that in Eq. (23) for $\bar{\alpha}_n=\alpha_n{}^*$. The (complex) phase shift is again exponentially damped for $j\to\infty$, as would be expected for a finite-range interaction [cf. Ref. 24]. On the other hand, for $j\ll\mathrm{Re}\alpha_n$, $\mathrm{Re}\bar{\alpha}_n$, the phase shift grows exponentially, and will lead to a unitarity violation $|S_{ii}|\gg1$ unless $\mathrm{Re}\bar{\alpha}_n>\mathrm{Re}\alpha_n$. [It is possible also that the unitarity violation is prevented by the contribution of lower-lying or abnormal trajectories.]

[33] We interpret the exchange amplitudes in a generalized sense. It may be necessary, for example, to include contributions from moving cuts as well as poles. See, for example, Ref. 10.

[34] The Cheng representation does not hold in its present form for the off-diagonal matrix elements. However, for such important processes as π^-p charge-exchange scattering, the scattering amplitude can be expressed in terms of the diagonal matrix elements using isospin invariance. The nonrigorous representation in Eq. (44) gives a useful form for the off-diagonal matrix elements in more general situations.

[35] See, for example, N. N. Khuri and B. M. Udgaonkar, Phys. Rev. Letters **10**, 172 (1963). More than one value of ξ may appear, as in the case of πN scattering discussed by these authors. However, a single effective value ξ is probably sufficient for practical purposes (Ref. 32).

The behavior of S_{ii} near a resonance $j\sim\alpha_n$ is determined primarily by the first term in Eq. (26). We can distinguish several cases.

(i) If the resonance is elastic, $\bar{\alpha}_n=\alpha_n^*$, and the discussion parallels that of the preceding section. In this case, the phase shift is real, $|S_{ii}|=1$, and the point representing S_{ii} on an Argand diagram moves rapidly around the unit circle in the counterclockwise direction as the position of the pole $\alpha_n(s)$ moves past the integer value j. [$\mathrm{Re}\alpha_n(s)$ increases with s in the resonance region.]

(ii) If the resonance is inelastic, the zero $\bar{\alpha}_n$ of S_{ii} in the j plane moves toward the real axis, $|\mathrm{Im}\bar{\alpha}_n|<\mathrm{Im}\alpha_n$. The single-factor representation for S_{ii} will have a magnitude less than unity provided $(\mathrm{Re}\bar{\alpha}_n-\mathrm{Re}\alpha_n)^2+\mathrm{Im}\bar{\alpha}_n^2<\mathrm{Im}\alpha_n^2$. Proper behavior in the j plane is also assured if $\mathrm{Re}\bar{\alpha}_n>\mathrm{Re}\alpha_n$, as noted above. If $\bar{\alpha}_n$ remains in the lower half j plane, the track of S_{ii} in an Argand diagram circles the origin counterclockwise as the resonant energy is passed, but remains inside the unit circle. The phase of S_{ii} again increases by $\sim 2\pi$ through the resonance region.

(iii) If the zero $\bar{\alpha}_n$ moves into the upper half j plane, again with the unitarity restrictions satisfied, the phase of S_{ii} does not change greatly as the resonance is passed. However, the magnitude of S_{ii} decreases markedly for $\bar{\alpha}_n\sim j$, and the resonance appears as a small counterclockwise loop on the track of S_{ii}. The loop does not circle the origin.

The foregoing behavior of the S matrix near a resonance is of course familiar from reaction theory. The power of the representation in Eqs. (24)–(26) consists in the fact that it determines the behavior of the partial-wave amplitudes as functions of j as well as s.

It is desirable at this point to extract the smooth asymptotic part of the phase shift in Eq. (25), as in the modified Cheng representation for potential scattering.[22] The resonance fluctuations in the S matrix will then be localized as functions of j or s. In accordance with our previous assumptions, the asymptotic form of the scattering amplitude is given by the Regge-pole exchange model. Both t- and u-channel exchanges may be present. The smooth high-energy limits of the partial-wave S-matrix elements or phase shifts can therefore be determined by partial-wave projection once the exchange amplitudes are given. To complete the transition from Eq. (25) to a modified Cheng representation for S_{ii}, it is then necessary only to identify that part of the asymptotic phase shift which is to be associated with each Regge pole.

Although clear in principle, the foregoing construction leads to some difficulties in practice. First, the correct parametrization of the Regge-pole exchange amplitudes is generally not known except for near-forward or near-backward angles. The resulting uncertainties in the partial-wave amplitudes are fortunately rather small for the higher partial waves. Since the prominent high-energy resonances are also associated with large values of j,[1] extraction of the background phase shift for these states is probably not a serious problem. The uncertainties are much greater with respect to resonances on lower-lying Regge trajectories.

A second problem is associated with the necessity of identifying the contributions associated with individual Regge poles. We shall assume that the exchange amplitude can be adequately represented in the relevant region as[36]

$$A_{\mathrm{exch}}(s,t)=\frac{1}{\sqrt{s}}\int dt'\frac{\sigma(s,t')}{t'-t}\,, \tag{27}$$

where $t=2p^2(x-1)$, with p the 3-momentum of the particles in the center-of-mass system. Then

$$(S_{ii})_{\mathrm{exch}}=1+\frac{i}{2p\sqrt{s}}\int dt'\sigma(s,t')Q_j\left(1+\frac{t'}{2p^2}\right). \tag{28}$$

The second term in Eq. (28) will be small at high energies,[37] and may be identified with the complex phase of S_{ii}. We may therefore follow the procedure in the second paper of Ref. 22 using this asymptotic approximation for the phase, and obtain as the modified Cheng representation for S_{ii}

$$S_{ii}(j,s)=e^{i\phi}\prod_n(\tilde{S}_{ii})_n\,, \tag{29}$$

where the complex quantity $i\phi$ is given by the second

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[36] This representation is of course equivalent to a dispersion relation in t for the exchange amplitude. The existence of such a dispersion relation is not obvious if $\alpha(t)\rightarrow\infty$ for $t\rightarrow\infty$, as suggested by current data. It is actually sufficient for our purposes that Eq. (27) give a reasonable approximation to $A_{\mathrm{exch.}}(s,t)$ in the energy range of interest. Such representations are readily obtained. For example, a ρ exchange amplitude of the form

$$\beta(t)(s/s_0)^{\alpha(t)}[1-e^{-i\pi\alpha}]/\sin\pi\alpha$$

is well approximated for $|t|\lesssim 2m_\rho^2$ as

$$B(t)\left(\frac{s}{s_0}\right)^{\alpha(0)}e^{\frac{1}{2}i\pi[1-\alpha(0)]}\frac{1-\frac{1}{4}i\pi[\alpha(t)-\alpha(0)]}{1+\frac{1}{4}i\pi[\alpha(t)-\mathrm{g}(0)]}\times\{(m_\rho^2-t)[1-t\alpha'(0)\ln(s/s_0)]\}^{-1}.$$

This approximation is obtained by extracting the known zeros of β, and using the product representation for the trigonometric functions. The function $B(t)$, which contains the remaining t dependence of the amplitude, is essentially constant for the models which have been proposed for ρ exchange in πN scattering. The foregoing approximation is of the desired form over the region in which $\alpha(t)$ is well approximated as $\alpha(t)=\alpha(0)+t\alpha'(0)$. More precise approximations are easily constructed.

[37] We assume that the high-energy behavior of the scattering amplitude is determined by the exchange of the Pomeranchuk trajectory. In this case, the second term in Eq. (28) decreases logarithmically for $s\rightarrow\infty$.

term in Eq. (28), and

$$(\hat{S}_{ii})_n=\exp\left\{\int_{-\infty}^{\bar{\alpha}_n(s)}\frac{\exp[(\lambda-j)\xi(s)]}{\lambda-j}d\lambda-\int_{-\infty}^{\alpha_n(s)}\frac{\exp[(\lambda-j)\xi(s)]}{\lambda-j}d\lambda-\frac{i}{2p\sqrt{s}}\frac{\exp[-(j+n)\xi(s)]}{j+n}\times\int dt'\sigma(s,t')P_{n-1}\left(1+\frac{t'}{2p^2}\right)\right\}. \quad (30)$$

The parameter $\xi(s)$ is determined by the position of the cuts in t associated with two-particle intermediate states.[22]

A final problem is associated with the exponential increase of the phase shifts for small j if α_n becomes large and $\text{Re}\bar{\alpha}_n\leq\text{Re}\alpha_n$. As noted in the preceding section, this problem can be circumvented for practical purposes by subtracting the S-wave phase shift as in Eqs. (18) and (19). We will not consider the details.

The complete high-energy scattering amplitude including both the Regge-pole-exchange and resonance terms can now be approximated following Eqs. (21) and (22):

$$A(s,t)\to A_{\text{exch}}(s,t)+(2i\sqrt{s})^{-1}\sum_j(2j+1)e^{i\phi}(\hat{S}-1)P_j(x). \quad (31)$$

It is plausible that, with the asymptotic phases extracted, only a few Regge trajectories need be included in the evaluation of $\hat{S}$. In particular, the inelasticity associated with low-lying trajectories which do not contribute to the resonance amplitude, is included in the factor $e^{i\phi}$.

The expressions in Eqs. (26) and (28)–(31) provide a generally satisfactory version of the RPR model for the treatment of high-energy resonances in elastic scattering. In particular, the present model does not suffer from the difficulties noted in the Introduction.

B. Inelastic Scattering

As was noted previously, the foregoing representation is applicable only to the diagonal elements of the S matrix, that is, to elastic scattering.[34] It is consequently of interest to obtain an alternative representation which is applicable to the full S matrix. The representation which we shall discuss is nonrigorous. However, it incorporates the essential features of the modified Cheng representation, and, moreover, is simpler both to interpret physically, and to use in actual calculations, than that given above.

A natural approach to the many-channel problem is through the eigen-S matrix. In the neighborhood of a resonance, S may be written as the sum of a slowly varying background matrix B, and a matrix R which contains the resonance pole in j (or s), $R\propto(j-\alpha)^{-1}$,

$$S=B+R. \quad (32)$$

The S matrix can be diagonalized by an orthogonal transformation,

$$S=O^TDO, \quad (33)$$

where

$$O^TO=OO^T=1 \quad (34)$$

and

$$D=(e^{2i\delta}). \quad (35)$$

Since the eigenvalues of S are simple phase factors, it might be supposed that they could be expressed in terms of the s-channel Regge poles using a Cheng or modified Cheng representation. This is not the case: unless the background matrix B is a multiple of the unit matrix, the eigenvalues of S will contain cuts as well as poles in the j plane, and the Cheng integral will contain extra contributions which cannot be expressed simply in terms of the Regge-trajectory functions.[38] A related problem arises from the fact that the eigenphases cannot cross except in special circumstances.[39] Thus, if a resonance occurs in one of the eigenchannels, the rapid increase of the corresponding phase 2δ by $\sim2\pi$ will in general carry the resonant phase past one or more others. The phases do not cross. Rather, the resonant behavior is transferred from one eigenchannel to the next, with the corresponding phases interchanging their roles. As a consequence, the elements of the coupling matrix O change rapidly near the "crossing point." This typical "level-crossing" behavior is of course associated with the branch points of S, which lie rather close to the resonance poles.[39] Since these complications are absent for the physical S matrix, which has only dynamical singularities in the j plane, it is evident that they are in a sense spurious as far as the representation of physical quantities is concerned.

The problems associated with the eigenphase representation of S can be avoided in the neighborhood of a resonance using a method discussed recently by McVoy.[39] As noted above, the level-crossing problems are absent if the background matrix is a multiple of the unit matrix. Since the matrix B presumably varies slowly as a function of j through the resonance region, and R becomes small away from resonance, B must be approximately unitary, $B^\dagger B\sim1$. Thus, B can be diagonalized (at least approximately) by an orthogonal matrix O_B,

$$B=O_B{}^TD_BO_B. \quad (36)$$

[38] This is easily seen, for example, by considering a two-channel problem. The corresponding results for the eigen-S matrix considered as a function of s are discussed in detail by K. W. McVoy, Ref. 39.

[39] K. W. McVoy, lecture notes for the International Course on Nuclear Physics at Trieste, 1966, ICTP Paper No. SMR 3/21 (unpublished). C. J. Goebel and K. W. McVoy, Phys. Rev. 164, 1932 (1967). The results quoted for the j-plane behavior of S are strictly analogous to those given by McVoy for the behavior of S as a function of s.

A modified S matrix with a unit background can thus be defined as

$$\mathcal{S}=D_B^{-1/2}O_BSO_B^TD_B^{-1/2}=1+D_B^{-1/2}O_BRO_B^TD_B^{-1/2}. \quad (37)$$

The matrix $\mathcal{S}$ is unitary, and can be diagonalized by a second orthogonal transformation O_R,

$$\mathcal{S}=O_R^T[1+\Delta_R]O_R=1+O_R^T\Delta_RO_R, \quad (38)$$

where Δ_R is the eigenmatrix of the modified resonance term. If the resonance in question corresponds to an isolated pole, only one element of Δ_R will be nonzero. The corresponding eigenvalue of $\mathcal{S}$ will be a simple phase factor, $1+\Delta_R=e^{2i\delta}$; all other eigenvalues are unity. Because of this unit background, the level-crossing problems characteristic of the eigenvalues of S are not present for $\mathcal{S}$.[39]

It is plausible that the resonant eigenphase of $\mathcal{S}$ can be represented near the resonance by a modified Cheng-type integral. It is not necessary that the representation be exact: the resonant contribution to the S matrix is expected to be significant only for $j\sim\alpha_n$. The modified Cheng representation has this characteristic. Thus, we will assume that the resonant term in the eigenrepresentation of $\mathcal{S}$ can be approximated by the usual product of terms of the form given in Eqs. (4) and (5). Note in particular that the end points of the integrals are at $\alpha_n(s)$ and $\alpha_n^*(s)$, since the eigenphase is real.

The extraction of the asymptotic contribution of the Regge poles to $\mathcal{S}$ causes some difficulty, in that the asymptotic phase shift is not given directly in terms of the exchange amplitude in the channel of interest. All channels are involved. On the other hand, the results of Sec. II suggest that the proper result is[40]

$$\mathcal{S}=\prod_n \mathfrak{S}_n, \quad (39)$$

with

$$\mathfrak{S}_n=\exp\left\{\int_{-\infty}^{\alpha_n^*(s)}\frac{\exp[(\lambda-j)\xi(s)]}{\lambda-j}d\lambda-\int_{-\infty}^{\alpha_n(s)}\frac{\exp[(\lambda-j)\xi(s)]}{\lambda-j}d\lambda-2i\,\mathrm{Im}\alpha_n(s)\frac{\exp[-(j+n)\xi(s)]}{j+n}\right\}. \quad (40)$$

The last term removes the asymptotic contributions of the first term for large j and for large s, provided in the second case that $\alpha_n(s)\to -n$ for $s\to\infty$. The first two terms can again be approximated quite well for physically interesting values of the parameters by the expression in Eq. (17). It should perhaps be recalled that the apparent exponential increase of the phase shifts for small j and increasing α_n can be eliminated if necessary by removing the $j=0$ term as in Eq. (18), and writing $\mathcal{S}$ as $\mathcal{S}(j,s)=\mathcal{S}(0,s)\prod_n[\mathfrak{S}_n(j,s)/\mathfrak{S}_n(0,s)]$.

An approximate representation for the S matrix may now be obtained by expressing S in terms of $\mathfrak{S}$:

$$S=B+A^T\Delta_RA, \quad (41)$$

where the complex matrix A is given by

$$A=O_RD_B^{1/2}O_B \quad (42)$$

and

$$\Delta_R=\mathfrak{S}-1. \quad (43)$$

In this form, the Regge-pole exchange amplitudes give the smooth high-energy contribution to B; only the terms relevant to the particular reaction in question need be known. The resonant terms appear in S in a form very similar to that, for example, in Eq. (20). In particular, Δ_R is multiplied by a factor containing the sum of the background phases in the entrance and exit channels. The factorization property of the residues of the s-channel Regge poles is also evident. Except for the background phase factors, the elements of the A matrix are essentially the square roots of the partial widths for the decay of the resonance into the various channels. For a resonance in channel j, $A_j\propto e^{i\delta_k}\Gamma_{jk}^{1/2}$. We may therefore expect the elements of A to decrease rapidly in magnitude as the energy and the number of channels which communicate with the resonant channel increase. It is plausible also that the elements of A decrease with decreasing j; close collisions are expected to be less elastic than grazing collisions, and the angular momentum barrier is lower.[14]

The form of the RPR model for particle reactions which follows from the foregoing discussion and the representation of Eq. (41) for S is

$$A(s,t)=A_{\mathrm{exch}}(s,t)+\left(\frac{p'}{2ip\sqrt{s}}\right)\sum_j(2j+1)\beta(j,s)(\mathfrak{S}-1)P_j(x). \quad (44)$$

The usual two-body phase space and flux factors have been extracted from the complex function $\beta(j,s)$; the final 3-momentum in the center-of-mass system is denoted by p', the initial 3-momentum by p. The function $\beta(j,s)$ acts as an effective elasticity factor. At a fixed energy high enough that many channels are open, the magnitude of β is expected to increase smoothly from a rather small value for $j\sim 0$, and approach 1 as $j\to\infty$ [cf. Ref. 14 and the remarks following Eqs. (25) and (26)]. However, it may be possible for practical purposes to neglect this j dependence and use an effective value appropriate to any resonant contribution to the sum. The error introduced with respect to the low partial waves is reduced by their low statistical weights, while the high partial-wave amplitudes are exponentially damped by the angular momentum barrier penetration factors.

[40] Note that the asymptotic phase of $\mathfrak{S}$ is zero. The actual background phase has been extracted in the construction of this matrix.

IV. CONCLUSIONS

We have seen that it is possible to formulate a precise analog for potential scattering of the relativistic RPR model which has proved so useful in the analysis of high-energy scattering data.[1–9] The complications arising from the many-channel aspect of relativistic scattering render the model somewhat less precise in that situation. However, satisfactory versions of the RPR model can still be formulated. The results are given in Eq. (31) for elastic scattering, and, in a somewhat more convenient form, applicable to both elastic and inelastic scattering, in Eq. (44). The theoretical uncertainties in these results are probably not of great practical consequence. The important advantage of the present results compared to the simpler early versions of the RPR model is the elimination of the double-counting problem noted in the Introduction, and the possibility of treating entire sequences of resonances as correlated states, parametrized by the Regge trajectory and residue functions.

As is clear from our derivation, the RPR model is basically an intermediate-to-high energy approximation. The proper high-energy behavior of the scattering amplitude is built into the representation.[41] The model will be useful for the description of resonance phenomena if there exists an energy region in which the smooth average behavior of the partial-wave scattering amplitudes is adequately described by the exchange terms, yet one or more s-channel Regge poles continue to move to the right in the j plane.[42] Although such a region need not exist, the results of Abbe *et al.*[22,28] for potential scattering, and the success of the simpler versions of the RPR model,[1,2] are most encouraging. On the other hand, the model will certainly fail at low energies as contributions to the exchange amplitude from low-lying t-channel trajectories, or the t-channel background integral, become important. It should be emphasized, finally, that the approximations which have been made in our discussion are those appropriate to the s channel; the RPR amplitude is not suitable as given for continuation to other channels.

We conclude by noting several situations in which the modifications in the RPR model may be especially important.

(i) As was noted by Barger and Cline,[1] the tails of distant s-channel Breit-Wigner resonances contribute a large background to the amplitude for backward π^+p scattering. [This background is much reduced for π^-p scattering by cancellation between the $T=\frac{1}{2}$ and $T=\frac{3}{2}$ resonance amplitudes.] Since the resonance terms contribute significantly to the scattering amplitude only for $j\sim\alpha$, this unphysical background will be absent in the modified RPR model.

(ii) A similar background problem exists in some calculations of the $\pi^-p\rightarrow\pi^0n$ charge-exchange scattering[4–7]; the calculated polarization is particularly sensitive to this background. Contributions from very inelastic high-energy resonances are also important for the high-energy polarization.[8,9] The polarization should be recalculated using the modified RPR model including entire Regge trajectories. This would constitute a rather clear test for the explanation of the observed polarization in terms of s-channel resonances.

(iii) The fact that the resonant terms in Eq. (44) are modified by the background phase can lead in principle to apparent shifts in the masses of resonances as observed by different methods, for example, in total cross sections or in differential cross sections or polarizations. The apparent shape of a trajectory may also be changed by these mass shifts; the low-energy break in the N_γ trajectory as determined by Barger and Cline[1] may be a spurious effect of this sort.

(iv) Finally, the parametrization of the resonant amplitudes in terms of the smooth Regge trajectory and residue functions provides a method of extracting information on highly excited states from data on total or differential cross sections which is much more powerful than the standard method using sums of uncorrelated Breit-Wigner terms.

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[41] The rate at which the resonance contributions vanish relative to the Regge-pole exchange amplitudes is not completely clear. For single-channel potential scattering, the extra term in $A(s,t)$ vanishes as $1/\sqrt{s}$ relative to $A_B(s,t)$ [cf. Eqs. (15) and (23)]. However, in the relativistic many-channel problem, the rapid decrease of the factors β in Eq. (44) as new channels open can lead to a much sharper diminution of the resonant contributions at high energies. This is probably the case, for example, for πN scattering (Ref. 14). In other cases, it may be necessary to extract more of the smooth high-energy contribution of the s-channel Regge poles than is done in Eq. (40) before the resonant amplitudes are clearly separated in their energy dependence from the exchange amplitudes for low-lying t-channel Regge poles.

[42] It should be observed in this connection that high-energy scattering amplitudes are generally dominated by the contributions of high partial waves. Since these are determined primarily by the long-range exchanges, they may be expected to satisfy the conditions necessary for the validity of the RPR model. That low partial waves violate these conditions may be relatively unimportant as a practical matter because of their low statistical weights.