PHYSICAL REVIEW VOLUME 166, NUMBER 5 25 FEBRUARY 1968

Representations of the Lie Group of Strong-Coupling Theory and the Procedure of Contraction

S. M. VAIDYA
Tata Institute of Fundamental Research, Bombay, India
(Received 29 August 1967)

A physically interesting representation for baryon isobars for the group of strong-coupling theory $[SU(2)\otimes SU(3)]\times T_{24}$ was obtained by Goebel and others as an induced representation. It is shown that the Goebel representation cannot be obtained by a straightforward contraction on $SU(6)$, but that it can be obtained by contraction on $SU(2)\otimes SU(6)$.

I. INTRODUCTION

RECENTLY, the strong-coupling theory came once again into prominence with emphasis on the Lie group structure of the static models.[1–8] The group of strong coupling is $G=K\times T$, where K is the group of the invariance, e.g., $SU(2)\otimes SU(2)$ or $SU(2)\otimes SU(3)$, etc. and T is the Abelian group generated by the meson source currents which commute in the limit of strong coupling, and various unitary irreducible representations of this Lie group give the possible isobar spectra. The number of isobars for any irreducible representation is infinite since the group concerned is noncompact and therefore all the unitary representations are necessarily infinite-dimensional.

For symmetric pseudoscalar meson theory, as was shown in CGS, the Lie group G is $[SU(2)\otimes SU(2)]\times T_9$. We use $\otimes$ to denote direct product and $\times$ to denote semidirect product, in agreement with CGS. Some representations of this group were obtained by CGS by using the procedure of group contraction on $SU(4)$. The physically interesting nucleon isobar series (B) has $i=j=\frac{1}{2}, \frac{3}{2}, \cdots$ and arises as the limiting sequence of representations of $SU(4)$ characterized by $\{f,0,0\}$ when $f\to\infty$ through odd integers. There is another interesting series (Y), the hyperon isobar series, which has $j=i\pm\frac{1}{2}=\frac{1}{2}, \frac{3}{2}, \cdots$. The representation for the series (Y) was obtained by Singh and Udgaonkar[3] by exploiting the equivalence of the strong coupling with bootstrap condition. The representation for the hyperon series cannot be obtained by a straightforward contraction procedure on $SU(4)$ for the obvious reason that any representation of $SU(4)$ has $|i-j|=$integer. It can, however, be obtained by using contraction on $SU(2)\otimes SU(4)$ as shown by Babu *et al.*[4] Both (B) and (Y) have also been obtained as induced representations.[5–8]

When the group of invariance is enlarged to $SU(2)\otimes SU(3)$ the corresponding group of strong coupling is $[SU(2)\otimes SU(3)]\times T_{24}$. The isobar belong to one of its representations will be characterized by $j(a,b)$ where j is the spin of the isobar and (a,b) denotes the $SU(3)$ multiplet.[9] Of physical interest would be the most economical isobar series which contains in it the nucleon isobar series (B). It would therefore be natural to demand that the $SU(3)$ multiplet (a,b) has the appropriate nucleon isobar in it, i.e., $j(a,b)$ be such that it has an $i=j$, $Y=1$ member with $j=$half-odd integer. Written out explicitly, only those $j(a,b)$ occur for which $j_{\max}\geqslant j\geqslant j_{\min}$, where

$$j_{\max}=\tfrac{1}{3}(a+2b)+\tfrac{1}{2} \quad \text{if} \quad a>b$$
$$=\tfrac{1}{3}(2a+b)-\tfrac{1}{2} \quad \text{if} \quad a\leqslant b,$$

and

$$j_{\min}=\tfrac{1}{3}(a-b)+\tfrac{1}{2} \quad \text{if} \quad a\geqslant b$$
$$=\tfrac{1}{3}(b-a)-\tfrac{1}{2} \quad \text{if} \quad a<b.$$

We would further like all $SU(3)$ multiplets (a,b) with $(a-b)\equiv 0 \bmod 3$ to be represented once and only once with the exception of (0,0) the $SU(3)$ singlet, which cannot have an $i=j$ member with $j=$half-odd integer. We denote such a representation by P. Such a representation was in fact obtained by Goebel[5] as an induced representation. (The sequence was also conjectured by Capps[10] from a bootstrap argument.) The question we would like to ask is whether it is possible to obtain the isobar series (P) by contraction on $SU(6)$. We answer the question in the negative and then show how this series can be obtained by contraction on $SU(2)\otimes SU(6)$.

II. CONTRACTION OF $SU(6)$ AND $SU(2)\otimes SU(6)$ TO $[SU(2)\otimes SU(3)]\times T_{24}$

The generators of $SU(6)$ satisfy[11] (with $i, j=1, \cdots 3$ and $\alpha, \beta, \gamma=1, \cdots 8$)

$$[J_i,J_j]=i\varepsilon_{ijk}J_k, \tag{1}$$

$$[J_i,F_\alpha]=0, \tag{2}$$

$$[F_\alpha,F_\beta]=if_{\alpha\beta\gamma}F_\gamma, \tag{3}$$

[1] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965), to be referred to as CGS.
[2] V. Singh, Phys. Rev. **144**, 1275 (1966).
[3] V. Singh and B. M. Udgaonkar, Phys. Rev. **149**, 1164 (1966).
[4] P. Babu, A. Rangwala, and V. Singh, Phys. Rev. **157**, 1322 (1967).
[5] C. J. Goebel, Phys. Rev. Letters **16**, 1130 (1966).
[6] T. Cook and B. Sakita, J. Math. Phys. **8**, 708 (1967).
[7] C. J. Goebel, in Proceedings of the 1965 Midwest Conference on Theoretical Physics (unpublished).
[8] C. J. Goebel, in *Non-Compact Groups in Particle Physics*, edited by Y. Chow (W. A. Benjamin, Inc., New York, 1966).

[9] For $SU(3)$ we shall always use the (p,q) notation which corresponds to the Young diagram $\{p+q, q\}$. For $SU(6)$, however, we use the Young diagram notation. For $SU(2)$ we just give the spin.
[10] R. Capps, Phys. Rev. Letters **13**, 536 (1964).
[11] See, for example, A. Pais, Rev. Mod. Phys. **38**, 215 (1966).

$$[J_i,A_{j\alpha}]=i\varepsilon_{ijk}A_{k\alpha}\,, \tag{4}$$

$$[F_\alpha,A_{i\beta}]=if_{\alpha\beta\gamma}A_{i\gamma}\,, \tag{5}$$

$$[A_{i\alpha},A_{j\beta}]=\tfrac{1}{2}\delta_{ij}f_{\alpha\beta\gamma}F_\gamma+i\varepsilon_{ijk}(\tfrac{1}{3}\delta_{\alpha\beta}J_k+d_{\alpha\beta\gamma}A_{k\gamma})\,. \tag{6}$$

If we now introduce $A_{i\alpha}=\mathcal{A}_{i\alpha}/\epsilon$ and take the limit as $\epsilon\to 0$ keeping $\mathcal{A}_{i\alpha}$ finite, we obtain the algebra of $[SU(2)\otimes SU(3)]\times T_{24}$. Relations (1)–(3) remain unchanged, while in relations (4) and (5) $A_{i\alpha}$ is changed to $\mathcal{A}_{i\alpha}$. Relation (6) changes to

$$[\mathcal{A}_{i\alpha},\mathcal{A}_{j\beta}]=\epsilon^2(\tfrac{1}{2}i)\delta_{ij}f_{\alpha\beta\gamma}F_\gamma+i\epsilon^2\varepsilon_{ijk}(\tfrac{1}{3}\delta_{\alpha\beta}J_k)+i\epsilon\varepsilon_{ijk}d_{\alpha\beta\gamma}\mathcal{A}_{k\gamma}\underset{\epsilon\to 0}{\longrightarrow}0\,.$$

Consider[12] now $SU(2)\otimes SU(6)$. We add three more generators J_i' to the set (1)–(6) such that

$$[J_i',J_j']=i\varepsilon_{ijk}J_k'\,, \tag{7}$$

$$[J_i',J_j]=0\,, \tag{8}$$

$$[J_i',F_\alpha]=0\,, \tag{9}$$

$$[J_i',A_{j\alpha}]=0\,. \tag{10}$$

Let us introduce $\mathcal{J}_i=J_i+J_i'$. Clearly,

$$[\mathcal{J}_i,\mathcal{J}_j]=i\varepsilon_{ijk}\mathcal{J}_k\,, \tag{11}$$

$$[\mathcal{J}_i,F_\alpha]=0\,, \tag{12}$$

$$[\mathcal{J}_i,A_{j\alpha}]=[J_i+J_i',\,A_{j\alpha}]=i\varepsilon_{ijk}A_{k\alpha}\,, \tag{13}$$

$$[A_{i\alpha},A_{j\beta}]=\tfrac{1}{2}i\delta_{ij}f_{\alpha\beta\gamma}F_\gamma+i\varepsilon_{ijk}d_{\alpha\beta\gamma}A_{k\gamma}+i\mathcal{E}_{ijk}\tfrac{1}{3}\delta_{\alpha\beta}\mathcal{J}_k-\mathcal{J}_k')\,.$$

Taking the limit $\epsilon\to 0$ with $A_{i\alpha}=\mathcal{A}_{i\alpha}/\epsilon$, where $\mathcal{A}_{i\alpha}$ is finite, we again obtain the Lie algebra of the strong-coupling group, $[SU(2)\otimes SU(3)]\times T_{24}$.

In order that the representation of $[SU(2)\otimes SU(3)]\times T_{24}$ so obtained by contraction from $SU(6)$ [or from $SU(2)\otimes SU(6)$] be faithful we must, of course, take sequence of representations of $SU(6)$ [or of $SU(2)\otimes SU(6)$] of higher and higher dimensions and finally take the limit as the dimension goes to infinity.[13] The question we now ask is whether there exists a sequence of representstions of $SU(6)$ [or $SU(2)\otimes SU(6)$] such that in the limit it provides the representation P.

III. IMPOSSIBILITY OF OBTAINING REPRESENTATION "P" BY CONTRACTION FROM $SU(6)$

Let $\{f_1,f_2,f_3,f_4,f_5\}$ characterize a particular representation of $SU(6)$. In this section we shall show that for any choices of f_1, f_2, f_3, f_4, f_5 (except $\{3,0,0,0,0\}$) this representation contains constituents $j(a,b)$ which violate the condition (C):

$$j_{\max}\geqslant j\geqslant j_{\min}\quad (C)\,,$$

where $j_{\max}$ and $j_{\min}$ are given in Sec. I.

In order to prove the above assertion, we have to consider the $j(a,b)$ decomposition of any representation $\{f_1,f_2,f_3,f_4,f_5\}$ of $SU(6)$. Fortunately, however, we do not need to carry out the complete decomposition, for it suffices to consider only the constituent of $\{f_1,f_2,f_3,f_4,f_5\}$ with the highest spin. This is given by[14]

$$J=\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5)\,,\quad (a,b)=(f_1-f_2,\ f_2-f_3)\otimes(f_4-f_5,\ f_5)\,.$$

There are six cases to discuss and we consider them one by one.

Case 1: $f_1\neq 0$, $f_2=f_3=f_4=f_5=0$. Here

$$J=\tfrac{1}{2}f_1\,,\quad (a,b)=(f_1,0)\otimes(0,0)=(f_1,0)\,.$$

Now $\frac{1}{2}f_1$ $(f_1,0)$ satises condition (C) if

$$j_{\max}=j_{\min}=\tfrac{1}{6}(2f_1+3)=J=\tfrac{1}{2}f_1\,,$$

i.e., if $f_1=3$. Thus only $\{3,0,0,0,0\}$ can give rise to constituents which satisfy the condition (C). In fact, the decomposition in this case is the familiar one: $\{3,0,0,0,0\}=\frac{3}{2}(3,0)\oplus\frac{1}{2}(1,1)$ and the condition (C) is clearly satisfied.

Case 2: $f_1\geqslant f_2\neq 0$, $f_3=f_4=f_5=0$.

$$J=\tfrac{1}{2}(f_1+f_2)\,,\quad (a,b)=(f_1-f_2,\ f_2)\otimes(0,0)=(f_1-f_2,\ f_2)\,.$$

Now we have to consider separately the three possibilities $f_1-f_2>f_2$, $f_1-f_2=f_2$, and $f_1-f_2<f_2$.

(a) If $f_1-f_2>f_2$, $j_{\max}=\frac{1}{3}[(f_1-f_2)+2f_2]+\frac{1}{2}$ and condition (C) fails unless $\frac{1}{3}[(f_1-f_2)+2f_2]\geqslant\frac{1}{2}(f_1+f_2)$, i.e., $3\geqslant f_1+2f_2>4f_2$, which is impossible.

(b) If $f_1-f_2=f_2$, $j_{\max}=f_2-\frac{1}{2}$ and condition (C) fails unless $\frac{1}{2}(2f_2-1)\geqslant\frac{1}{2}(f_1+f_2)$, i.e., $0\geqslant f_2+1$, which is impossible.

(c) If $f_1-f_2<f_2$, $j_{\max}=\frac{1}{3}[f_2+2(f_1-f_2)]-\frac{1}{2}$ and again condition (C) fails unless $\frac{1}{3}[f_2+2(f_1-f_2)]-\frac{1}{2}\geqslant\frac{1}{2}(f_1+f_2)$, i.e., $f_1\geqslant 5f_2+3$, which is impossible since $f_1<2f_2$.

Case 3: $f_1\geqslant f_2\geqslant f_3\neq 0$, $f_4=f_5=0$.

$$J=\tfrac{1}{2}(f_1+f_2+f_3)\,,\quad (a,b)=(f_1-f_2,\ f_2-f_3)\otimes(0,0)=(f_1-f_2,\ f_2-f_3)\,.$$

Again we have to discuss separately the three possibilities $f_1-f_2>f_2-f_3$, $f_1-f_2=f_2-f_3$, and $f_1-f_2<f_2-f_3$.

(a) If $f_1-f_2>f_2-f_3$,

$$j_{\max}=\tfrac{1}{3}[(f_1-f_2)+2(f_2-f_3)]+\tfrac{1}{2}$$

and condition (C) fails unless $j_{\max}\geqslant\frac{1}{2}(f_1+f_2+f_3)$, i.e., $3\geqslant f_1+f_2+7f_3$, which is impossible.

[12] $SU(6)\times O(3)$ was considered in a different context by K. T. Mahanthappa and E. C. G. Sudarshan, Phys. Rev. Letters **14**, 163 (1965).

[13] Innonu and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **39**, 510 (1953).

[14] A. M. Perelomov, V. S. Popov, and I. A. Malkin, Yadernaya Fiz. **1**, 533 (1965) [English transl.: Soviet J. Nucl. Phys. **2**, 382 (1966)].

(b) If $f_1-f_2=f_2-f_3$, $j_{\max}=(f_1-f_2)-\frac{1}{2}$ and condition (C) fails unless $j_{\max}\geqslant\frac{1}{3}(f_1+f_2+f_3)$, i.e., $0\geqslant f_2+2f_3+1$, which is impossible.

(c) If $f_1-f_2<f_2-f_3$,

$$j_{\max}=\tfrac{1}{3}[2(f_1-f_2)+(f_2-f_3)]-\tfrac{1}{2}$$

and condition (C) fails unless $j_{\max}\geqslant\frac{1}{2}(f_1+f_2+f_3)$, i.e., $2f_2-f_3>f_1\geqslant 5f_2+5f_3+3$ or $0>3f_2+6f_3+3$, which is impossible.

Case 4: $f_1\geqslant f_2\geqslant f_3\geqslant f_4\neq 0$, $f_5=0$.

$$J=\tfrac{1}{2}(f_1+f_2+f_3-f_4),$$
$$(a,b)=(f_1-f_2,\ f_2-f_3)\otimes(f_4,0)$$
$$=(f_1-f_2+f_4,\ f_2-f_3)+\text{other terms if any.}$$

We again consider three possibilities:

$$f_1-f_2+f_4(>,=,<)f_2-f_3.$$

(a) If $f_1-f_2+f_4>f_2-f_3$,

$$j_{\max}=\tfrac{1}{3}[(f_1-f_2+f_4)+2(f_2-f_3)]+\tfrac{1}{2}$$

and condition (C) fails unless $j_{\max}\geqslant\frac{1}{2}(f_1+f_2+f_3-f_4)$, i.e., $3\geqslant f_1+f_2+7f_3-5f_4\geqslant f_1+f_2+2f_3$, which is impossible.

(b) If $f_1-f_2+f_4=f_2-f_3$, $j_{\max}=(f_2-f_3)-\frac{1}{2}$ and condition (C) fails unless $j_{\max}\geqslant\frac{1}{2}(f_1+f_2+f_3-f_4)$, i.e., $0\geqslant f_1-f_2+3f_3-f_4+1$ or $0\geqslant f_2+2f_3-2f_4+1\geqslant f_2+1$.

(c) If $f_1-f_2+f_4<f_2-f_3$,

$$j_{\max}=\tfrac{1}{3}[(f_2-f_3)+2(f_1-f_2+f_4)]-\tfrac{1}{2}$$

and condition (C) fails unless $j_{\max}\geqslant\frac{1}{2}(f_1+f_2+f_3-f_4)$, i.e., $2f_2-f_3-f_4>f_1\geqslant 5f_2+5f_3+3-7f_4$ or $0>3f_2+6f_3-6f_4+3\geqslant 3f_2+3$, which is impossible.

Case 5: $f_1=f_2\geqslant f_3\geqslant f_4\geqslant f_5\neq 0$.

$$J=\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5),$$
$$(a,b)=(0,\ f_2-f_3)\otimes(f_4-f_5,\ f_5)$$
$$=(f_4-f_5,\ f_2-f_3+f_5)+\text{other terms if any.}$$

Here again the three possibilities $f_4-f_5(>,=,<)f_2-f_3+f_5$ must be discussed separately.

(a) If $f_4-f_5>f_2-f_3+f_5$,

$$j_{\max}=\tfrac{1}{3}[(f_4-f_5)+2(f_2-f_3+f_5)]+\tfrac{1}{2}$$

and condition (C) fails unless

$$j_{\max}\geqslant\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5)$$

i.e., $5(f_3+f_4)>5(f_2+2f_5)\geqslant 9f_2+14f_3-10f_4-6$ or $15f_4+6>9f_2+9f_3$, i.e., $2>f_2$. This is impossible since the only choice compatible with this is $f_1=f_2=f_3=f_4=f_5=1$, which contradicts $f_4-f_5>f_2-f_3+f_5$.

(b) If $f_4-f_5=f_2-f_3+f_5$, $j_{\max}=(f_4-f_5)-\frac{1}{2}$ and condition (C) fails unless $j_{\max}\geqslant\frac{1}{2}(f_1+f_2+f_3-f_4-f_5)$, i.e., $4f_4\geqslant 4f_2+2f_3+1$ which is impossible since

$$f_2\geqslant f_3\geqslant f_4.$$

(c) If $f_4-f_5<f_2-f_3+f_5$,

$$j_{\max}=\tfrac{1}{3}[2(f_4-f_5)+(f_2-f_3+f_5)]-\tfrac{1}{2}$$

and condition (C) fails unless

$$j_{\max}\geqslant\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5),$$

i.e., $5f_4+2f_5\geqslant 5f_2+4f_3+3$ or $0>2f_2+3$, which is also impossible.

Case 6: $f_1>f_2\geqslant f_3\geqslant f_4\geqslant f_5\neq 0$

$$J=\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5),$$
$$(a,b)=(f_1-f_2,\ f_2-f_3)\otimes(f_4-f_5,\ f_5)$$
$$=(f_1-f_2+f_4-f_5,\ f_2-f_3+f_5)$$
$$\oplus(f_1-f_2+f_4-f_5-1,\ f_2-f_3+f_5-1)$$
$$+\text{other terms if any.}$$

We shall now show that $\frac{1}{2}(f_1+f_2+f_3-f_4-f_5)(f_1-f_2+f_4-f_5-1,\ f_2-f_3+f_5-1)$ does not satisfy condition (C). As before we discuss the three possibilities $f_1-f_2+f_4-f_5-1(>,=,<)f_2-f_3+f_5-1$ separately.

(a) If $f_1-f_2+f_4-f_5-1>f_2-f_3+f_5-1$,

$$j_{\max}=\tfrac{1}{3}[2(f_2-f_3+f_5-1)+(f_1-f_2+f_4-f_5-1)]+\tfrac{1}{2}$$

and condition (C) fails unless

$$j_{\max}\geqslant\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5),$$

i.e., $5f_4+5f_5-f_2-7f_3-3\geqslant f_1>2f_2-f_3-f_4+2f_5$ or $6f_4+3f_5>3f_2+6f_3+3$, which is impossible.

(b) If $f_1-f_2+f_4-f_5-1=f_2-f_3+f_5-1$,

$$j_{\max}=(f_2-f_3+f_5-1)-\tfrac{1}{2}$$

and condition (C) fails unless

$$j_{\max}\geqslant\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5),$$

i.e., $2f_4\geqslant 2f_2+f_5+2$, which is impossible.

(c) If $f_1-f_2+f_4-f_5-1<f_2-f_3+f_5-1$,

$$j_{\max}=\tfrac{1}{3}[(f_2-f_3+f_5-1)+2(f_1-f_2+f_4-f_5-1)]-\tfrac{1}{2}$$

and condition (C) fails unless

$$j_{\max}\geqslant\tfrac{1}{2}(f_1+f_2+f_3-f_4-f_5),$$

i.e., $2f_2-f_3-f_4+2f_5>f_1\geqslant 5f_2-5f_3-7f_4-f_4+9$ or $6f_4+3f_5>3f_2+6f_3+9$, which is impossible.

This completes all the possible cases.

IV. REPRESENTATION P BY CONTRACTION FROM $SU(2)\otimes SU(6)$

We have seen in Sec. III that no sequence of representations of $SU(6)$ can contract to produce the representation (P) of $[SU(2)\otimes SU(3)]\times T_{24}$. The reason for this was that every representation of $SU(6)$ (except $\{3,0,0,0,0\}$, i.e., $\{2n+3, n, n, n, n\}$ with $n=0$) contained constituents which violated the condition (C). Consider now the representation given by $\{2n+3, n, n, n, n\}$. The constituents with the highest spin here are

$$\tfrac{1}{2}(2n+3)(n+3,0)\otimes(0,n)=\tfrac{1}{2}(2n+3)\sum_{i=0}^{n}(n+3-i,n-i).$$

Though $\frac{1}{2}(2n+3)(n+3,\ n)$ satisfies the condition (C) and hence is an admissible constituent for (P), none of the other constituents, $\frac{1}{2}(2n+3)\sum_{i=1}^{n}(n+3-i,\ n-i)$, satisfies the condition (C). One checks easily that these inadmissible constituents are precisely those which occur in the decomposition of $(1)\{2n+1,\ n-1,\ n-1,\ n-1,\ n-1\}$. Since a representation of $[SU(2)\otimes SU(3)]\times T_{24}$ obtained by contraction from a sequence of representations of $SU(6)$ need not be irreducible, one might hope that a representation obtained by contraction from the sequence $\{2n+3, n, n, n, n\}$ is reducible, and that one irreducible part is just the one obtained by contraction from the sequence $(1)\{2n+1,\ n-1,\ n-1,\ n-1,\ n-1\}$ the remaining irreducible part being just the representation (P). Unfortunately this is not the case and one has to consider more complicated combinations of representations of $SU(6)$ in order to recover the isobar series of interest to us.

Consider the following reducible (direct sum) representation σ_n of $SU(6)$.

$$\sigma_n\colon \{2n+3, n, n, n, n\}\oplus\{2n+1, n-1, n-1, n-1, n-1\}\oplus\{2n-1, n-2, n-2, n-2, n-2\}.$$

Let us denote by Σ_n the representation $(1)\{2n+1,\ n-1,\ n-1,\ n-1,\ n-1\}$ of $SU(2)\otimes SU(6)$. Let us denote by $\tilde{\sigma}_n$, the reduction of σ_n with respect to $SU(2)\otimes SU(3)$, and by $\tilde{\Sigma}_n$, the reduction of Σ_n with respect to $SU(2)\otimes SU(3)$. We can show that (see Appendix for a sketch of the derivation of this result)

$$\tilde{\sigma}_n=\tilde{\Sigma}_n+\tilde{R}_n,$$

where

$$\begin{aligned}\tilde{R}_n=&\sum_{m=0}^{I(\frac{1}{2}n)}\sum_{i=0}^{m}\sum_{r=0}^{2i+1}\tfrac{1}{2}[2(m+i-r)+3](3m+3-i,\ 2i)\\&+\sum_{m=0}^{I(\frac{1}{2}(n-1))}\sum_{i=0}^{m}\sum_{r=0}^{2i+2}\tfrac{1}{2}[2(m+i-r)+5](3m+4-i,\ 2i+1)\\&+\sum_{i=1}^{n+1}\sum_{r=0}^{i-1}\tfrac{1}{2}[2(i-r)-1](i,i)\\&+\sum_{m=1}^{I(\frac{1}{2}(n+2))-1}\sum_{i=0}^{m-1}\sum_{r=0}^{2(m+i)-1}\tfrac{1}{2}[2(2m-i-r)-1]\\&\times(2m-1-2i,\ 2m+2+i)\\&+\sum_{m=1}^{I(\frac{1}{2}(n+1))}\sum_{i=0}^{m-1}\sum_{r=0}^{2(m+i)-2}\tfrac{1}{2}[2(2m-i-r)-3]\\&\times(2m-2-2i,\ 2m+1+i),\end{aligned}$$

where $I(x)\equiv$ integral part of x.

Clearly, as shown in Sec. II, both σ_n and Σ_n will converge to a representation of $[SU(2)\otimes SU(3)]\times T_{24}$. In the limit $n\to\infty$, since both $\tilde{\sigma}_n$ and $\tilde{\Sigma}_n$ are basis for the representation of $[SU(2)\otimes SU(3)]\times T_{24}$, $\tilde{R}_n$ is also a basis for the representation of $[SU(2)\otimes SU(3)]\times T_{24}$. We can easily see that $\tilde{R}_n$ is nothing but the basis for the representation P which was obtained by Goebel. For example consider, the first term in $\tilde{R}_n$, viz.,

$$\sum_{m=0}^{I(\frac{1}{2}n)}\sum_{i=0}^{m}\sum_{r=0}^{2i+1}\tfrac{1}{3}[2(m+i-r)+3](3m+3-i,\ 2i).$$

Here it is clear that $3m+3-i>2i$. Hence

$$j_{\max}=\tfrac{1}{3}[(3m+3-i)+2(2i)]+\tfrac{1}{2}=\tfrac{1}{2}[2(m+i)+3]$$

which corresponds to $r=0$ in the above sum and

$$j_{\min}=\tfrac{1}{3}[(3m+3-i)-(2i)]-\tfrac{1}{2}=\tfrac{1}{2}[2(m-i)+1]$$

which corresponds to $r=2i+1$ in the above sum. The condition (C) is thus satisfied for this term. It is not difficult to see that the remaining four terms also satisfy the condition (C). It is also not difficult to see that in $\tilde{R}_n$ (as $n\to\infty$) each (a,b) such that $(a-b)=0 \bmod 3$ is present once and only once with the exception of (0,0), which does not occur. First of all it is clear that for all the terms $(a-b)=0 \bmod 3$. Further, the first term contains all $a>b$ with b even, and the second term contains all $a>b$ with b odd. The third term gives all $a=b$, while the fourth term gives all $a<b$ with a odd, and the last term gives all $a<b$ with a even.

ACKNOWLEDGMENTS

Sincere thanks are due to Professor B. M. Udgaonkar and Dr. A. Rangwala for numerous discussions.

APPENDIX

We sketch a derivation of the result

$$\tilde{\sigma}_n=\tilde{\Sigma}_n+\tilde{R}_n.$$

We have

$$\begin{aligned}\sigma_n=&\{2n+3, n, n, n, n\}\\&\otimes\{2n+1, n-1, n-1, n-1, n-1\}\\&\oplus\{2n-1, n-2, n-2, n-2\}.\end{aligned}$$

Denoting the representation conjugate to $\{f_1,f_2,f_3,f_4,f_5\}$ by $\{f_1,f_2,f_3,f_4,f_5\}^*$, we note that

$$\begin{aligned}&\{n+3,\ 0,\ 0,\ 0,\ 0\}\otimes\{n,0,0,0,0\}^*\\&=\{n+3,\ 0,\ 0,\ 0,\ 0\}\otimes\{n,n,n,n,n\}\\&=\{2n+3,\ n,\ n,\ n,\ n\}\\&\quad+\{2n+1,\ n-1,\ n-1,\ n-1,\ n-1\}\\&\quad+\cdots+\{3,0,0,0,0\}.\end{aligned}$$

Hence

$$\begin{aligned}S_n\equiv&\{2n+3,\ n,\ n,\ n,\ n\}\\=&\{n+3,\ 0,\ 0,\ 0,\ 0\}\otimes\{n,0,0,0,0\}^*\\&-\{n+2,\ 0,\ 0,\ 0,\ 0\}\otimes\{n-1,\ 0,\ 0,\ 0,\ 0\}^*.\end{aligned}$$

Let us now denote by $\tilde{S}_n$ the decomposition of S_n with respect to $SU(2)\otimes SU(3)$. Then using the notation of

Hagen and Macfarlane,[15] $\tilde{S}_n$ can be given in terms of homogeneous product sums "h" appropriate to $SU(6)$:

$$\tilde{S}_n = h_{n+3}h_n^* - h_{n+2}h_{n-1}^*,$$

where[15]

$$h_n = \sum_{k=0}^{I(\frac{1}{2}n)} \tfrac{1}{2}(n-2k)(n-2k,\, k)$$

and

$$h_n^* = \sum_{k=0}^{I(\frac{1}{2}n)} \tfrac{1}{2}(n-2k)(k,\, n-2k),$$

$$I(x) = \text{integral part of } x.$$

Thus

$$\tilde{S}_n = \sum_{k=0}^{I(\frac{1}{2}(n+3))} \tfrac{1}{2}(n+3-2k)(n+3-2k,\, k)$$

$$\otimes \sum_{l=0}^{I(\frac{1}{2}n)} \tfrac{1}{2}(n-2l)(l,\, n-2l) - \sum_{k=0}^{I(\frac{1}{2}(n+2))} \tfrac{1}{2}(n+2-2k)$$

$$\times (n+2-2k,\, k) \otimes \sum_{l=0}^{I(\frac{1}{2}(n-1))} \tfrac{1}{2}(n-1-2l)(l,\, n-1-2l).$$

[15] C. R. Hagen and A. J. Macfarlane, J. Math. Phys. 6, 1355 (1965).

The reduction of the $SU(2)$ part, e.g., $\frac{1}{2}(n+3-2k) \otimes \frac{1}{2}(n-2)$ causes no problem. The reduction of the $SU(3)$ part can be achieved by using the elegant formula of Coleman[16] which we quote below.

$$(n,m)\otimes(n',m') = \sum_{i=0}^{\min(n,m')} \sum_{j=0}^{\min(m,n')} (n-i, n'-j; m-j, m'-i),$$

where

$$(n,n'; m,m') = (n+n',\, m+m') \oplus \sum_{i=1}^{\min(n,n')} (n+n'-2i,\, m+m'+i) + \sum_{j=1}^{\min(m,m')} (n+n'+i,\, m+m'-2i).$$

Proceeding in this fashion, one obtains $\tilde{\sigma}_n$ and $\tilde{\Sigma}_n$. After a straightforward but very lengthy and laborious calculation, one then obtains the expression $\tilde{R}_n$ as quoted.

[16] S. Coleman, in *Proceedings of the Seminar in High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965).

PHYSICAL REVIEW VOLUME 166, NUMBER 5 25 FEBRUARY 1968

Regge-Pole Exchange and Direct-Channel Resonances in Models for High-Energy Scattering Amplitudes*

LOYAL DURAND III
Department of Physics, University of Wisconsin, Madison, Wisconsin

(Received 28 September 1967)

The behavior of the forward and backward πN scattering amplitudes for momenta of 1–5 BeV/c has been analyzed recently using models in which Breit-Wigner amplitudes describing direct-channel resonances are added to a background amplitude given by the Regge-pole-exchange model. Although remarkably successful in practice, the model has severe theoretical limitations, especially with regard to the treatment of the tails of the resonant terms, double counting of the background contributions, and the restriction to the Breit-Wigner approximation for sets of isolated resonances. The theory of Regge-pole–plus–resonance (RPR) models is examined in detail for both single-channel potential scattering and the many-channel relativistic case. A modified RPR model is developed in which (i) the double-counting problems are eliminated, and (ii) direct-channel resonances are described in terms of their Regge-trajectory functions. There is no difficulty with the tails of the resonant amplitudes in this formulation of the RPR model. Moreover, the contributions of the entire set of resonances on a given Regge trajectory can be included in the scattering amplitude. The relevance of these modifications of the RPR model to past analyses of πN scattering is discussed briefly.

I. INTRODUCTION

It has become clear in the past year that a remarkably successful description of πN scattering for laboratory momenta of 1–5 BeV/c can be obtained by adding appropriate direct-channel resonance terms to the Regge-pole-exchange amplitudes deduced from fits to high-energy scattering cross sections. The rationale for such models is simple: the Regge-pole-exchange amplitude is used to represent the smooth average behavior of the complete amplitudes, while the resonant terms take into account the large deviations of a few partial-wave amplitudes from that average behavior. Models

* Work supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-881, No. COO-881-120.