Absorption Model and $U(6,6)$ for the Reactions $p\bar{p} \to Y\bar{Y}$†

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(Received 29 May 1967; revised manuscript received 3 November 1967)

The $U(6.6)$ symmetry is used to write down peripheral amplitudes for the processes $p\bar{p} \to (K,K^*)Y\bar{Y}$. The requirements of unitarity are approximately enforced on the amplitudes so derived by the inclusion of absorptive corrections. Assuming a value of $G_{\pi NN}^2/4\pi$ and adopting a prescription for dealing with the mass splitting in $U(6,6)$, we find satisfactory agreement for the angular distributions in all cases and obtain satisfactory absolute values for $\Lambda\bar{\Lambda}$ and $\Sigma^+\bar{\Sigma}^-$, but not $\Lambda\bar{\Sigma}^0$. The energy variation of the cross section is satisfactorily reproduced for $\Lambda\bar{\Lambda}$ and $\Sigma^+\bar{\Sigma}^-$ in the intermediate momentum range from 3.0 to 5.7 GeV/c; but not for $\Lambda\bar{\Sigma}^0$.

I. INTRODUCTION

IN this work we are primarily interested in ($\frac{1}{2}^+$) hyperon-antihyperon pair production from proton-antiproton interactions at high energy. There are six such $Y\bar{Y}$ channels:

$$\Lambda\bar{\Lambda},\ \Lambda\bar{\Sigma}^0,\ \Sigma^+\bar{\Sigma}^-,\ \Sigma^0\bar{\Sigma}^0,\ \Sigma^-\bar{\Sigma}^+,\ \Xi^-\bar{\Xi}^+.$$

Experimental information on these reactions is available at various energies from the CERN[1] and Brookhaven[2] groups between 3.0- and 7.0-GeV/c incident antiproton momenta.

The most striking feature of the observations is the extreme forward peaking of the angular distributions. Typically, the majority of the events are concentrated in the region $1.0 > \cos\theta > 0.8$, where θ is the center-of-mass $\bar{p},\bar{Y}$ scattering angle. This feature is very suggestive of a peripheral production mechanism, and further evidence for this view may be obtained from an examination of the various production cross sections.

The $\Sigma^-\bar{\Sigma}^+$ and $\Xi^-\bar{\Xi}^+$ cross sections are very much smaller than the others, and these two configurations are precisely those which cannot be reached from the initial state with the exchange of an $I=\frac{1}{2}$ $S=1$ meson. We are led therefore to consider the production of $\Lambda\bar{\Lambda}$, $\Lambda\bar{\Sigma}^0$, $\Sigma^+\bar{\Sigma}^-$, and $\Sigma^0\bar{\Sigma}^0$ pairs by the exchange of either $K(0^-)$ or $K^*(1^-)$.

The possibility of constructing a peripheral model for these processes has been discussed by various authors: (i) Early pole calculations[3,4] pointed to K as the intermediate particle. (The amplitude for K exchange approximates to the form $t/(m_K^2-t)$, where t is the square of the four-momentum transfer and $t<0$ for the physical region; this clearly represents a backward peaking, and was rejected as in conflict with experiment.) (ii) Durand and Chiu[5] drew attention to the importance of including corrections for the effect of absorption into competing channels in the initial and final states. (A more complete discussion of these "absorptive effects" is given in Sec. III.) These authors include these effects in a discussion of $\bar{p}p \to \Lambda\bar{\Lambda}$, and confirmed the dominant role of K^* exchange against K exchange, finding the latter to be *too* forwardly peaked. (iii) Cohen-Tannoudji and Navelet[6] repeated the calculation of Durand and Chiu, taking spin fully into account. In this case, K exchange was found to be *insufficiently* forwardly peaked. (iv) All the above investigations assumed a γ_μ-type coupling at the $K^*\bar{N}\Lambda$ vertex. Hogaasen and Hogaasen[7] generalized the K^*-exchange absorption model to include an admixture of a Pauli-type term $\sigma_{\mu\nu}q_{\nu'}$ and concluded that if the Pauli-type term was more than approximately the same size as the Dirac-type term, the model would no longer fit the data.

All of these calculations admit of a contribution from K exchange.

This discussion illustrates the considerable freedom of maneuver one has in peripheral calculations, which arises from three main sources: (i) there is often a wide variety of particles to be considered as intermediary; (ii) there are often alternative couplings possible at the vertices; and (iii) the values of coupling constants which enter the calculations (e.g., $g_{K\Lambda N}$, $g_{K\Sigma N}$, $g_{K^*\Lambda N}$, $g_{K^*\Sigma N}$) may usually be chosen freely.

Rather than use an arbitrary mixture of K- and K^*-exchange matrix elements which would be varied at will to fit the experimental data, it seems preferable to

† The research reported in the document has been sponsored in part by the Air Force Office of Scientific Research OAR through the European Office Aerospace Research, U. S. Air Force.

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[1] B. Musgrave *et al.*, Nuovo Cimento **35**, 735 (1965); R. Bock *et al.*, in *Proceedings of the Twelfth Annual Conference on High-Energy Physics, Dubna, 1964* (Atomizadt, Moscow, 1965).

[2] C. Baltay *et al.*, Stanford Nuclear Structure Meeting, 1963 (unpublished); C. Baltay *et al.*, in *Proceedings of the Twelfth Annual Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965); Phys. Rev. **140**, B1027 (1965).

[3] H. D. D. Watson, Nuovo Cimento **29**, 1338 (1963).

[4] D. Bessis, C. Itzykson, and M. Jacob, Nuovo Cimento **27**, 376 (1963).

[5] L. Durand, III, and Yam Tsi Chiu, Phys. Rev. Letters **12**, 399 (1964).

[6] G. Cohen-Tannoudji and H. Navelet, Nuovo Cimento **37**, 1511 (1965).

[7] H. Hogaasen and J. Hogaasen, Nuovo Cimento **40**, 560 (1965).

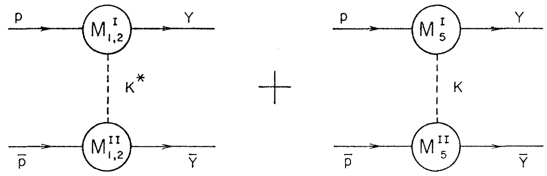

FIG. 1. The one-particle-exchange diagrams for the annihilation process $p+\bar{p} \rightarrow Y+\bar{Y}$.

assume some higher symmetry in which the Lagrangian and the coupling constants are *uniquely* determined, and to compare the model against experiment. In this work we assume the $U(6,6)$ symmetry,[8] which is not in conflict with physical unitarity, and we impose unitarity requirements in terms of the absorption model. Earlier calculations by the authors[9] on the $\Lambda\bar{\Lambda}$ channel at high momentum (3.7 GeV/c) have given a fair description of the magnitude and angular distribution of the differential cross section. As we shall see, the form of the Y-N interaction predicted by $U(6,6)$ is realistic and we expect the main features of this interaction to be preserved in any alternative theory.

II. $U(6,6)$ INTERACTION

The $U(6,6)$ interaction Lagrangian is given by

$$L=G(J_5\varphi_5+J_\mu\varphi_\mu)\,, \tag{2.1}$$

where φ_5 and φ_μ are the pseudoscalar and vector nonets and G is the "$U(6,6)$ coupling constant." The $U(6,6)$ prediction for those parts of the pseudoscalar and vector currents relevant to the interaction of the eightfold baryons with the 0^- and 1^- mesons are

$$J_5=\left(1+\frac{2m}{S}\right)\frac{P^2}{4m^2}(\bar{N}\gamma_5N)_{D+(2/3)F}\,, \tag{2.2}$$

$$J_\mu=\frac{P_\mu}{2m}\left(1+\frac{q^2}{2mV}\right)(\bar{N}N)_F+\left(1+\frac{2m}{V}\right)\left(\bar{N}\frac{r_\mu}{4m^2}N\right)_{D+(2/3)F}\,, \tag{2.3}$$

where N is the baryon of mass m, q is the momentum transfer, S is the pseudoscalar-meson mass, and V is the vector-meson mass. In the $U(6,6)$ symmetry, $S=V=\mu$ (the "meson mass"), but we wish to keep open the possibility of setting S and V to be different, and of thus splitting the mesons into the pseudoscalar and vector nonets. P_μ and r_μ are the conventional forms for "electric" and "magnetic interactions," rather than the Dirac and Pauli terms, i.e., γ_μ and $\sigma_{\mu\nu}q_\nu$; the coefficients of $P_\mu/2m$ and $r_\mu/4m^2$ are the conventional Sachs[10] form factors F_C and F_M, respectively.

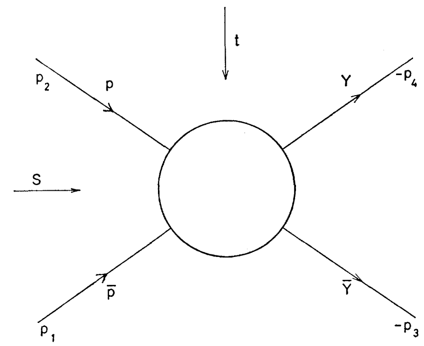

FIG. 2. The production $p+\bar{p}\rightarrow Y+\bar{Y}$ in the s channel, or the elastic scattering $p+\bar{Y}\rightarrow p+\bar{Y}$ in the t channel.

The basis of our model is an amplitude corresponding to Eq. (2.1), which contains a contribution from each of the K and K^* poles (Fig. 1). We see that, apart from the question of the choice of S and V, there is no arbitrariness in the model. [The $U(6,6)$ coupling constant G must be chosen so as to reproduce the known value for $G_{\pi NN}^2/4\pi$.]

It should be emphasized that, *a priori*, one might be hopeful of satisfactory results. First, as is well known, the $U(6,6)$ symmetry is badly broken, but we might hope that, in the present case, the effect of this breaking might not be too important, as we are dealing with reactions only involving particles from the same $SU(3)$ multiplets, namely the baryon octet. The mas differences between p (938 MeV), Λ (1115 MeV), and Σ (1195 MeV) are small, while the difference between the K mass (494 MeV) and that of K^* (891 MeV) is at least much less than the full variation of the meson masses, e.g., from the π (135 MeV) to φ (1019 MeV). Secondly, the $U(6,6)$ predictions for the three-point function are very much better than those for the four-point one [11] and, in the present calculation, we use only $U(6,6)$ expressions for specific vertices.

Finally, the currents as given in Eqs. (2.2) and (2.3) by $U(6,6)$ resemble very much a form which one expects on general grounds. The vector current contains a large magnetic-type term, of the order 3 or 4 times that of the electric term, and there exists considerable evidence for a vector-meson interaction of this form, e.g., for ρ in electromagnetic form-factor data or from the Stodolsky-Sakurai model.[12] Again, any breaking of the symmetry with choices of $S\neq V$ will emphasize the scalar coupling relative to the vector one. We therefore consider the currents given to be of a realistic form.

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[9] H. D. D. Watson and J. H. R. Migneron, Phys. Letters **19**, 424 (1965).

[10] R. G. Sachs, Phys. Rev. **126**, 2256 (1962).

[11] R. Delbourgo *et al.*, *Seminar on High-Energy Physics and Elementary Particles, Trieste* (International Atomic Energy Agency, Vienna, 1965), p. 455.

[12] L. Stodolsky and J. J. Sakurai, Phys. Rev. Letters **11**, 90 (1963).

The explicit form of the currents for the processes under consideration can be found in Appendix A. The matrix element for the diagrams of Fig. 1 with the exchange of pseudoscalar and vector particles [with masses at the physical poles, i.e., $\mu(K)=495$ MeV and $\mu(K^*)=891$ MeV] is given by

$$\begin{aligned}T=&M_1^{\mathrm{II}}(q^2)M_1^{\mathrm{I}}(q^2)\bar{v}(p_1)\frac{P_\mu'}{2m}v(-p_3)\frac{g_{\mu\nu}-q_\mu q_\nu/\mu^2}{\mu^2(K^*)-t}\bar{u}(-p_4)\frac{P_\nu}{2m}u(p_2)\\&+M_1^{\mathrm{II}}(q^2)M_2^{\mathrm{I}}(q^2)\bar{v}(p_1)\frac{P_\mu'}{2m}v(-p_3)\frac{g_{\mu\nu}-q_\mu q_\nu/\mu^2}{\mu^2(K^*)-t}\bar{u}(-p_4)\gamma_\nu u(p_2)\\&+M_2^{\mathrm{II}}(q^2)M_1^{\mathrm{I}}(q^2)\bar{v}(p_1)\gamma_\mu v(-p_3)\frac{g_{\mu\nu}-q_\mu q_\nu/\mu^2}{\mu^2(K^*)-t}\bar{u}(-p_4)\frac{P_\nu}{2m}u(p_2)\\&+M_2^{\mathrm{II}}(q^2)M_2^{\mathrm{I}}(q^2)\bar{v}(p_1)\gamma_\mu v(-p_3)\frac{g_{\mu\nu}-q_\mu q_\nu/\mu^2}{\mu^2(K^*)-t}\bar{u}(-p_4)\frac{V_\nu}{2m}u(p_2)\\&+M_5^{\mathrm{II}}(q^2)M_5^{\mathrm{I}}(q^2)\bar{v}(p_1)\gamma_5 v(-p_3)\frac{1}{\mu^2(K)-t}\bar{u}(-p_4)\gamma_5 u(p_2)\,,\end{aligned}\tag{2.4}$$

where p_1, p_2 and $-p_3$, $-p_4$ are the 4-momenta of the incoming antiproton and proton and outgoing antibaryon and baryon, respectively (Fig. 2). P_μ and P_μ' are the sum of fermion momenta at the particle and antiparticle vertices, respectively. The relation between the F's and the M's is given in Appendix A. The superscripts I and II of the form factors refer to the upper and lower vertices of the diagrams of Fig. 1. We consider m as the average mass of the four baryons in each particular channel: $m_1=m_2=m_3=m_4=m$, i.e., the mass difference between the proton and the other baryons is negligible, as is reasonable at high energy considered in our case. In the center-of-mass system[13]

$$\begin{aligned}s&=(p_1+p_2)^2=4E^2=W^2\,,\\t&=(p_1+p_3)^2=-2p^2(1-\cos\theta)\,,\\u&=(p_1+p_4)^2=-2p^2(1+\cos\theta)\,,\end{aligned}\tag{2.5}$$

where E and p are the energy and momentum of each particle; θ is defined as the scattering angle between the incoming proton and the outgoing baryon.

In order to include absorptive corrections in the diagrams of Fig. 1, the matrix element T is diagonalized in the helicity representation of Jacob and Wick.[14] We define

$$\phi_i=\langle\lambda_B\lambda_{\bar{B}}|T|\lambda_p\lambda_{\bar{p}}\rangle\,,\tag{2.6}$$

where the λ's label the helicities of the particles and the index i specifies the helicity dependence[15] (see Table I). For the $\Lambda\bar{\Lambda}$, $\Sigma^+\bar{\Sigma}^-$, and $\Sigma^0\bar{\Sigma}^0$ channels, the restrictions imposed by parity and charge-conjugation invariance reduce the number of amplitudes from 16 to 6, which we define in Table I. The $\Lambda\bar{\Sigma}^0$ and $\Sigma^0\bar{\Lambda}$ channels, however, are described by a set of 8 independent amplitudes, as charge conjugation only relates the $\Lambda\bar{\Sigma}^0$ and $\Sigma^0\bar{\Lambda}$ channels which are physically distinguishable.[15] Our normalization is such that the differential cross section for unpolarized particles in the initial state is given by

$$\frac{d\sigma}{d\Omega}=\frac{1}{(16\pi E)^2}\tfrac{1}{4}\sum_i|\varphi_i|^2\,,\tag{2.7}$$

where the summation is over the 16 helicity amplitudes.

The next step consists in the evaluation of the independent helicity amplitudes whose general form is given by Eq. (2.4). We choose our coordinate system with the initial antiproton moving along the positive z axis and the outgoing antibaryon moving in the x-z plane at an angle θ to the z axis. The Dirac spinors in the center-of-mass system are written as direct products of Pauli's spinors:

$$\begin{aligned}v_{\lambda_1}&=\frac{1}{(E+m)^{1/2}}\begin{bmatrix}p\\-2\lambda_1(E+m)\end{bmatrix}\chi_{-\lambda_1}\,; & v_{\lambda_3}&=\frac{1}{(E+m)^{1/2}}\begin{bmatrix}p\\-2\lambda_3(E+m)\end{bmatrix}e^{-i\sigma_y\theta/2}\chi_{-\lambda_3}\,,\\u_{\lambda_2}&=\frac{1}{(E+m)^{1/2}}\begin{bmatrix}E+m\\2\lambda_2 p\end{bmatrix}\chi_{-\lambda_2}\,; & u_{\lambda_4}&=\frac{1}{(E+m)^{1/2}}\begin{bmatrix}E+m\\2\lambda_4 p\end{bmatrix}e^{-i\sigma_y\theta/2}\chi_{-\lambda_4}\,,\end{aligned}\tag{2.8}$$

[13] We use the metric $g_{\mu\nu}=(1,\,-1,\,-1,\,-1)$.
[14] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).
[15] The extraneous two amplitudes for the $\Lambda\bar{\Sigma}^0$ process are defined in terms of helicities as:

$$\phi_7=\langle+\tfrac{1}{2}+\tfrac{1}{2}|T|-\tfrac{1}{2}+\tfrac{1}{2}\rangle$$

and

$$\phi_8=\langle+\tfrac{1}{2}-\tfrac{1}{2}|T|-\tfrac{1}{2}-\tfrac{1}{2}\rangle.$$

where χ_λ is an eigenstate of $\frac{1}{2}\sigma_z$ and the representation is such that γ_0 is Hermitian, γ_i anti-Hermitian, and $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$. (The Dirac spinors are normalized so that $\bar{u}u=-\bar{v}v=2m$.)

The contributions to each φ_i from the pseudoscalar- and vector-exchange diagrams (thus giving rise to interference effects) must be added to give the total $U(6,6)$ amplitudes ($x\equiv\cos\theta$):

$$\varphi_1=\frac{1}{\mu^2(K^*)-t}\left\{\frac{1}{4m^2}M_1^{\rm II}(q^2)M_1^{\rm I}(q^2)[4m^2(2E^2+p^2(1+x))(1+x)]\right.$$
$$+(1/2m)[M_1^{\rm II}(q^2)M_2^{\rm I}(q^2)+M_2^{\rm II}(q^2)M_1^{\rm I}(q^2)][4m(E^2+p^2)(1+x)]$$
$$\left.+M_2^{\rm II}(q^2)M_2^{\rm I}(q^2)[8p^2+2m^2(1+x)]\right\}, \quad (2.9a)$$

$$\varphi_2=\frac{1}{\mu^2(K^*)-t}\left\{\frac{1}{4m^2}M_1^{\rm II}(q^2)M_1^{\rm I}(q^2)[-4E^2(2E^2+p^2(1+x))(1-x)]\right.$$
$$\left.+(1/2m)[M_1^{\rm II}(q^2)M_2^{\rm I}(q^2)+M_2^{\rm II}(q^2)M_1^{\rm I}(q^2)][-4E^2m(1-x)]+M_2^{\rm II}(q^2)M_2^{\rm I}(q^2)[-2m^2(1-x)]\right\}$$
$$+\frac{1}{\mu^2(K)-t}M_5^{\rm II}(q^2)M_5^{\rm I}(q^2)[-2p^2(1-x)], \quad (2.9b)$$

$$\varphi_3=\frac{1}{\mu^2(K^*)-t}\left\{\frac{1}{4m^2}M_1^{\rm II}(q^2)M_1^{\rm I}(q^2)[4m^2(2E^2+p^2(1+x))(1+x)]\right.$$
$$+(1/2m)[M_1^{\rm II}(q^2)M_2^{\rm I}(q^2)+M_2^{\rm II}(q^2)M_1^{\rm I}(q^2)][4m(E^2+p^2)(1+x)]$$
$$\left.+M_2^{\rm II}(q^2)M_2^{\rm I}(q^2)[2(E^2+p^2)(1+x)]\right\}, \quad (2.9c)$$

$$\varphi_4=\frac{1}{\mu^2(K^*)-t}\left\{\frac{1}{4m^2}M_1^{\rm II}(q^2)M_1^{\rm I}(q^2)[4E^2(2E^2+p^2(1+x))(1-x)]\right.$$
$$\left.+(1/2m)[M_1^{\rm II}(q^2)M_2^{\rm I}(q^2)+M_2^{\rm II}(q^2)M_1^{\rm I}(q^2)][4E^2m(1-x)]+M_2^{\rm II}(q^2)M_2^{\rm I}(q^2)[2m^2(1-x)]\right\}$$
$$+\frac{1}{\mu^2(K)-t}M_5^{\rm II}(q^2)M_5^{\rm I}(q^2)[-2p^2(1-x)], \quad (2.9d)$$

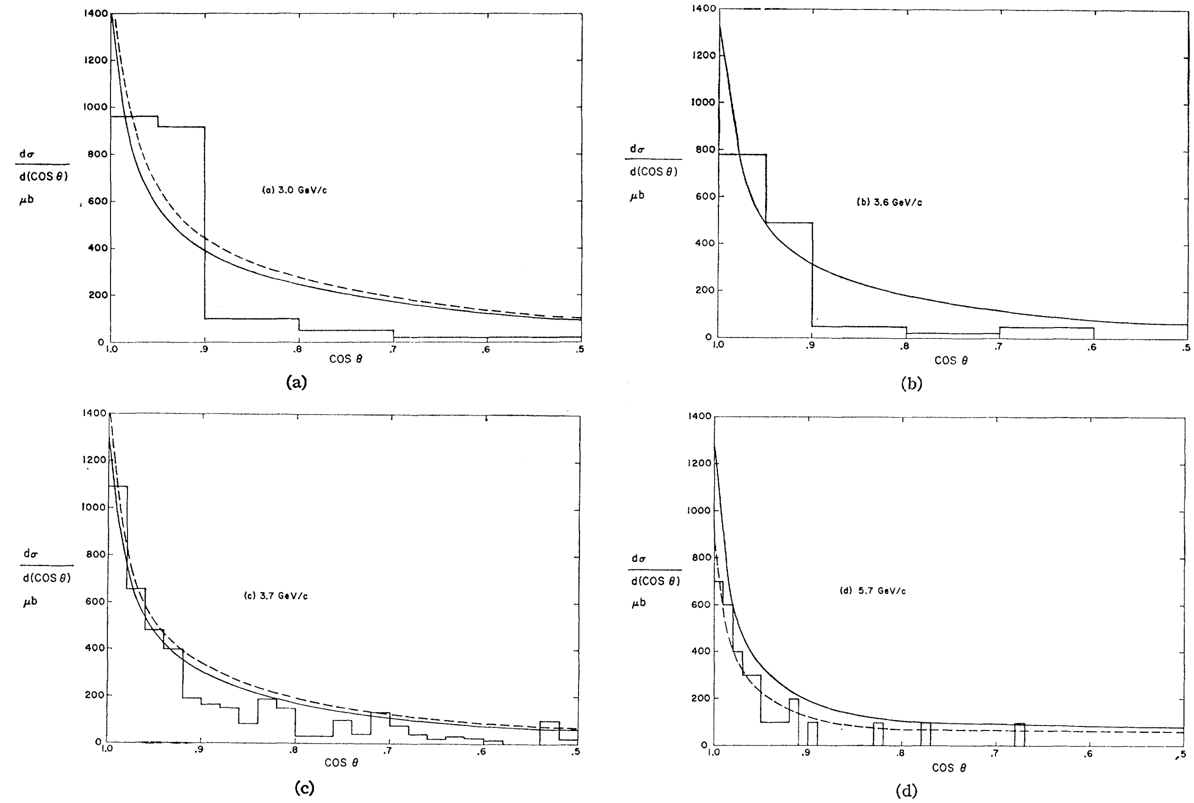
$$\varphi_5=\frac{1}{\mu^2(K^*)-t}\left\{\frac{1}{4m^2}M_1^{\rm II}(q^2)M_1^{\rm I}(q^2)[4mE(2E^2+p^2(1+x))\sin\theta]\right.$$
$$+(1/2m)M_1^{\rm II}(q^2)M_2^{\rm I}(q^2)[4E(E^2+p^2)\sin\theta]+(1/2m)M_2^{\rm II}(q^2)M_1^{\rm I}(q^2)[4m^2E\sin\theta]$$
$$\left.+M_2^{\rm II}(q^2)M_2^{\rm I}(q^2)[2mE\sin\theta]\right\}, \quad (2.9e)$$

$$\varphi_6=-\varphi_5. \quad (2.9f)$$

The remaining two amplitudes[15] required for the description of the $\Lambda\bar{\Sigma}^0$ and $\Sigma^0\bar{\Lambda}$ channels are

$$\varphi_7=\frac{1}{\mu^2(K^*)-t}\left\{\frac{1}{4m^2}M_1^{\rm II}(q^2)M_1^{\rm I}(q^2)[-4mE(2E^2+p^2(1+x))\sin\theta]\right.$$
$$+(1/2m)M_1^{\rm II}(q^2)M_2^{\rm I}(q^2)[-4m^2E\sin\theta]+(1/2m)M_2^{\rm II}(q^2)M_1^{\rm I}(q^2)[-4E(E^2+p^2)\sin\theta]$$
$$\left.+M_2^{\rm II}(q^2)M_2^{\rm I}(q^2)[-2mE\sin\theta]\right\}, \quad (2.9g)$$

$$\varphi_8=\varphi_7. \quad (2.9h)$$



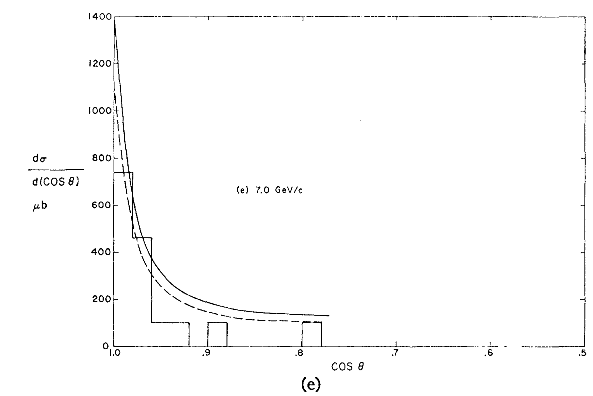


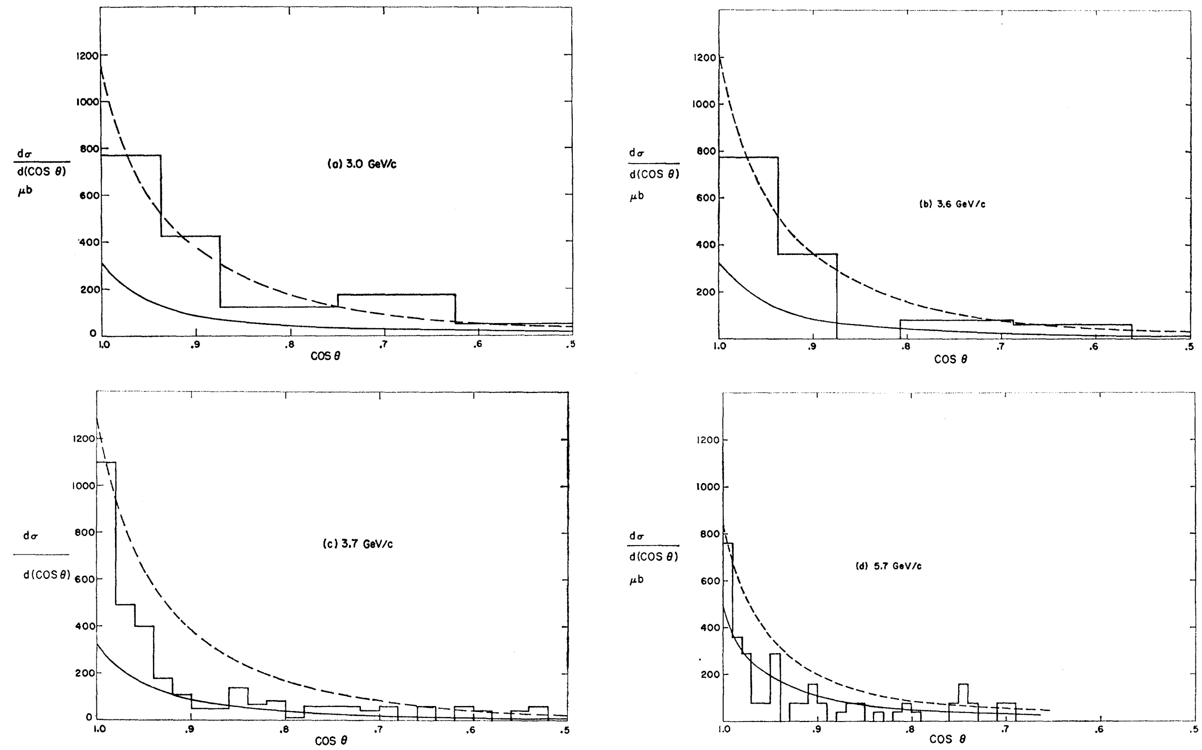
FIG. 3. $p+\bar{p}\rightarrow\Lambda+\bar{\Lambda}$ at 3.0, 3.6, 3.7, 5.7, and 7.0 GeV/c. Data of Refs. 1 and 2. The full curves are the predictions of the model, while the broken curves have been normalized to correctly reproduce the number of events in the first experimental interval.

III. ABSORPTION EFFECTS

We now consider how the Born amplitudes as given above [see Eq. (2.9)] are modified when the effects of the strong elastic interactions in the initial and final states are taken into account.

The inclusion of such effects to give an "absorption model" was originally proposed by Sopkovich,[16] and has been widely discussed recently in several reviews of the peripheral model,[16] where further references and discussions can be found. We confine ourselves to give a summary of the procedure. The physical idea behind the model is that the opaque core which one particle displays to another at high energies in elastic scattering and the consequent diffraction peak ought to show up in inelastic processes. These effects are just statistical manifestations of the very many open competing channels, the presence of which reflects on the amplitudes in any one channel through unitarity.

[16] N. J. Sopkovich, Nuovo Cimento **39**, 186 (1962); J. D. Jackson, in Proceedings of the 1966 Stony Brook Conference on High-Energy Two-Body Reactions (unpublished); J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965). J. D. Jackson, in *High Energy Physics, Proceedings of the 1965 Les Houches Summer School* (Gordon and Breach Science Publishers, Inc., New York, 1966). A. C. Hearn and S. D. Drell, in High-Energy Physics (Academic Press Inc., New York, to be published) (SLAC-PUB-176, ITP-200, 1966). Uri Maor, in Proceedings of the Conference on Nuclear and Particle Physics, University of Glasgow, 1966 (CERN/TC/PHYSICS 66–26) (unpublished).



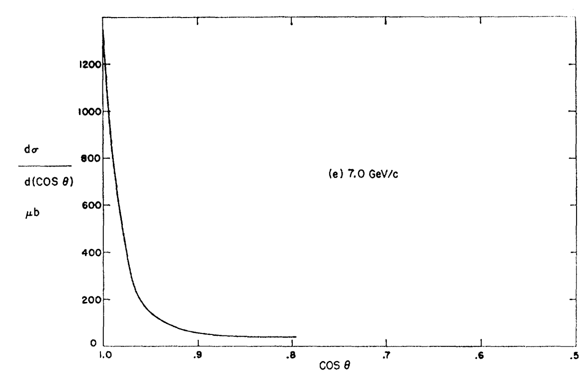


FIG. 4. $p+\bar{p}\rightarrow\Lambda+\bar{\Sigma}^0$ and $\Sigma^0+\bar{\Lambda}$ at 3.0, 3.6, 3.7, 5.7, and 7.0 GeV/c. Data of Refs. 1 and 2. The full curves are the predictions of the model, while the broken curves have been normalized to correctly reproduce the number of events in the first experimental interval.

We use a prescription given by one of the authors[17] in a previous publication, according to which the corrected or unitarized partial wave $T'^j(W)$ is related to the Born amplitude $T^j(W)$ by

$$T_{\beta\alpha}'^j=\tfrac{1}{2}[S_{\beta\beta}^jT_{\beta\alpha}^j+T_{\beta\alpha}^jS_{\alpha\alpha}^j], \qquad (3.1)$$

where α and β label the channels, and $S_{\alpha\alpha}^j$ and $S_{\beta\beta}^j$ are the S-matrix elements for elastic scattering in the initial and final states. This formula may be simply derived using the K-matrix formalism and an approximation scheme in which only the unitarity effects of the presumably large elastic scattering amplitudes are retained, while the *direct* effects of the typically small inelastic amplitudes are neglected. The result is valid relativistically, and is independent of any restriction on the range of the various forces. The formal differences between this treatment and the usual one (Sopkovich[16]) which gives

$$T_{\beta\alpha}'^j=(\sqrt{S_{\beta\beta}^j})T_{\beta\alpha}^j(\sqrt{S_{\alpha\alpha}^j}) \qquad (3.1')$$

are discussed further in Ref. 17. The two formulas become identical when the initial- and final-state elastic scattering parameters are set identical, as in the conventional approach to the application of the absorption model.

[17] H. D. D. Watson, Phys. Letters 17, 72 (1965).

TABLE I. Helicity amplitudes for the reaction $p\bar{p}\rightarrow Y\bar{Y}$.

$\lambda_4\lambda_3(Y\bar{Y})$ \ $\lambda_2\lambda_1(p\bar{p})$	$++$	$+-$	$-+$	$--$
$++$	φ_1	φ_5	$-\varphi_5$	φ_2
$+-$	φ_6	φ_3	φ_4	$-\varphi_6$
$-+$	$-\varphi_6$	φ_4	φ_3	φ_6
$--$	φ_2	φ_5	$-\varphi_5$	φ_1

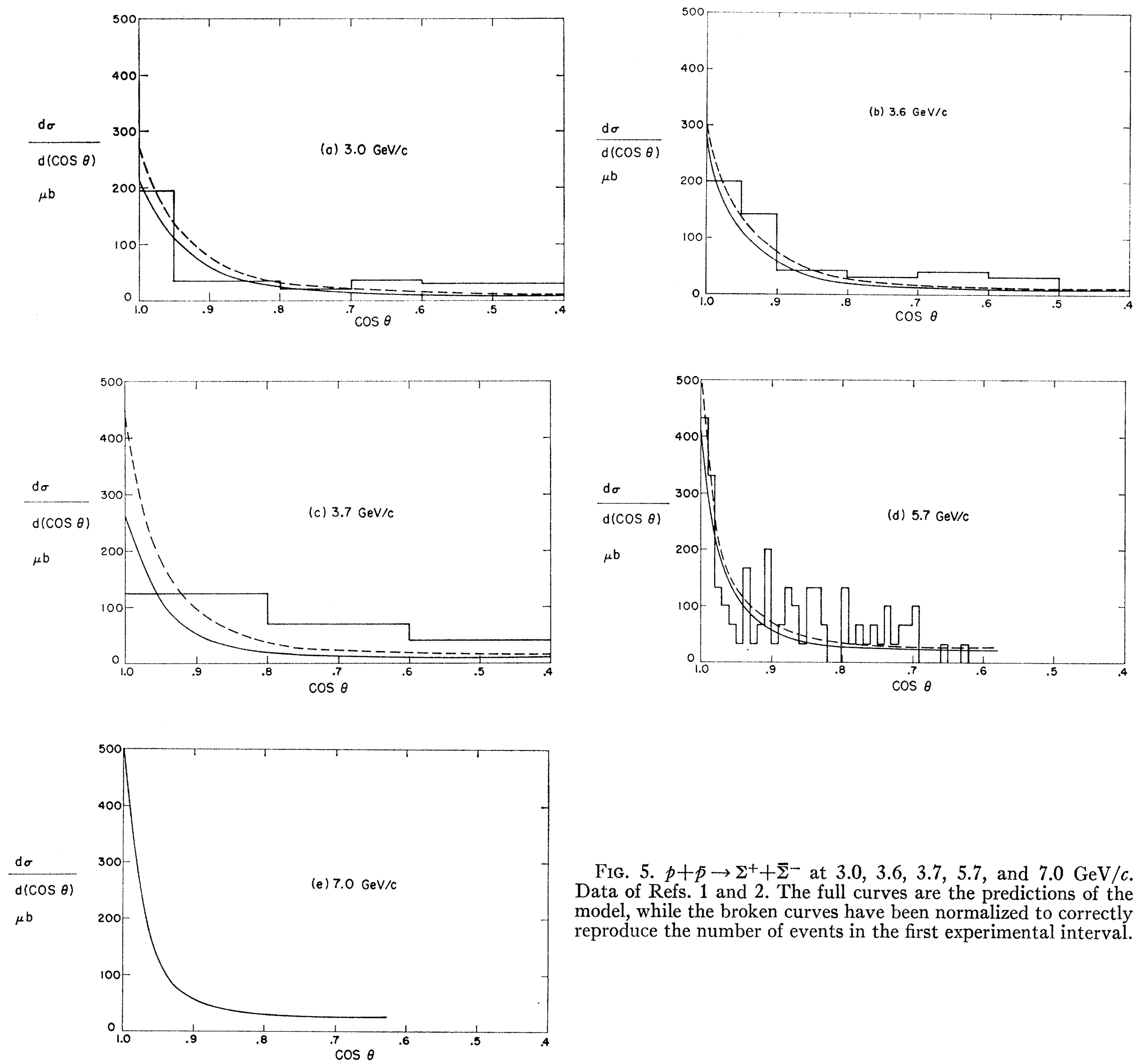


FIG. 5. $p+\bar{p}\rightarrow\Sigma^{+}+\bar{\Sigma}^{-}$ at 3.0, 3.6, 3.7, 5.7, and 7.0 GeV/c. Data of Refs. 1 and 2. The full curves are the predictions of the model, while the broken curves have been normalized to correctly reproduce the number of events in the first experimental interval.

In the application of Eq. (3.1), the forms for $S_{\alpha\alpha}{}^{j}$ and $S_{\beta\beta}{}^{j}$ are taken from experiment. In practice, one usually has no information on the form of the elastic scattering amplitude in the final state. In the absence of such information, we assume all $Y\bar{Y}$ elastic amplitudes to be identical to the $p\bar{p}$ amplitudes, which is a plausible approximation. As noted above, the general feature of elastic interactions at high energy is the diffraction peak, and this is independent, to a large extent, of the nature of the particles involved. In the present case all particles involved are spin-$\frac{1}{2}$ baryon-antibaryon pairs, and one would be surprised if the elastic amplitudes were very different.

Equation (3.1) is readily generalized to include spin. Inserting indices to label the spin helicity representation, we have

$$\langle\beta_1\beta_2|T_{\beta\alpha}{}'^{j}|\alpha_1\alpha_2\rangle =\tfrac{1}{2}\Big[\sum_{\beta_1'\beta_2'}\langle\beta_1\beta_2|S_{\beta\beta}{}^{j}|\beta_1'\beta_2'\rangle\langle\beta_1'\beta_2'|T_{\beta\alpha}{}^{j}|\alpha_1\alpha_2\rangle + \sum_{\alpha_1'\alpha_2'}\langle\beta_1\beta_2|T_{\beta\alpha}{}^{j}|\alpha_1'\alpha_2'\rangle\langle\alpha_1'\alpha_2'|S_{\alpha\alpha}{}^{j}|\alpha_1\alpha_2\rangle\Big]. \quad (3.2)$$

We make the following assumptions regarding the form of the elastic scattering amplitudes:

(i) As noted above, we take $S_{\alpha\alpha}{}^{j}=S_{\beta\beta}{}^{j}$.

(ii) We assume $S_{\alpha\alpha}{}^{j}$ only involves non-helicity-changing amplitudes. (There is no evidence, at these energies, for important helicity-changing terms and, intuitively, we indeed expect these to be absent.)

(iii) We parameterize $p\bar{p}$ elastic scattering in terms of a Gaussian model of radius ν, setting $\langle\alpha_1\alpha_2|S^j|\alpha_1\alpha_2\rangle = 1 - Ce^{-j(j+1)/\nu^2p^2}$. In this, we have taken the two independent non-helicity-changing amplitudes to be equal. The parameters ν $[\equiv \nu(E)]$ and C are obtained from experiment[18] and, typically, $\nu^{-1}\sim 200$ MeV; the parameter C is found to be equal to unity.

With these assumptions, Eq. (3.2) greatly simplifies; the new helicity partial-wave amplitudes $T_i'^j(W)$, where i specifies the helicity dependence (see Table I), is

$$T_i'^j(W) = [1 - e^{-j(j+1)/\nu^2p^2}]T_i^j(W), \tag{3.3}$$

which gives the unitarized or corrected amplitudes $T_i'^j(W)$ in terms of the Born amplitudes $T_i^j(W)$.

We have shown in Sec. II how the six independent helicity amplitudes $\varphi_i(i=1\cdots 6)$ for the reactions $p\bar{p}\to Y\bar{Y}$ can be obtained from the $U(6,6)$ symmetry. These can be decomposed in the helicity representation of Jacob and Wick as

$$\varphi_1 = \langle +\tfrac{1}{2}+\tfrac{1}{2}|T|+\tfrac{1}{2}+\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}+\tfrac{1}{2}|T_1^j(W)|+\tfrac{1}{2}+\tfrac{1}{2}\rangle d_{00}^j(\theta),$$

$$\varphi_2 = \langle +\tfrac{1}{2}+\tfrac{1}{2}|T|-\tfrac{1}{2}-\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}+\tfrac{1}{2}|T_2^j(W)|-\tfrac{1}{2}-\tfrac{1}{2}\rangle d_{00}^j(\theta),$$

$$\varphi_3 = \langle +\tfrac{1}{2}-\tfrac{1}{2}|T|+\tfrac{1}{2}-\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}-\tfrac{1}{2}|T_3^j(W)|+\tfrac{1}{2}-\tfrac{1}{2}\rangle d_{11}^j(\theta),$$

$$\varphi_4 = \langle +\tfrac{1}{2}-\tfrac{1}{2}|T|-\tfrac{1}{2}+\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}-\tfrac{1}{2}|T_4^j(W)|-\tfrac{1}{2}+\tfrac{1}{2}\rangle d_{-11}^j(\theta),$$

$$\varphi_5 = \langle +\tfrac{1}{2}+\tfrac{1}{2}|T|+\tfrac{1}{2}-\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}+\tfrac{1}{2}|T_5^j(W)|+\tfrac{1}{2}-\tfrac{1}{2}\rangle d_{10}^j(\theta),$$

$$\varphi_6 = \langle +\tfrac{1}{2}-\tfrac{1}{2}|T|+\tfrac{1}{2}+\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}-\tfrac{1}{2}|T_6^j(W)|+\tfrac{1}{2}+\tfrac{1}{2}\rangle d_{01}^j(\theta),$$

and similarly for the other two helicity amplitudes for the processes $p\bar{p}\to\Lambda\bar{\Sigma}^0$ and $p\bar{p}\to\Sigma^0\bar{\Lambda}$:

$$\varphi_7 = \langle +\tfrac{1}{2}+\tfrac{1}{2}|T|-\tfrac{1}{2}+\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}+\tfrac{1}{2}|T_7^j(W)|-\tfrac{1}{2}+\tfrac{1}{2}\rangle d_{-10}^j(\theta),$$

$$\varphi_8 = \langle +\tfrac{1}{2}-\tfrac{1}{2}|T|-\tfrac{1}{2}-\tfrac{1}{2}\rangle = \sum_j (2j+1)\langle +\tfrac{1}{2}-\tfrac{1}{2}|T_8^j(W)|-\tfrac{1}{2}-\tfrac{1}{2}\rangle d_{01}^j(\theta),$$

[18] O. Czyzewski *et al.*, in *Proceedings of the Sienna International Conference on Elementary Particles and High-Energy Physics, 1963*, edited by G. Bernardini and G. P. Puppi (Societa Italiana di Fisica, Bologna, 1963), Vol. I. p. 252.

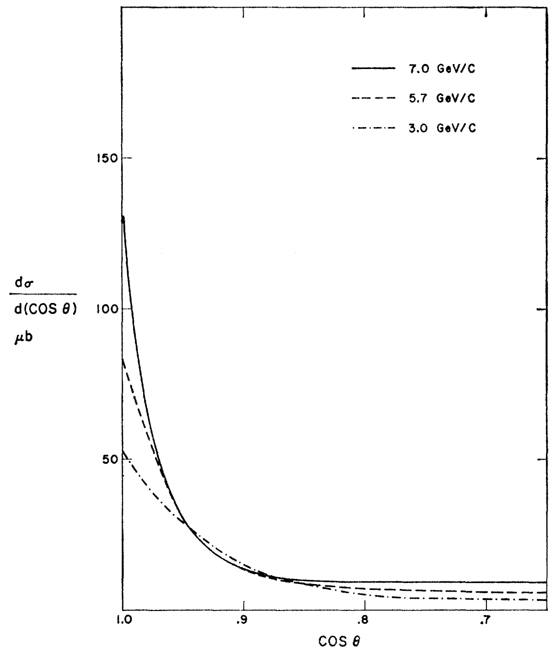


FIG. 6. $p+\bar{p}\to\Sigma^0+\bar{\Sigma}^0$. The predictions of the model at 3.0, 5.7, and 7.0 GeV/c.

or, in general,

$$\varphi_i = \sum_j (2j+1)T_i^j(W)d_{\lambda\mu}^j(\theta), \tag{3.4}$$

where

$$\lambda = \lambda_2 - \lambda_1, \mu = \lambda_4 - \lambda_3.$$

Conversely, making use of the orthogonality of the $d_{\lambda\mu}^j(\theta)$ functions

$$\int_{-1}^{1} d_{\lambda\mu}^j(\theta)d_{\lambda\mu}^{j'}(\theta)d(\cos\theta) = [2/(2j+1)]\delta_{jj'},$$

we obtain the partial-wave helicity amplitudes

$$T_i^j(W) = \frac{1}{2}\int_{-1}^{1} \varphi_i d_{\lambda\mu}^j(\theta)d(\cos\theta). \tag{3.5}$$

The unitarized scattering amplitudes can then be written in the form [remembering Eq. (3.3)]:

$$\varphi_j' = \sum_j (2j+1)[1 - e^{-j(j+1)/\nu^2p^2}]T_i^j(W)d_{\lambda\mu}^j(\theta). \tag{3.6}$$

The explicit form of the T_i^j and φ_i' in terms of Legendre polynomials are given in Appendix B. The differential unpolarized cross section according to the absorption model is obtained from Eq. (2.7) by substituting φ_i' for φ_i.

IV. COMPARISON WITH EXPERIMENT

In comparing the model with experiment, there is considerable ambiguity as to the treatment of the masses in $U(6,6)$. As far as the position of the poles is concerned, these ought to be assigned at points corresponding to the physical masses of the exchanged particles.

The question of the choice of S and V, the pseudoscalar and vector masses, is less easily settled. We have taken an *ad hoc* choice of $SU(3)$ masses, $S=417$ MeV., $V=850$ MeV, the mean mass of the pseudoscalar and vector nonets, respectively. This prescription has been been used elsewhere with success.[11] The present point of view has the advantage that no further ambiguity arises when we wish to relate the $U(6,6)$ coupling constant to the πNN coupling constant, using

$$G_{\pi NN}=G[1+(2m/S)](5/3). \tag{4.1}$$

More explicitly, the determination of the D:F ratio and the relative contribution of the pseudoscalar- and vector-exchange mechanisms are uniquely predicted by the $U(6,6)$ theory. Indeed, we find our results very sensitive to a variation of the relative contribution of K and K^* exchanges, and in general, to alternate choices of the masses in $U(6,6)$.

The predictions of the model are shown against experiment in Figs. 3 to 6, for the various channels and at various energies. The solid curve gives the predicted result correct in absolute magnitude, assuming a value of 14.9 for $G_{\pi NN}{}^2/4\pi$. For the dashed curves, the results have been normalized to correctly reproduce the observed number of events in the first experimental interval to facilitate comparison.

It will be seen that in all cases the angular distribution is well reproduced over the whole energy range. In addition, in the intermediate energy range under consideration, the absolute magnitude of the differential cross section is approximately correct for $\Lambda\bar{\Lambda}$ and $\Sigma^+\bar{\Sigma}^-$, but for $\Lambda\bar{\Sigma}^0$ (and $\Sigma^0\bar{\Lambda}$) the predicted value is too small by a factor of about 4; this particular feature of the model results from a destructive interference of the D and $\frac{2}{3}F$ couplings at the $K^+\bar{p}\Sigma^0$ vertex.

In discussing the energy variation of the cross section, we recall the large experimental errors of the data. This makes it difficult to draw definite conclusions about the energy dependence. For $\Lambda\bar{\Lambda}$, the theoretical cross section seems to decrease more slowly than the experimental results between 3.0 and 5.7 GeV/c, where it starts rising. For $\Sigma^+\bar{\Sigma}^-$, the experimental energy variation is much weaker than for $\Lambda\bar{\Lambda}$. The theoretical cross section falls within the limit of experimental error up to 5.7 GeV/c, where again it starts to rise. In the case of $\Lambda\bar{\Sigma}^0$ (and $\Sigma^0\bar{\Lambda}$), the energy variation of the model is wrong, the theoretical cross section showing a slight increase, whereas experiment shows a sharp decrease between 3.0 and 5.7 GeV/c. In Fig. 6, we give the predictions of the model for $p\bar{p}\rightarrow\Sigma^0\bar{\Sigma}^0$, where the model predicts a rise of the cross section, in disagreement with experiment. We are encouraged by the fact that agreement is best in the experimentally most accessible channels.

As expected, the model is relatively successful in predicting the energy dependence of the cross section for $\Lambda\bar{\Lambda}$ and $\Sigma^+\bar{\Sigma}^-$ in the intermediate momentum range 3.0 to 5.7 GeV/c. As the energy increases, the vector-exchange contribution takes over, leading to the well-known breakdown of vector-exchange models at high energy.

V. CONCLUSION

These calculations show good agreement with experiment as to the angular dependence of the produced particles in all cases and good agreement in absolute magnitude for $\Lambda\bar{\Lambda}$ and $\Sigma^+\bar{\Sigma}^-$ in the energy range under consideration, but not for $\Lambda\bar{\Sigma}^0$ (and $\Sigma^0\bar{\Lambda}$), which is too small by a factor of four. These results are determined from $G_{\pi NN}$ and can be considered as very satisfactory.

We wish to emphasize that the predictions of the model depend critically on the use of $U(6,6)$ symmetry at vertices. Other two-body final states such as $n\bar{n}$, $N^*\bar{N}^*$, and $Y^*\bar{Y}^*$ are also determined by the same model. Good agreement with experiment for $p\bar{p}\rightarrow n\bar{n}$ has been achieved in the momentum range 3.0 to 9.0 GeV/c.[19] Calculations are in progress to extend the predictions of the model to the $N^*\bar{N}^*$ and $Y^*\bar{Y}^*$ channels.[20]

ACKNOWLEDGMENTS

We are much indebted to Professor P. T. Matthews for assistance and encouragement in this work and to many members of the Theoretical Physics Department of Imperial College for very fruitful discussions. We wish to thank Dr. B. R. French, CERN, for providing us with experimental results at 5.7 GeV/c prior to publication. One of us (J. H. R. M.) would like to express his gratitude to the National Research Council of Canada for financial support during the execution of this work.

APPENDIX A: EXPLICIT FORM OF THE CURRENTS

The vector interaction as given by Eq. (2.3) can be written as

$$J_\mu=F_C(P_\mu/2m)\bar{N}N+F_M\bar{N}(r_\mu/4m^2)N, \tag{A1}$$

where F_C and F_M are the Sachs[10] form factors. [The symmetric and antisymmetric couplings have been implicitly included in (A1).] For subsequent algebraic work, it is more convenient to rewrite J_μ in terms of P_μ and γ_μ in the conventional manner:

$$J_\mu=M_1(q^2)(P_\mu/2m)\bar{N}N+M_2(q^2)\bar{N}\gamma_\mu N, \tag{A2}$$

where the relations between M_1, M_2 and F_C, F_M are given by

$$M_1=F_C-F_M,$$
$$M_2=[1-(q^2/4m^2)]F_M. \tag{A3}$$

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[19] J. H. R. Migneron and K. Moriarty, Phys. Rev. Letters 18, 978 (1967).
[20] J. H. R. Migneron and K. Moriarty (to be published).

The explicit forms of $M_1(q^2)$ and $M_2(q^2)$ for the required couplings of mesons with baryons in the annihilation $p+\bar{p}\to Y+\bar{Y}$ are (the $\pi^0\bar{p}p$ coupling constant has been normalized to unity):

$K^*\bar{\Lambda}p$:

$$M_1(q^2)=\sqrt{3}(2m/V)[1-(q^2/4m^2)]\,, \tag{A4}$$

$$M_2(q^2)=-\sqrt{3}[1+(2m/V)][1-(q^2/4m^2)]\,. \tag{A5}$$

$K^{*-}\bar{\Sigma}^0p$:

$$M_1(q^2)=-\tfrac{2}{3}[2+(m/V)(1+(3q^2/4m^2))]\,, \tag{A6}$$

$$M_2(q^2)=\tfrac{1}{3}[1+(2m/V)][1-(q^2/4m^2)]\,. \tag{A7}$$

$\bar{K}^{*0}\bar{\Sigma}^+p$:

$$M_1(q^2)=-\tfrac{2}{3}\sqrt{2}[2+(m/V)(1+(3q^2/4m^2))]\,, \tag{A8}$$

$$M_2(q^2)=\tfrac{1}{3}\sqrt{2}[1+(2m/V)][1-q^2/4m^2]\,. \tag{A9}$$

(For simplicity of writing, we have omitted the γ_μ in the couplings.) The pseudoscalar coupling of mesons with baryons can be written in a similar way (γ_5's omitted):

$K^-\bar{\Lambda}p$:

$$M_5(q^2)=-\sqrt{3}[1+(2m/S)][1-(q^2/4m^2)]\,; \tag{A10}$$

$K^-\bar{\Sigma}^0p$:

$$M_5(q^2)=\tfrac{1}{3}[1+(2m/S)][1-(q^2/4m^2)]\,; \tag{A11}$$

$\bar{K}^0\bar{\Sigma}p$:

$$M_5(q^2)=\tfrac{1}{3}\sqrt{2}[1+(2m/S)][1-(q^2/4m^2)]\,. \tag{A12}$$

APPENDIX B: PARTIAL-WAVE HELICITY AMPLITUDES $T_i^j(W)$

Equations (3.4) and (3.5) have to be evaluated for different j values. Since we deal with high-energy reactions, large j values will contribute significantly. It is therefore necessary to find some way of computing readily the $d_{\lambda\mu}^j(\theta)$.

We show how the $d_{\lambda\mu}^j(\theta)$ functions may be written quite generally in terms of Legendre polynomials. (The resulting expressions contain no derivatives of Legendre polynomials.) Written in terms of Jacobi polynomials,[21] the $d_{\lambda\mu}^j(\theta)$ functions assume the form

$$d_{\lambda\mu}^j(\theta)=\left[\frac{(j+\lambda)!(j-\lambda)!}{(j+\mu)!(j-\mu)!}\right]^{1/2}(\cos\tfrac{1}{2}\theta)^{\lambda+\mu} \times(\sin\tfrac{1}{2}\theta)^{\lambda-\mu}P_{j-\lambda}{}^{\lambda-\mu,\lambda+\mu}(\cos\theta)\,. \tag{B1}$$

The $d_{\lambda\mu}^j(\theta)$ function satisfies the relations

$$d_{\lambda\mu}^j(\theta)=(-1)^\mu d_{\lambda\mu}^j(\theta)={}^{\lambda-}d_{-\mu-\lambda}^j(\theta)\,. \tag{B2}$$

For our particular values of λ and μ, Eq. (B1) gives ($x\equiv\cos\theta$)

$$\begin{aligned}
d_{00}^j(\theta)&=P_j{}^{0,0}(x)\,,\\
d_{11}^j(\theta)&=\cos^2(\tfrac{1}{2}\theta)P_{j-1}{}^{0,2}(x)\,,\\
d_{-11}^j(\theta)&=\sin^2(\tfrac{1}{2}\theta)P_{j-1}{}^{2,0}(x)\,,\\
d_{10}^j(\theta)&=\left(\frac{j+1}{j}\right)^{1/2}\cos\tfrac{1}{2}\theta\,\sin\tfrac{1}{2}\theta P_{j-1}{}^{1,1}(x)\,.
\end{aligned} \tag{B3}$$

The other $d_{\lambda\mu}^j(\theta)$ functions are obtained from Eqs. (B2) and (B3).

We note first that

$$P_j{}^{00}(\cos\theta)=P_j(\cos\theta)\,. \tag{B4}$$

Any Jacobi polynomial $P_j{}^{m,n}$ can be written as a linear combination of Legendre polynomials by making use of the two recurrence relations[22]

$$\tfrac{1}{2}(2+\alpha+\beta+2n)(x+1)P_n{}^{\alpha,\beta+1}(x) =(n+1)P_{n+1}{}^{\alpha,\beta}(x)+(1+\beta+n)P_n{}^{\alpha,\beta}(x)\,, \tag{B5}$$

$$\tfrac{1}{2}(2+\alpha+\beta+2n)(x-1)P_n{}^{\alpha+1,\beta}(x) =(n+1)P_{n+1}{}^{\alpha,\beta}-(1+\alpha+n)P_n{}^{\alpha,\beta}(x)\,. \tag{B6}$$

For the polynomials occuring in Eq. (B3), one obtains in this way

$$\begin{aligned}
P_{j-1}{}^{0,2}(x)&=\frac{2}{(2j+1)(1+x)^2}[jP_{j+1}(x)+(2j+1)P_j(x)+(j+1)P_{j-1}(x)]\,,\\
P_{j-1}{}^{2,0}(x)&=\frac{2}{(2j+1)(1-x)^2}[jP_{j+1}(x)-(2j+1)P_j(x)+(j+1)P_{j-1}(x)]\,,\\
P_{j-1}{}^{1,1}(x)&=\frac{2j}{(2j+1)(1-x^2)}[P_{j-1}(x)-P_{j+1}(x)]\,.
\end{aligned} \tag{B7}$$

[21] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc. New York, 1963); A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

[22] E. D. Rainville, *Special Functions* (The Macmillan Company, New York, 1960), pp. 264–265.

Interserting (B3) and (B7) into Eq. (3.5), the partial-wave helicity amplitudes are written as

$$\begin{aligned}
T_1{}^j(W)&=\frac{1}{2}\int_{-1}^{1}dx\ \varphi_1 P_j(x)\,,\\
T_2{}^j(W)&=\frac{1}{2}\int_{-1}^{1}dx\ \varphi_2 P_j(x)\,,\\
T_3{}^j(W)&=\frac{1}{2(2j+1)}\int_{-1}^{1}dx\ \frac{\varphi_3}{1+x}[jP_{j+1}(x)+(2j+1)P_j(x)+(j+1)P_{j-1}(x)]\,,\\
T_4{}^j(W)&=\frac{1}{2(2j+1)}\int_{-1}^{1}dx\ \frac{\varphi_4}{1-x}[jP_{j+1}(x)-(2j+1)P_j(x)+(j+1)P_{j-1}(x)]\,,\\
T_5{}^j(W)&=\frac{[j(j+1)]^{1/2}}{2(2j+1)}\int_{-1}^{1}dx\ \frac{\varphi_5}{(1-x^2)^{1/2}}[P_{j-1}(x)-P_{j+1}(x)]\,,\\
T_6{}^j(W)&=\frac{[j(j+1)]^{1/2}}{2(2j+1)}\int_{-1}^{1}dx\ \frac{\varphi_6}{(1-x^2)^{1/2}}[P_{j+1}(x)-P_{j-1}(x)]\,.
\end{aligned}\tag{B8}$$

The partial-wave helicity amplitudes $T_7{}^j(W)$ and $T_8{}^j(W)$ (necessary for the $\Lambda\bar{\Sigma}^0$ and $\Sigma^0\bar{\Lambda}$ channels) take a form identical to $T_6{}^j(W)$ with the obvious change of subscripts.

In a similar way, the independent unitarized scattering amplitudes (3.6) can finally be written in terms of Legendre polynomials[23]:

$$\begin{aligned}
\varphi_1'&=\sum_{j=0}(2j+1)T_1'^j(W)P_j(x)\,,\\
\varphi_2'&=\sum_{j=0}(2j+1)T_2'^j(W)P_j(x)\,,\\
\varphi_3'&=\sum_{j=1}\frac{T_3'^j(W)}{1+x}\frac{1}{2j+1}[jP_{j+1}(x)+(2j+1)P_j(x)+(j+1)P_{j-1}(x)]\,,\\
\varphi_4'&=\sum_{j=1}\frac{T_4'^j(W)}{1-x}\frac{1}{2j+1}[jP_{j+1}(x)-(2j+1)P_j(x)+(j+1)P_{j-1}(x)]\,,\\
\varphi_5'&=\sum_{j=1}\frac{T_5'^j}{(1-x^2)^{1/2}}[j(j+1)]^{1/2}[P_{j-1}(x)-P_{j+1}(x)]\,,\\
\varphi_6'&=\sum_{j=1}\frac{T_6'^j(W)}{(1-x^2)^{1/2}}[j(j+1)]^{1/2}[P_{j+1}(x)-P_{j-1}(x)]\,.
\end{aligned}\tag{B9}$$

[23] The explicit calculations were performed up to $j=30$.