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Maximum Number of Collisions for Three Point Particles*

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A simple method of obtaining $n_{\max}$, the maximum number of binary collisions for three point particles with masses m_1, m_2, and m_3 interacting through zero-range forces, is presented. The result is that $n_{\max}$ is the smallest integer greater than or equal to x, where $\cos(\pi/x)=[m_1m_2/(m_1+m_3)(m_2+m_3)]^{1/2}$. (Here m_3 is the smallest mass, and $x\geq 3$.) Also, the magnitudes of the momenta after any number of classically allowed collisions are simply expressed in terms of the initial momenta.

WE present below a simple method of obtaining $n_{\max}$, the maximum number of classical binary collisions allowed by energy and momentum conservation for three point particles of arbitrary mass. Previously, only an upper bound for $n_{\max}$ has been obtained.[1] It is of some interest to have $n_{\max}$ in solving the Schrödinger equation using the Faddeev equations because the nth iterated kernel of these equations is not compact for $n\leq n_{\max}$. It is also interesting because the nonrelativistic conditions for Landau rescattering singularities are identical to the conditions for classically realizable rescattering of point particles. Hence, those singularities involving only three particles appear in diagrams in which n binary collisions occur, where $n\leq n_{\max}$.

We work with three particles with masses m_1, m_2, and m_3 and with momenta $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. We shall use the following linear combinations of the momenta in the center-of-momentum system:

$$\mathbf{p}_1=(m_3\mathbf{k}_2-m_2\mathbf{k}_3)/[2m_2m_3(m_2+m_3)]^{1/2},$$
$$\mathbf{q}_1=[-\mathbf{k}_1(m_2+m_3)+m_1(\mathbf{k}_2+\mathbf{k}_3)]/[2m_1(m_2+m_3)(m_1+m_2+m_3)]^{1/2}. \quad (1)$$

Two other sets of momenta, $(\mathbf{p}_2,\mathbf{q}_2)$ and $(\mathbf{p}_3,\mathbf{q}_3)$, are obtained by a cyclic interchange of subscripts in (1). The nonrelativistic kinetic energy H_0 is easily written as

$$H_0=k_1^2/2m_1+k_2^2/2m_2+k_3^2/2m_3=p_i^2+q_i^2,\quad i=1,2,3. \quad (2)$$

These three sets of momenta are linearly dependent on each other, as can be seen from

$$\mathbf{p}_1=-A_3\mathbf{p}_2-B_3\mathbf{q}_2, \quad (3)$$

$$\mathbf{q}_1=B_3\mathbf{p}_2-A_3\mathbf{q}_2 \quad (4)$$

and

$$\mathbf{p}_2=-A_3\mathbf{p}_1+B_3\mathbf{q}_1, \quad (5)$$

$$\mathbf{q}_2=-B_3\mathbf{p}_1-A_3\mathbf{q}_1, \quad (6)$$

where

$$A_3=[m_1m_2/(m_1+m_3)(m_2+m_3)]^{1/2},$$
$$B_3=[m_3(m_1+m_2+m_3)/(m_1+m_3)(m_2+m_3)]^{1/2}, \quad (7)$$

$$A_3^2+B_3^2=1. \quad (8)$$

Note that these relations depend on only one parameter containing the masses. We shall use superscripts in parentheses to indicate the number of collisions that have previously occurred; for example, $\mathbf{p}_1^{(4)}$ is the relative momentum of particles 2 and 3 after the fourth collision.

Because we consider only point particles with zero-range interactions, all momentum vectors except the initial and final ones must be collinear. If any collision deflects the momentum vectors of the two colliding particles away from this collinear axis, clearly no further scattering can occur. This restriction is not valid in the case of hard-sphere scattering where additional collisions are allowed.[2]

We first consider the sequence of collisions in which particle 3 bounces back and forth between the two other particles. (Later, we show that the maximum number of collisions occurs when m_3 is the smallest mass.) In each of these elastic collisions, the relative momentum of the two particles involved changes sign; i.e.,

$$\mathbf{p}^{(i+1)}=-\mathbf{p}^{(i)}. \quad (9)$$

If we look at the momentum coordinates before the final collision, we see that the existence of this collision is determined by the sign of the relative momentum of the two particles involved. Hence, one condition for maximum rescattering is that this momentum be zero. We now consider the time-reversed sequence of events and start with $\mathbf{p}^{(0)}=0$ and $\mathbf{q}^{(0)}$ of arbitrary magnitude. Upon following the sequence of successive scatterings, we arrive at a configuration in which two particles are receding from the third and also from each other. Obviously no further scatterings are possible. The limiting condition for the existence of the final collision is again that the relative momentum of the two particles involved be zero.

Consider the initial system in which particles 1 and 3 have almost zero relative momentum after their first

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collision. Let particle 3 be the particle in the middle on a collision course with particle 2. We choose the positive direction of our scattering axis to be the direction of the initial vector $\mathbf{q}_2^{(0)}$. Let $\mathbf{j}$ be a unit vector in this direction. The conditions for no further scatterings in the $(\mathbf{q}_1,\mathbf{p}_1)$ system are

$$\mathbf{p}_1 = c_1\mathbf{j}, \quad c_1 \geq 0 \tag{10}$$

$$\mathbf{q}_1 = d_1\mathbf{j}, \quad d_1 \geq 0 \tag{11}$$

and in the $(\mathbf{p}_2,\mathbf{q}_2)$ system

$$\mathbf{p}_2 = c_2\mathbf{j}, \quad c_2 \geq 0 \tag{12}$$

$$\mathbf{q}_2 = d_2\mathbf{j}, \quad d_2 \leq 0. \tag{13}$$

We now express $p^{(n)}$ and $q^{(n)}$ in terms of $p^{(0)}$ and $q^{(0)}$ by induction. From (3), (4), and (9), we have

$$p_1^{(1)} = -A_3(-p_2^{(0)}) - B_3 q_2^{(0)} = \mathrm{Re}(A_3+iB_3)p_2^{(0)} - \mathrm{Im}(A_3+iB_3)q_2^{(0)}, \tag{14}$$

$$q_1^{(1)} = B_3(-p_2^{(0)}) - A_3 q_2^{(0)} = -\mathrm{Im}(A_3+iB_3)p_2^{(0)} - \mathrm{Re}(A_3+iB_3)q_2^{(0)}, \tag{15}$$

where the expression A_3+iB_3 is introduced to simplify notation. From (5), (6), and (9), we have

$$p_2^{(2)} = -A_3(-p_1^{(1)}) + B_3 q_1^{(1)} = \mathrm{Re}[(A_3+iB_3)^2]p_2^{(0)} - \mathrm{Im}[(A_3+iB_3)^2]q_2^{(0)}, \tag{16}$$

$$q_2^{(2)} = -B_3(-p_1^{(1)}) - A_3 q_1^{(1)} = \mathrm{Im}[(A_3+iB_3)^2]p_2^{(0)} + \mathrm{Re}[(A_3+iB_3)^2]q_2^{(0)}. \tag{17}$$

From (3), (4), and (9) again, we have

$$p_1^{(3)} = \mathrm{Re}[(A_3+iB_3)^3]p_2^{(0)} - \mathrm{Im}[(A_3+iB_3)^3]q_2^{(0)}, \tag{18}$$

$$q_1^{(3)} = -\mathrm{Im}[(A_3+iB_3)^3]p_2^{(0)} - \mathrm{Re}[(A_3+iB_3)^3]q_2^{(0)}. \tag{19}$$

If we assume that (14) and (15) are true for the superscripts and powers increased by n, then we can show that (16) through (19) are true for the superscripts and powers increased by n. Hence

$$p^{(n)} = \mathrm{Re}[(A_3+iB_3)^n]p_2^{(0)} - \mathrm{Im}[(A_3+iB_3)^n]q_2^{(0)}, \quad n=1,2,3,\cdots \tag{20}$$

$$q_1^{(n)} = -\mathrm{Im}[(A_3+iB_3)^n]p_2^{(0)} - \mathrm{Re}[(A_3+iB_3)^n]q_2^{(0)}, \quad n=1,3,5,\cdots \tag{21}$$

$$q_2^{(n)} = \mathrm{Im}[(A_3+iB_3)^n]p_2^{(0)} + \mathrm{Re}[(A_3+iB_3)^n]q_2^{(0)}. \quad n=2,4,6,\cdots. \tag{22}$$

Setting $p_2^{(0)}=0$, (11) and (13) require

$$\mathrm{Re}[(A_3+iB_3)^n] \leq 0, \quad n=1,2,3,\cdots. \tag{23}$$

For $n=1$, (23) cannot be satisfied. For $n=2$, Eqs. (12) and (13) cannot be satisfied with the lightest particle in the middle. Hence three collisions are always possible.

Let us consider the equalities in (10) and (12). Both may be written as

$$\mathrm{Im}[(A_3+iB_3)^n] = 0. \tag{24}$$

Solving (23) and (24) for A_3 given n is equivalent to finding the cosine of a complex unit vector whose nth power is -1. The solution is

$$A_3^{(n)} = \cos(\pi/n + 2m\pi). \tag{25}$$

All solutions except $m=0$ can be excluded by requiring the masses be positive and that the $(n-1)$th collision be allowed; i.e., $A_3^{(n)}>0$ and $A_3^{(n)}>A_3^{(n-1)}$. Hence the solution is

$$A_3^{(n)} = \cos(\pi/n). \tag{26}$$

If $A_3 > A_3^{(n)}$, at least one more collision is allowed since the final two-body relative momentum is then finite and of the appropriate sign. $A_3^{(n)}$ is thus the limiting case of $n_{\max}=n+1$ collisions. If $A_3=A_3^{(n)}$, only $n_{\max}=n$ collisions are allowed since the final two-body relative momentum is then zero. Hence four collisions are not allowed for three equal masses since $A_3=A_3^{(3)}=\cos\frac{1}{3}\pi=\frac{1}{2}$.

We may also solve (26) for $n_{\max}$ given A_3. In this case, $n_{\max}$ is the smallest integer greater than or equal to x, where $x=\pi/\cos^{-1}A_3$.

We now set $m_3=1$ and plot the results in the x, y plane, where $x=1/m_1$ and $y=1/m_2$. The boundaries of the regions with different values of $n_{\max}$ are given by $[A_3^{(n)}]^2=1/(1+x)(1+y)$, where $A_3^{(n)}$ is the nth solution of (26). These curves are hyperbolas with asymptotes $x=-1$ and $y=-1$. Some typical values of $A_3^{(n)}$ and the coordinates of the point $1/m_1=1/m_2$ are given in Table I. A few of the curves are drawn in Fig. 1.

Since the boundary for $n_{\max}=4$ goes through the points $(x,y)=(1,0)$ and $(0,1)$ we see that for two equal-mass particles and one heavier particle, only four binary scatterings are allowed. This is the case for electron-hydrogen scattering. For K-d scattering, $n_{\max}=4$; for π-d scattering, $n_{\max}=7$; and for electron-deuteron scattering, $n_{\max}=31$. Of course, if $m_3=0$, $n_{\max}=\infty$.

By examining all sequences of scattering, we now show that the maximum number of binary collisions

TABLE I. Typical maximum values of the mass ratio A_3 [Eq. (7)] and minimum values of $1/m_1=1/m_2$ for a given $n_{\max}$.

$n_{\max}$	A_3	$1/m_1=1/m_2$
4	0.707	0.414
5	0.809	0.236
6	0.866	0.155
7	0.901	0.110
10	0.951	0.051
15	0.978	0.022
20	0.988	0.012
25	0.992	0.008
30	0.994	0.0055

occurs for the previously considered sequence of scattering in which the lightest particle scatters back and forth between the two heavier particles. In order to include the physical processes in which the particles can change their order on the axis, we must generalize (9) to allow for the possibility of forward scattering for each collision in the scattering sequence; i.e.,

$$\mathbf{p}^{(i+1)}=\mathbf{p}^{(i)}. \tag{27}$$

Using cyclic permutations of Eqs. (3)–(8), we obtain

$$p_j^{(n)}=\{\mathrm{Re}[(A_1+iB_1)^{N_1}(A_2+iB_2)^{N_2}(A_3+iB_3)^{N_3}]p_2^{(0)} -\mathrm{Im}[(A_1+iB_1)^{N_1}(A_2+iB_2)^{N_2} \times(A_3+iB_3)^{N_3}]q_2^{(0)}\}(-1)^{N_F} \tag{28}$$

and

$$q_j^{(n)}=(-)^n\,\mathrm{Im}[(A_1+iB_1)^{N_1}(A_2+iB_2)^{N_2} \times(A_3+iB_3)^{N_3}]p_2^{(0)}+(-)^n\,\mathrm{Re}[(A_1+iB_1)^{N_1} \times(A_2+iB_2)^{N_2}(A_3+iB_3)^{N_3}]q_2^{(0)}, \tag{29}$$

where N_1 is the number of transformations between the "2" and the "3" system. N_2 and N_3 are similarly defined.

$n=$ total number of transformations; $j=1, 2,$ or 3. (30)

N_F is the number of forward scatterings.

Starting with the "2" coordinate system, it can be shown that if N_1, N_2, and N_3 are all odd or all even, then $j=2$. If N_1 is odd (even) while N_2 and N_3 are even (odd), then $j=3$. If N_1 and N_2 are even (odd) while N_3 is odd (even), then $j=1$. The combinations $(N_1,N_2,N_3)=$ (even, odd, even) and $(N_1,N_2,N_3)=$ (odd, even, odd) are not allowed when the initial coordinate system is the "2" system.

The conditions for no further scattering are similar to Eqs. (10)–(13). The limiting conditions for no further scattering are

$$\mathrm{Im}[(A_1+iB_1)^{N_1}(A_2+iB_2)^{N_2}(A_3+iB_3)^{N_3}]=0 \tag{31}$$

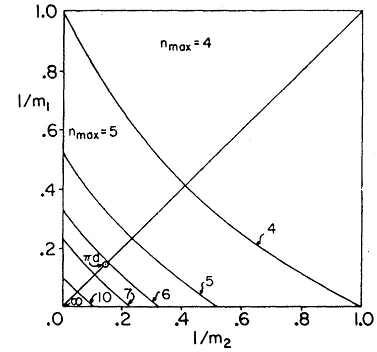

FIG. 1. Regions of maximum number of binary collisions in the $1/m_1$ versus $1/m_2$ plane for $m_3=1$. Points on the curves belong to the region with the lower $n_{\max}$. For the point where all three masses are equal, $n_{\max}=3$.

and

$$\mathrm{Re}[(A_1+iB_1)^{N_1}(A_2+iB_2)^{N_2}(A_3+iB_3)^{N_3}]\leq 0. \tag{32}$$

The problem of finding the allowed values of N_1, N_2, and N_3, given the masses of the three particles, reduces to finding all sets of (N_1,N_2,N_3) such that

$$N_1\cos^{-1}A_1+N_2\cos^{-1}A_2+N_3\cos^{-1}A_3\leq\pi. \tag{33}$$

If m_3 is the smallest mass, $\cos^{-1}A_3$ is smaller than $\cos^{-1}A_1$ and $\cos^{-1}A_2$. Clearly the maximum number of scatterings allowed by (33) then occurs when $N_1=N_2=0$ and N_3 attains its maximum. In this case, (33) reduces to (26). Hence, maximum scattering occurs when the smallest mass particle bounces between the other two masses.

In general, $p_j^{(n)}$ in (28) is not zero, allowing one more collision. In this case, Eqs. (28) and (29) still express the magnitudes of the final momenta in terms of the magnitudes of the initial momenta. The final $\mathbf{q}$ is still either parallel or antiparallel to $\mathbf{q}_2$, and the direction of the final $\mathbf{p}$ is arbitrary, as is the direction of the initial $\mathbf{p}_2$.