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Conspiracy and Evasion: Property of Regge Poles*

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It is shown that if standard methods are used to apply the Regge-pole theory to relativistic problems in which the external particles have nonzero spin, then there exist constraint equations which enforce relationships between the residue and trajectory functions of the participating poles in the region $t\sim 0$. The constraint equations follow directly from general quantum-mechanical principles and it is therefore essential to satisfy them. Moreover, the number of constraint equations increases roughly as the fourth power of the spin of the external particles. The structure of the constraint equations also differs radically according to whether the t-channel process has equal-mass particles in both the initial and the final state or unequal-mass particles in both the initial and the final state. A separate treatment of the various situations is given. Several examples are worked out in detail: $\pi N\to\pi N$, $\pi N^*\to\pi N^*$, $\pi\rho\to\pi\rho$, $NN\to NN$, and $\rho\rho\to\rho\rho$. The discussion of the general case in which the external particles have arbitrary spins requires a slight extension of previously given methods for the Reggeization of processes with spin. A very simple, and completely general scheme for the Reggeization, and for the classification of the Regge poles involved, is given. Finally, a discussion is given of the fundamental underlying group-theoretical origin of the constraint equations, and it is suggested that the necessity to satisfy them artificially represents a *weakness* in our present methods of applying Regge-pole theory to relativistic processes.

I. INTRODUCTION

In the past year it has begun to become rather apparent that our present methods of applying Regge's original ideas, developed within the context of potential scattering, to more realistic situations involving the relativistic scattering of particles with arbitrary masses and spins, suffers from a certain naïveté. The appearance of several papers[1] pointing out paradoxes, pseudoparadoxes and downright idiosyncrasies in the predictions of and requirements on the Regge-pole theory has borne witness to this situation.

In the present paper, we shall show that there is a class of constraints, the existence of which follows directly from the general principles of quantum mechanics, which are *automatically* satisfied in any "decent" theory but which appear in the Regge theory in a very complicated and restrictive guise. These constraints enforce relationships among the residues and/or the trajectories of quite dissimilar Regge poles, and lead to very powerful experimental predictions. It was our original motivation to examine and explore the phenomenological consequences of these constraints but our present feeling is that the requirements are so artifical, and even arbitrary, that it is much more likely that their existence is simply a manifestation of a weakness in our standard method of Reggeizing relativistic problems.

Two classic examples of these constraint conditions have already appeared in the literature.

(i) In backward πN scattering, in which the exchanged Regge poles are fermions, Gribov *et al.*[2] showed that the Regge poles have to occur in pairs of opposite parity, with trajectory functions $\alpha_{\pm}(u)$ which intersect at $u=0$.

(ii) In the theory of nucleon-nucleon scattering it has been known for a long time[3] that some of the t-channel helicity amplitudes are related to each other as $t\to 0$, namely, in the notation of Ref. 3,

$$\bar{f}_1-z_t\bar{f}_4-\bar{f}_3\sim t \quad \text{as} \quad t\to 0. \tag{1}$$

Since the $\bar{f}_i$ receive contributions from various types of Regge pole, Eq. (1) enforces a relationship amongst the trajectory functions and residue functions of dissimilar Regge poles. The consequences of Eq. (1) were partially analyzed by Volkov and Gribov[4] some time ago, but for inexplicable reasons their work seems to have passed unnoticed. More recently, Durand[5] has reexamined Eq. (1) and discussed alternative methods of satisfying it.

In both the above examples, the results quoted are reached by means of a study of the relationship between the Regge-pole trajectory and residue functions, and the invariant functions (A, B in πN; F_S, F_V, F_T, F_A, F_P in NN) used in the epxression for the scattering matrix of the process concerned. It is the assumed analyticity of these invariant functions, i.e., their nonsingular behavior at $t=0$ or $u=0$,[6] which leads to the above results.

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[1] M. L. Goldberger and C. Edward Jones, Phys. Rev. Letters **17**, 105 (1966); D. Z. Freedman, C. E. Jones, and J. M. Wang, Phys. Rev. **155**, 1645 (1967).

[2] V. N. Gribov, L. Okun', and I. Ya. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **45**, 1114 (1963) [English transl.: Soviet Phys.—JETP **18**, 769 (1964)].

[3] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[4] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)].

[5] Loyal Durand, III, Phys. Rev. Letters **18**, 58 (1967).

[6] In a theory with exchange of mass-zero particles, this would not be true.

Since the Regge poles which contribute to the above process also contribute to a vast number of other processes, it is of great importance to inquire as to the existence of analogous equations of constraint in other processes, and as to whether the conditions imposed on the Regge-pole functions in satisfying, say Eq. (1), are sufficient for, or even *compatible with*, the conditions demanded by these further constraint equations.

There is little hope of discovering or studying constraint equations in general by the methods used in the πN and NN cases since to begin with one has little idea of how to go about formulating the decomposition of the scattering matrix into invariant functions. However, the clue to an alternative method emerges when one realizes that the particular combination of t-channel amplitudes occurring in Eq. (1) has a very simple significance. It is just that combination of t-channel amplitudes which for $t\approx 0$ is equal to one of the s-channel $NN \rightarrow NN$ helicity-flip amplitudes, i.e.,

$$\bar{f}_1 - z_t \bar{f}_4 - \bar{f}_3 \approx \phi_4 \quad \text{for} \quad t\approx 0\,, \tag{2}$$

where

$$\phi_4 = \langle +\tfrac{1}{2} - \tfrac{1}{2} | \phi | -\tfrac{1}{2} + \tfrac{1}{2} \rangle$$

in the notation of Ref. 3. The behavior demanded in Eq. (1) is then just a consequence of the kinematical requirement that for $\theta \rightarrow 0$, (where θ is the s-channel c.m. scattering angle)

$$\begin{aligned} \phi_4 &\propto \sin^2(\tfrac{1}{2}\theta) \\ &\propto t\,. \end{aligned} \tag{3}$$

The constraint (1) is thus an immediate consequence of the conservation of angular momentum and of the fact that for processes of the type $m_1+m_2 \rightarrow m_1+m_2$, θ is proportional to t for small θ. In this form the generalization to processes involving particles of arbitrary spin is fairly straightforward, and will be dealt with fully in Sec. IV.[7] We mention here only that one finds a vast number of constraint equations; their number going up roughly as the fourth power of the spin, for example, in processes like fermion-fermion scattering.

In processes analogous to backward (u small) πN scattering, which we prefer to describe as forward (t small) processes of the type $m_1+m_2 \rightarrow m_2+m_1$, $(m_1 \neq m_2)$, the point $\theta=0$ does not coincide, at finite energies, with the point $t=0$, so that the above method is not directly applicable. Nevertheless, one finds that a very similar approach, using the properties of the crossing matrix, suffices. One also finds constraints analogous to (1) at $t \sim 0$ in all processes of type $m_1+m_2 \rightarrow m_2+m_1$, $m_1 \neq m_2$ even when the exchanged Regge pole is bosonic. The precise description of this situation and the differences between the fermionic and bosonic cases are dealt with in Sec. III.

Since the same Regge poles which contribute to processes like $m_1+m_2 \rightarrow m_1+m_2$ will also contribute to processes of the type $m_1+m_2 \rightarrow m_2+m_1$ one is again faced with the question of compatability: Are the properties enforced on the Regge-pole functions by constraints arising in the first type of process automatically compatible with the properties enforced in the latter type of process? The answer to this question depends to some extent on the fact that the constraint equations do not lead to unique statements about the Regge trajectories and residues, and there is an element of arbitrariness involved in our choice as to what properties we shall consider acceptable.

Basically we shall divide the solutions of the constraint equations into three main types.

(a) *Conspiratorial:* If the constraint equations are satisfied by demanding a relationship between the *trajectories* of different Regge poles, (i.e., poles with different internal quantum numbers), we shall say there is a conspiracy among them.

(b) *Evasive:* If the constraint equations can be satisfied without demanding a relationship amongst trajectories, but simply by enforcing certain conditions on the *residue* functions, then we shall say that a conspiracy is evaded, and shall refer to the situation as evasion.

(c) *Daughterlike:* If the constraint equations are satisfied by demanding the existence of sequences of Regge poles with the same internal quantum numbers but different trajectory functions then the solution will be called daughterlike.[8]

There always exists the possibility that the Regge poles can satisfy the constraint equations by decoupling themselves completely from the process at $t=0$. This is a rather unacceptable situation which we shall refer to as trivial evasion.

In this paper we show the following:

(i) For bosonic poles or poles of even fermion number it is never *necessary* to have a conspiracy. The conditions, which lead to the necessity for the existence of pairs of opposite parity fermion trajectories, are automatically satisfied by the requirements of the factorization theorem when the poles are bosonic or of even fermion number (Sec. III).

(ii) The constraint conditions analogous to (2), for the case of arbitrary spin, do not require a conspiracy and can always be satisfied by a nontrivial evasive solution. This result is valid *to all orders* in s, and leads to s-channel amplitudes whose contribution from each Regge pole has a behavior in t as $t \rightarrow 0$ which is factorizable for all s (Sec. IV).

(iii) In any process the leading term as $s \rightarrow \infty$ of the contribution of each Regge pole to the s-channel

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[7] In the course of preparation of this manuscript we received a copy of a paper by E. Abers and V. Teplitz [Phys. Rev. **158**, 1365 (1967)] which applies rather similar methods to the question of the Reggeization of field theories.

[8] In a conspiratorial or daughterlike situation it may also be necessary to enforce some conditions on the residue functions.

helicity amplitudes has the behavior

$$f_{cd;ab}^{(s)} \propto t^{\frac{1}{2}(|a-c|+|b-d|)} \quad \text{as} \quad t \to 0$$

in the absence of conspiracy.

Although conspiracy is not necessary, there is no obvious reason why it might not nevertheless occur. We have been unable to solve the general problem of compatibility for conspiratorial solutions. It does seem, however, from looking at examples, that the standard conspiratorial solution used in the NN problem[4] will not work in other cases. This, we feel, is very reasonable since if there are conspiracies the NN process ought not to be a good place to find them since only three of the four possible types of Regge pole can occur in nucleon-nucleon scattering. In this paper we have looked only at conspiracies of the parity doublet type, and the relevant behavior of $f_{cd;ab}^{(s)}$ is given in Eq. (68).

The question of daughter trajectories is not dealt with in detail here, and a brief discussion of their role in equal-mass constraints, where they can provide an alternative to the evasive solution, is given in Sec. VI. A critique of the group-theoretic approach[9] to the constraints at $t=0$ is given. The proof of the uniqueness of the Lorentz-pole hypothesis[10] is questioned, and the general problem of group theory for unequal-mass processes is touched upon.

From the various solutions there follow very striking experimental consequences. For the evasive type, large numbers of amplitudes vanish or get related to each other for $t \approx 0$. For the conspiratorial and daughter types, one is predicting the existence of hitherto unidentified Regge trajectories which ought to manifest themselves as particles and resonances. Thus if one continues to take the Regge-pole model seriously, one can hope to distinguish between the types of solution experimentally.

In what follows, we shall make use of the concept of the kinematically normal behavior (k.n.b.) of an amplitude. It is defined as the most singular behavior an amplitude may have consistent with the finiteness of all experimental parameters. It is this behavior which is yielded for example, by Wang's analysis of singularity-free helicity amplitudes.[11] In practice, however, in any dynamical theory the actual behavior of amplitudes may differ from their k.n.b. (For example, k.n.b. is not consistent with the factorization theorem.) Whenever this is so, there will always be some experimental consequence. Some analysis in this direction has been carried out in Wang's later paper.[11]

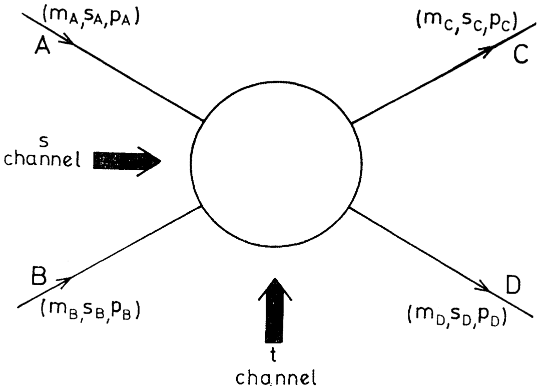

FIG. 1. Definition of the scattering channels.

In Sec. II we discuss the kinematics and crossing properties of an arbitrary mass scattering process. With only minor modifications, we shall follow the notation of Ref. 11.

In Secs. III–V we study the constraint equations for processes of the type $m_1+m_2 \to m_3+m_4$ with $m_1 \neq m_3$, $m_2 \neq m_4$; $m_1+m_2 \to m_1+m_2$ and $m_1+m_2 \to m_1+m_3$, $m_2 \neq m_3$, respectively. Several examples are worked out in detail.

In Sec. VI we consider the group-theoretic approach to the constraints at $t=0$ and in the final section (VII) we attempt to discuss at a more fundamental level the origin and cause of the constraints and their implications for the Regge theory as presently interpreted. We compare the situation at $t=0$ in the Regge-pole theory with the simpler case of elementary particle exchange. Some remarks are made which may be relevant to recent attempts to generalize the concept of Regge poles.[9]

II. KINEMATICS, CROSSING, AND REGGEIZATION

A. Kinematics

We consider a scattering process

$$A+B \to C+D$$

involving particles of arbitrary mass m_A, m_B, $\cdots$, arbitrary spin s_A, s_B, $\cdots$, and four-momenta p_A, p_B, $\cdots$. The physical process takes place in the s channel (see Fig. 1) and the t channel is defined as the process

$$\bar{D}+B \to C+\bar{A},$$

with the definitions

$$s=(p_A+p_B)^2$$

and

$$t=(p_A-p_C)^2. \tag{4}$$

The first particle on either side of a reaction formula is to be treated as "particle number 1," in the sense of Jacob and Wick,[12] when defining helicity amplitudes.

Following Wang[11] we define s-channel helicity amplitudes $f_{cd;ab}^{(s)}(s,t)$, which are related to the Jacob-Wick

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[9] A. Sciarrino and M. Toller, University of Rome, Internal note No. 108, 1966 (unpublished); M. Toller, Nuovo Cimento 37, 8631 (1965); D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **18**, 863 (1967).

[10] J. Finkelstein and J. M. Wang, University of California Radiation Laboratory Report No. UCRL-17500, 1967 (unpublished).

[11] Ling-Lie Chan Wang, Phys. Rev. **142**, 1187 (1966); **153**, 1664 (1967).

[12] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

helicity amplitudes as follows:

$$f_{cd;ab}^{(s)}(s,t)=2\pi(sp_{CD}/p_{AB})^{1/2}f_{cd;ab}^{(\text{Jacob-Wick})}. \quad (5)$$

Subscript a, b, $\cdots$ refer, of course, to the helicities of particles A, B, $\cdots$, etc., and p_{AB} and p_{CD} are the magnitudes of the c.m. momenta of particles A, B, and C, D.

In an analogous way we define t-channel helicity amplitudes $f_{c\bar{a};\bar{d}b}^{(t)}(t,s)$, where $\bar{a}$, $\bar{d}$ are the helicities of $\bar{A}$, $\bar{D}$ the antiparticles of A and D.

In the c.m. system of the s channel, with scattering angle θ_s, defined as the angle between A and C, we have

$$\cos\theta_s=(1/\mathcal{S}_{AB}\mathcal{S}_{CD})[2st+s^2-s\sum m^2 +(m_A^2-m_B^2)(m_C^2-m_D^2)], \quad (6)$$

with

$$\mathcal{S}_{ij}^2=[s-(m_i-m_j)^2][s-(m_i+m_j)^2] \quad (7)$$

and

$$\sum m^2=m_A^2+m_B^2+m_C^2+m_D^2. \quad (8)$$

We also have for the c.m. momenta

$$p_{ij}^2=(1/4s)\mathcal{S}_{ij}^2, \quad ji=AB \text{ or } CD. \quad (9)$$

The physical region for all three channels is given by

$$\phi(s,t)\geq 0, \quad (10)$$

where

$$\begin{aligned}\phi(s,t)=st(\textstyle\sum m^2-s-t)-s(m_B^2-m_D^2)(m_A^2-m_C^2)\\ -t(m_A^2-m_B^2)(m_C^2-m_D^2)-(m_A^2m_D^2-m_B^2m_C^2)\\ \times(m_A^2+m_D^2-m_B^2-m_C^2). \quad (11)\end{aligned}$$

In terms of ϕ we have

$$\sin\theta_s=2[s\phi(s,t)]^{1/2}/\mathcal{S}_{AB}\mathcal{S}_{CD}, \quad 0\leq\theta_s\leq\pi. \quad (12)$$

In the c.m. system of the t channel, with scattering angle θ_t defined as the angle between $\bar{D}$ and C we have

$$\cos\theta_t=(1/\mathcal{T}_{\bar{A}C}\mathcal{T}_{B\bar{D}})[2st+t^2-t\sum m^2 +(m_D^2-m_B^2)(m_C^2-m_A^2)], \quad (13)$$

with

$$\mathcal{T}_{ij}^2=[t-(m_i+m_j)^2][t-(m_i-m_j)^2] \quad (14)$$

and

$$p_{ij}^2=(1/4t)\mathcal{T}_{ij}^2, \quad ij=\bar{A}C \text{ or } B\bar{D}. \quad (15)$$

Finally,

$$\sin\theta_t=2[t\phi(s,t)]^{1/2}/\mathcal{T}_{\bar{A}C}\mathcal{T}_{B\bar{D}}. \quad (16)$$

B. Crossing

The helicity amplitudes of the s and t channels are related to each other via the crossing matrix of Trueman and Wick[13] as follows:

$$f_{cd;ab}^{(s)}=\sum_{\bar{a}'b'c'\bar{d}'}M_{ca;db}^{c'\bar{a}';\bar{d}'b'}f_{c'\bar{a}';\bar{d}'b'}^{(t)}, \quad (17)$$

where

$$\begin{aligned}M_{cadb}^{c'\bar{a}';\bar{d}'b'}=d_{c'c}^{s_C}(\chi_C)d_{\bar{a}'a}^{s_A}(\chi_A)\\ \times d_{\bar{d}'d}^{s_D}(\chi_D)d_{b'b}^{s_B}(\chi_B)\,; \quad (18)\end{aligned}$$

and the angles χ are given by[9]

$$\begin{aligned}\cos\chi_A=(1/\mathcal{S}_{AB}\mathcal{T}_{\bar{A}C})[-(s+m_A^2-m_B^2)(t+m_A^2-m_C^2)\\ -2m_A(m_C^2-m_A^2+m_B^2-m_D^2)],\\ \cos\chi_B=(1/\mathcal{S}_{AB}\mathcal{T}_{B\bar{D}})[(s+m_B^2-m_A^2)(t+m_B^2-m_D^2)\\ -2m_B^2(m_C^2-m_A^2+m_B^2-m_D^2)],\\ \cos\chi_C=(1/\mathcal{S}_{CD}\mathcal{T}_{\bar{A}C})[(s+m_C^2-m_D^2)(t+m_C^2-m_A^2)\\ -2m_C^2(m_C^2-m_A^2+m_B^2-m_D^2)],\\ \cos\chi_D=(1/\mathcal{S}_{CD}\mathcal{T}_{B\bar{D}})[-(s+m_D^2-m_C^2)(t+m_D^2-m_B^2)\\ -2m_D^2(m_C^2-m_A^2+m_B^2-m_D^2)],\end{aligned} \quad (19)$$

or, in terms of $\phi(s,t)$,

$$\begin{aligned}\sin\chi_A=2m_A\sqrt{\phi(s,t)}/\mathcal{S}_{AB}\mathcal{T}_{\bar{A}C}\,,\\ \sin\chi_B=2m_B\sqrt{\phi(s,t)}/\mathcal{S}_{AB}\mathcal{T}_{B\bar{D}}\,,\\ \sin\chi_C=2m_C\sqrt{\phi(s,t)}/\mathcal{S}_{CD}\mathcal{T}_{\bar{A}C}\,,\\ \sin\chi_D=2m_D\sqrt{\phi(s,t)}/\mathcal{S}_{CD}\mathcal{T}_{B\bar{D}}\,.\end{aligned} \quad (20)$$

We shall be interested in the behavior of θ_t, θ_s and the χ_i as $t\rightarrow 0$. It is clear from the above equations that this behavior will be a sensitive function of the masses involved, and we shall therefore handle the analysis in three distinct sections for the cases $m_A\neq m_C$, $m_B\neq m_D$; $m_A=m_C$, $m_B=m_D$; and $m_A\neq m_C$, $m_B=m_D$.

C. Reggeization

The Reggeization of arbitrary spin processes has already been discussed in the literature.[14,15] We shall essentially follow the method of Gell-Mann *et al.*,[14] though we shall make some minor generalizations to their formalism. To begin with, in order to classify the Regge poles, we introduce a new operator

$$R_{ij}=\eta_i\eta_jP_{ij}\,, \quad (21)$$

where P_{ij} is the exchange operator of Jacob and Wick[12] and η_j is the intrinsic parity factor for particle j, and construct normalized eigenstates of angular momentum J, parity P, R_{ij}, and helicity:

$$\begin{aligned}|J;\lambda_i\lambda_j;p,\rho\rangle=2[(1+\delta_{\lambda_i,\lambda_j})(1+\delta_{\lambda_i,-\lambda_j})]^{-1/2}\\ \times\{|J;\lambda_i\lambda_j\rangle+p\rho|J;-\lambda_j-\lambda_i\rangle+p\eta_i\eta_j(-1)^{s_i+s_j-v}\\ \times[|J;-\lambda_i-\lambda_j\rangle+p\rho|J;\lambda_j\lambda_i\rangle]\}\,, \quad (22)\end{aligned}$$

where

$$\begin{aligned}v&=0 \text{ for } J \text{ integral}\\ &=\tfrac{1}{2} \text{ for } J \text{ half-integral}\,,\end{aligned}$$

and

$$p=\pm1, \quad \rho=\pm1.$$

These states are correctly normalized to one, and have the property that

$$P|J;\lambda_i\lambda_j;p,\rho\rangle=p(-1)^{J-v}|J;\lambda_i\lambda_j;p,\rho\rangle \quad (23)$$

and

$$R_{ij}|J;\lambda_i\lambda_j;p\rho\rangle=\rho(-1)^{J-v}|J;\lambda_i\lambda_j;p,\rho\rangle. \quad (24)$$

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[13] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

[14] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

[15] F. Calogero, J. Charap, and E. Squires, Ann. Phys. (N. Y.) **25**, 325 (1963).

TABLE I. Allowed states $|J;\lambda_i\lambda_j;p,\rho\rangle$ for $F\bar{F}$, BB, and FF systems. Note that (i) $\lambda_i\lambda_j$ means $\lambda_i\neq\lambda_j$ and $\lambda_i\neq-\lambda_j$; (ii) states with $\lambda_i=\lambda_j=0$ are shown separately.

ρ	p	$F\bar{F}$		BB		FF	
+	+	$\lambda_i\lambda_j$ $\lambda_i\lambda_i$ $\lambda_i\ -\lambda_i$	$G(-1)^T=(-1)^J$	$\lambda_i\lambda_j$ $\lambda_i\lambda_i$ $\lambda_i\ -\lambda_i$ 00	$T+J$ even	$\lambda_i\lambda_j$ $\lambda_i\ -\lambda_i$	$T+J$ even
	−	$\lambda_i\lambda_j$ $\lambda_i\lambda_i$		$\lambda_i\lambda_j$ $\lambda_i\lambda_i$		$\lambda_i\lambda_j$	
−	+	$\lambda_i\lambda_j$	$G(-1)^T=-(-1)^J$	$\lambda_i\lambda_j$	$T+J$ odd	$\lambda_i\lambda_j$ $\lambda_i\lambda_i$	$T+J$ odd
	−	$\lambda_i\lambda_j$ $\lambda_i\ -\lambda_i$		$\lambda_i\lambda_j$ $\lambda_i\ -\lambda_i$		$\lambda_i\lambda_j$ $\lambda_i\lambda_i$ $\lambda_i\ -\lambda_i$	

Note that when $m_i=m_j$ we have

$$R_{ij}=P_{ij}\quad\text{for } BB \text{ and } FF \text{ systems}$$
$$=-P_{ij}\quad\text{for } F\bar{F} \text{ systems}.$$

Also we have that $R_{ij}=G(-1)^T$ or $C(-1)^T$ when these are applicable. Thus R_{ij} is a conserved operator. Since J is conserved we see that the quantum numbers p, and ρ are individually conserved.

(If C or G are not applicable, e.g., for fermionic poles, or unequal-mass cases, we simply ignore ρ and deal with states obtained from (22) by formally putting $\rho=0$).

The quantum numbers p, ρ together with baryon number, isotopic spin and signature (τ) provide a convenient labelling system for arbitrary Regge poles. In Table I we list the allowed helicity states for various values of p and ρ for the cases of $F\bar{F}$, BB, and FF. The situation for other systems, e.g., F_1F_2, where F_1 and F_2 are different fermions, is easily deduced. The restrictions arising from the generalized Pauli exclusion principle would no longer apply.

We now introduce modified t-channel helicity amplitudes[14]

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}=(\sqrt{2}\cos\tfrac{1}{2}\theta_t)^{-|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|}\times(\sqrt{2}\sin\tfrac{1}{2}\theta_t)^{-|\Lambda(\bar{d}b)-\Lambda(ca)|}f_{c\bar{a};\bar{d}b}{}^{(t)}\,,\qquad(25)$$

where $\Lambda(c\bar{a})=c-\bar{a}$, etc., and take combinations of them to form "parity symmetry conserving amplitudes."

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(p,\rho)=[(1+\delta_{\bar{d}b})(1+\delta_{\bar{d}-b})(1+\delta_{c\bar{a}})(1+\delta_{c-\bar{a}})]^{-1/2}\times\{\bar{f}_{c\bar{a};\bar{d}b}+p\rho\bar{f}_{-\bar{a}-c;\bar{d}b}+p\eta_C\eta_{\bar{A}}(-1)^{s_C+s_A-v}\times(-1)^{\Lambda(\bar{d}b)+\lambda_m}[\bar{f}_{-c-\bar{a};\bar{d}b}+p\rho\bar{f}_{\bar{a}c;d\bar{b}}]\}\,,\qquad(26)$$

where

$$\lambda_m=\max\{|\Lambda(\bar{d}b)|;|\Lambda(c\bar{a})|\}.$$

These amplitudes have a partial-wave expansion involving the functions $e^{J\pm}(\theta)$ introduced in Ref. 13.

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(p,\rho)=\frac{\pi\sqrt{t}}{(p_{\bar{D}B}p_{\bar{A}C})^{1/2}}\sum_J(2J+1)[\langle c\bar{a}|T^J(p,\rho)|\bar{d}b\rangle\times e_{\Lambda(\bar{d}b),\Lambda(c\bar{a})}{}^{J+}(\theta_t)+\langle c\bar{a}|T^J(-p,-\rho)|\bar{d}b\rangle\times e_{\Lambda(\bar{d}b),\Lambda(c\bar{a})}{}^{J-}(\theta_t)]\,,\qquad(27)$$

and are thus dominated by the Regge poles in $T^J(p,\rho)$ as $z_t=\cos\theta_t\to\infty$.

Equation (27), because of its good analytic properties, is the most suitable starting point for the Sommerfeld-Watson transform and the Reggeization procedure; and this is carried out exactly as described in Ref. 14 so we shall not discuss it any further here.

However, it is sometimes more convenient to deal directly with the unmodified t-channel helicity amplitudes, and we shall note here a very useful formula for the partial-wave expansion of certain linear combinations of them. For arbitrary coefficients A, B, C, D we have

$$Af_{c\bar{a};\bar{d}b}{}^{(t)}+Bf_{-c-\bar{a};\bar{d}b}{}^{(t)}+Cf_{-\bar{a}-c;\bar{d}b}{}^{(t)}+Df_{\bar{a}c;\bar{d}b}{}^{(t)}$$
$$=\frac{\pi\sqrt{t}}{(p_{\bar{A}C}p_{\bar{D}B})^{1/2}}\times\tfrac{1}{4}[(1+\delta_{\bar{a}c})(1+\delta_{\bar{a}-c})(1+\delta_{\bar{d}b})\times(1+\delta_{\bar{d}-b})]^{1/2}\times\sum_{p,\rho}\sum_J(2J+1)[(A+p\rho C)\times d_{\Lambda(\bar{d}b),-\Lambda(c\bar{a})}{}^J(\theta_t)+p\eta_C\eta_{\bar{A}}(-1)^{s_c+s_A-v}\times(B+p\rho D)d_{\Lambda(\bar{d}b),-\Lambda(c\bar{a})}{}^J(\theta_t)]\times\langle c\bar{a}|T^J(p,\rho)|\bar{d}b\rangle.\qquad(28)$$

The Reggeization of Eq. (28) then leads to the formula

$$Af_{c\bar{a};\bar{d}b}{}^{(t)}+Bf_{-c-\bar{a};\bar{d}b}{}^{(t)}+Cf_{-\bar{a}-c;\bar{d}b}{}^{(t)}+Df_{\bar{a}c;\bar{d}b}{}^{(t)}$$
$$=(-1)^{1+v+\Lambda(\bar{d}b)}\frac{\pi^2\sqrt{t}}{(p_{C\bar{A}}p_{\bar{D}B})^{1/2}}\frac{1}{4}[(1+\delta_{\bar{a}c})(1+\delta_{\bar{a}-c})\times(1+\delta_{\bar{d}b})(1+\delta_{\bar{d}-b})]^{1/2}\times\sum_{p,\rho,\tau}[2\alpha(p,\rho,\tau)+1]\times\zeta_\tau r_{c\bar{a};\bar{d}b}(p,\rho,\tau)[(A+p\rho C)d_{\Lambda(\bar{d}b),-\Lambda(c\bar{a})}{}^{\alpha(p,\rho,\tau)}(-z_t)+p\eta_C\eta_{\bar{A}}(-1)^{s_C+s_A-v}(B+p\rho D)\times d_{\Lambda(\bar{d}b),\Lambda(c\bar{a})}{}^{\alpha(p,\rho,\tau)}(-z_t)]\,,\qquad(29)$$

where the sum runs over all Regge poles. Here ζ_τ is the usual signature factor

$$\zeta_\tau=\frac{1+\tau e^{-i\pi[\alpha(p,\rho,\tau)-v]}}{2\sin\pi[\alpha(p,\rho,\tau)-v]}\,. \tag{30}$$

The residue functions $r(p,\rho,\tau;t)$ are the unmodified residues of $T^J(p,\rho)$ at the relevant poles in the J plane.

III. PROCESSES WITH $m_A\neq m_C$; $m_B\neq m_D$

Here the t-channel process is of the type (unequal-mass pair) → (unequal-mass pair) and we shall thus denote it as a UU process. We wish to investigate the behavior of the modified t-channel amplitudes $\bar{f}^{(t)}$ as $t\to 0$ in order to deduce properties of the Regge poles in this region.

It is assumed that the $\bar{f}^{(t)}(t,s)$ are functions whose analyticity in s is comparable with the Lehmann ellipse analyticity of the spinless case. If we define analogous modified helicity functions in the s channel as

$$\bar{f}_{cd;ab}{}^{(s)}=(\sqrt{2}\cos\tfrac{1}{2}\theta_s)^{-|\Lambda(ab)+\Lambda(cd)|}\times(\sqrt{2}\sin\tfrac{1}{2}\theta_s)^{-|\Lambda(ab)-\Lambda(cd)|}\times f_{cd;ab}{}^{(s)}\,, \tag{31}$$

then the $\bar{f}^{(s)}(s,t)$ will have a region of analyticity in t comparable with the Lehmann ellipse.

Using (25), (31), and the inverse of (17) we can relate the $\bar{f}^{(t)}$ to the $\bar{f}^{(s)}$ getting

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}=(\sqrt{2}\cos\tfrac{1}{2}\theta_t)^{-|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|}(\sqrt{2}\sin\tfrac{1}{2}\theta_t)^{-|\Lambda(\bar{d}b)-\Lambda(c\bar{a})|}\times M_{c'd';a'b'}{}^{c\bar{d}\,\bar{a}b}(\sqrt{2}\cos\tfrac{1}{2}\theta_s)^{|\Lambda(a'b')+\Lambda(c'd')|}\times(\sqrt{2}\sin\tfrac{1}{2}\theta_s)^{|\Lambda(a'b')-\Lambda(c'd')|}\bar{f}_{c'd';a'b'}{}^{(s)}\,. \tag{32}$$

Now as $t\to 0$ we see from (13) and (16) that

(i) If $m_A>m_C$; $m_B>m_D$ or $m_A<m_C$; $m_B<m_D$ then

$$\theta_t\propto t^{1/2}\,, \tag{33}$$

so that

$$\cos\tfrac{1}{2}\theta_t\to 1 \tag{34}$$

and

$$\sin\tfrac{1}{2}\theta_t\propto t^{1/2}.$$

(ii) If $m_A>m_C$; $m_B<m_D$ or vice versa, then

$$\theta_t-\pi\propto t^{1/2}\,, \tag{35}$$

so that

$$\cos\tfrac{1}{2}\theta_t\propto t^{1/2}$$

and

$$\sin\tfrac{1}{2}\theta_t\to 1\,.$$

All the other functions occurring in (32) tend to unexceptional limits as $t\to 0$. We thus find that for case (i)

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}=O(t^{-\frac{1}{2}|\Lambda(\bar{d}b)-\Lambda(c\bar{a})|})\,, \tag{36}$$

and for case (ii)

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}=O(t^{-\frac{1}{2}|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|})\,. \tag{37}$$

The behavior of the $\bar{f}^{(t)}$ as $t\to 0$ could of course be much less singular than

$$t^{-\frac{1}{2}|\Lambda(\bar{d}b)-\Lambda(c\bar{a})|}$$

or

$$t^{-\frac{1}{2}|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|}$$

since there could be subtle cancellations in the summation involved in (32). However, their behavior *cannot be more singular.* We shall refer to this most singular allowed behavior as the kinematically normal behavior (k.n.b.) of the amplitude.

From Eq. (26) we have for the parity-symmetry conserving amplitudes that

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(p)=F_1+p\eta_C\eta_{\bar{A}}(-1)^{s_C+s_A-v}\times(-1)^{\Lambda(ab)+\lambda_m}F_2 \tag{38}$$

and

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(-p)=F_1-p\eta_C\eta_{\bar{A}}(-1)^{s_C+s_A-v}\times(-1)^{\Lambda(ab)+\lambda_m}F_2\,, \tag{39}$$

where for case (i)

$$\begin{aligned}F_1&=O(t^{-\frac{1}{2}|\Lambda(\bar{d}b)-\Lambda(c\bar{a})|})\,,\\F_2&=O(t^{-\frac{1}{2}|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|})\,,\end{aligned} \tag{40}$$

and for case (ii),

$$\begin{aligned}F_1&=O(t^{-\frac{1}{2}|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|})\,,\\F_2&=O(t^{-\frac{1}{2}|\Lambda(\bar{d}b)-\Lambda(c\bar{a})|})\,.\end{aligned} \tag{41}$$

It is thus clear that unless

$$\Lambda(\bar{d}b)=\Lambda(c\bar{a})=0\,, \tag{42}$$

we will always have

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(p)=\pm\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(-p)\quad\text{as}\quad t\to 0. \tag{43}$$

The result can be stated more precisely as follows. Define

$$\begin{aligned}s_m&=\operatorname{sgn}[(m_A-m_C)(m_B-m_D)]\,,\\s_\Lambda&=\operatorname{sgn}(|\Lambda(\bar{d}b)-\Lambda(c\bar{a})|-|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|)\,,\\\Lambda&=\big|\,|\Lambda(\bar{d}b)-\Lambda(c\bar{a})|-|\Lambda(\bar{d}b)+\Lambda(c\bar{a})|\,\big|\\&=2\min\{|\Lambda(\bar{d}b)|\,;\,|\Lambda(c\bar{a})|\}\,.\end{aligned} \tag{44}$$

Then

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(p)=s_ms_\Lambda\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(-p)[1+O(t^{\frac{1}{2}\Lambda})]\,, \tag{45}$$

and the k.n.b. is

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(p)\propto t^{-\frac{1}{2}(|\Lambda(\bar{d}b)|+|\Lambda(c\bar{a})|)}\,. \tag{46}$$

Let us now examine the effect of this behavior on the parameters of the Regge poles. From (27) and the known properties of the $e^{J\pm}$ functions, we deduce that the k.n.b. for the residues of $T^J(p,\rho)$ is

$$r_{c\bar{a};\bar{d}b}(p,\rho)\propto t^{\lambda_m-\alpha(p,\rho)}t^{-\frac{1}{2}(|\Lambda(\bar{d}b)|+|\Lambda(c\bar{a})|+1)}\quad\text{as}\quad t\to 0. \tag{47}$$

[There is, of course, the well-known problem of how to use Regge theory at $t=0$ in UU-type processes, but all known methods, daughters or no daughters, lead to Eq. (47).]

Now suppose that as $s \to \infty$, $\bar{f}^{(t)}(p)$ is dominated by some Regge pole with trajectory function $\alpha(p,\rho)$. Then Eq. (45) demands either of the following:

($\mathcal{C}$) There is a *conspiracy*, i.e., there exists a second Regge pole with quantum number $-p$ (and therefore with opposite τP), whose trajectory is such that

$$\alpha(p) = \alpha(-p) \quad \text{at} \quad t=0, \tag{48}$$

and whose residues satisfy

$$r_{c\bar{a};\bar{d}b}(p) = s_m s_\Lambda r_{c\bar{a};\bar{d}b}(-p) \quad \text{at} \quad t=0. \tag{49}$$

($\mathcal{E}$) A conspiracy is *evaded*, i.e., there is no need for the second Regge pole, but in order to satisfy (45) the residues have to behave less singularly than their k.n.b. by a factor $t^{\frac{1}{2}\Lambda}$. In this case the Regge pole effectively decouples from all states with $\Lambda \neq 0$. It thus remains coupled to states which have

$$|\Lambda(\bar{d}b) - \Lambda(c\bar{a})| = |\Lambda(\bar{d}b) + \Lambda(c\bar{a})|, \tag{50}$$

i.e., states for which $\Lambda(\bar{d}b)$ and/or $\Lambda(c\bar{a}) = 0$.

Hence if there are no states satisfying (50), then a Regge pole seeking evasion would completely decouple itself from the process at $t=0$. But this is precisely the situation when the t channel has odd fermion number, e.g., is of the type $F+B \to B+F$. Thus fermionic poles, i.e., poles with odd fermion number, must conspire if they are to avoid total decoupling. This is just the result of Gribov *et al.*[2] in a more general guise.

Putting aside the extremely unpalatable alternative of total decoupling, we thus conclude that (a) fermionic poles must conspire and their residues can have k.n.b.; (b) bosonic (and even-fermion number[16]) poles have the choice of conspiring or avoiding conspiracy at the expense of decoupling partially from the process at $t=0$.

However, *this is not the final picture* since we have up to now ignored the consequences of the factorization theorem. If we concentrate on a given process then the factorization theorem imposes relationships amongst the various residues of a single Regge pole, in the form

$$r_{c_1\bar{a}_1;\bar{d}_1 b_1} r_{c_2\bar{a}_2;\bar{d}_2 b_2} = r_{c_1\bar{a}_1;\bar{d}_2 b_2} r_{c_2\bar{a}_2;\bar{d}_1 b_1}, \tag{51}$$

where c_1, c_2, $\cdots$, etc., refer to the different helicity states of particles C, $\cdots$, etc.

If we substitute into (51) the k.n.b. for the residues at $t=0$ we see that it is not satisfied in general. For we would require, e.g.,

$$\begin{aligned} \max\{|\Lambda(\bar{d}_1 b_1)|, |\Lambda(c_1\bar{a}_1)|\} + \max\{|\Lambda(\bar{d}_2 b_2)|, |\Lambda(c_2\bar{a}_2)|\} \\ = \max\{|\Lambda(c_1\bar{a}_1)|, |\Lambda(\bar{d}_2 b_2)|\} \\ + \max\{|\Lambda(\bar{d}_1 b_1)|, |\Lambda(c_2\bar{a}_2)|\}, \end{aligned} \tag{52}$$

which is not true in general. Thus the k.n.b. as given by Eq. (47) *is not compatible with factorization.*

To see what the most singular behavior is compatible with factorization, we consider the simple case $c_1 = \bar{d}_1$, $\bar{a}_1 = b_1$, $c_2 = \bar{d}_2$, $\bar{a}_2 = b_2$.

For the terms in (51) we now have

$$r_{c_1\bar{a}_1;c_1\bar{a}_1} r_{c_2\bar{a}_2;c_2\bar{a}_2} \propto t^{-1-2\alpha} \tag{53}$$

and

$$r_{c_1\bar{a}_1;c_2\bar{a}_2}{}^2 \propto t^{-(|\Lambda(c_1\bar{a}_1)|+|\Lambda(c_2\bar{a}_2)|+1)} t^{2[\max\{|\Lambda(c_1\bar{a}_1)|,|\Lambda(c_2\bar{a}_2)|\}-\alpha]};$$

and since this has to hold for all $c_1\bar{a}_1$, $c_2\bar{a}_2$ we see that the simplest consistent solution is to take

$$r_{c\bar{a};\bar{d}b} \propto t^{\frac{1}{2}\{|\Lambda(c\bar{a})|+|\Lambda(\bar{d}b)|-1-2\alpha\}} \quad \text{for bosonic poles} \tag{54}$$

and

$$r_{c\bar{a};\bar{d}b} \propto t^{\frac{1}{2}\{|\Lambda(c\bar{a})|+|\Lambda(\bar{d}b)|-2-2\alpha\}} \quad \text{for fermionic poles.} \tag{55}$$

Let us now compare this behavior with the k.n.b. given in (47). We have

$$r_{c\bar{a};\bar{d}b} \propto t^{\Lambda/2} \times \text{k.n.b.}, \quad \text{for bosons} \tag{56a}$$

and

$$r_{c\bar{a};\bar{d}b} \propto t^{\Lambda/2} t^{-1/2} \times \text{k.n.b.}, \quad \text{for fermions.} \tag{56b}$$

But Eq. (56a) gives precisely the behavior specified in $\mathcal{E}$ in order to evade a conspiracy. Thus for bosonic poles the above simple choice of residue behavior guarantees compatibility with evasion. *We thus conclude that there is no need for conspiracy in the case of bosonic or even-fermion number poles.*

On the other hand, for fermion poles the behavior demanded in (56b) is not strong enough to satisfy the requirements for evasion. We thus reinforce our earlier conclusion that *fermion poles must conspire;* or totally decouple from the process at $t=0$.

The above behavior leads to interesting experimental consequences.

To see these we consider what effect the behavior (54) and (55) has on the s-channel amplitudes. We have not been able to find any simple result for arbitrary s, but if we keep only the *leading term in s*, then from (19)

$$\begin{aligned} \cos\chi_A &\approx \pm 1 \mp t/|m_C{}^2 - m_A{}^2|, \\ \cos\chi_C &\approx \pm 1 \pm t/|m_C{}^2 - m_A{}^2|, \end{aligned} \tag{57}$$

according as $m_C{}^2 - m_A{}^2 \gtrless 0$; and

$$\begin{aligned} \cos\chi_B &\approx \pm 1 \pm t/|m_B{}^2 - m_D{}^2|, \\ \cos\chi_D &\approx \pm 1 \mp t/|m_B{}^2 - m_D{}^2|, \end{aligned} \tag{58}$$

according as $m_B{}^2 - m_D{}^2 \gtrless 0$. Thus if $m_C{}^2 - m_A{}^2 \gtrless 0$,

$$\begin{aligned} \chi_A &\approx (2t/|m_C{}^2 - m_A{}^2|)^{1/2}, \\ \chi_C &\approx (-2t/|m_C{}^2 - m_A{}^2|)^{1/2} \\ &\text{or} \\ \chi_A &\approx \pi - (2t/|m_C{}^2 - m_A{}^2|)^{1/2}, \\ \chi_C &\approx \pi - (-2t/|m_C{}^2 - m_A{}^2|)^{1/2}; \end{aligned} \tag{59}$$

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[16] In what follows, bosonic will also include even-fermion number.

and if $m_B{}^2-m_D{}^2\gtrless 0$,

$$\begin{aligned} &\chi_B\approx(-2t/|m_B{}^2-m_D{}^2|)^{1/2}, \\ &\chi_D\approx(2t/|m_B{}^2-m_D{}^2|)^{1/2} \\ \text{or} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad (60) \\ &\chi_B\approx\pi-(-2t/|m_B{}^2-m_D{}^2|)^{1/2}, \\ &\chi_D\approx\pi-(2t/|m_B{}^2-m_D{}^2|)^{1/2}. \end{aligned}$$

In what follows, it makes no difference whether the χ^s approach zero or π, so for simplicity, we shall consider the case with all $\chi^s\to 0$ as $t\to 0$.

Consider now $d_{\lambda\mu}{}^J(\epsilon)$ as $\epsilon\to 0$. Expanding about $\epsilon=0$, we have

$$d_{\lambda\mu}{}^J(\epsilon)=d_{\lambda\mu}{}^J(0)+\epsilon d_{\lambda\mu}{}^{(1)J}(0)+\frac{\epsilon^2}{2!}d_{\lambda\mu}{}^{(2)J}(0)+\cdots, \tag{61}$$

where

$$d_{\lambda\mu}{}^{(r)J}(0)=\frac{d^r}{d\epsilon^r}d_{\lambda\mu}{}^J(\epsilon)\,|_{\epsilon=0}. \tag{62}$$

Now

$$d_{\lambda\mu}{}^J(0)=\delta_{\lambda\mu}, \tag{63}$$

and by the definition of the $d_{\lambda\mu}{}^J$ functions,

$$\begin{aligned} \frac{d^{(r)}}{d\epsilon^r}d_{\lambda\mu}{}^J(\epsilon)&=\frac{d^r}{d\epsilon^r}\langle J,\lambda|e^{-i\epsilon J_y}|J,\mu\rangle \\ &=\langle\lambda|(-iJ_y)^r e^{-i\epsilon J_y}|\mu\rangle. \end{aligned}$$

So,

$$\begin{aligned} d_{\lambda\mu}{}^{(r)J}(0)&=\langle\lambda|(-iJ_y)^r|\mu\rangle \\ &=0 \quad \text{if } |\lambda-\mu|>r. \end{aligned} \tag{64}$$

Now substituting (54) into (29) we have, for $t\approx 0$, for bosonic poles

$$f_{c\bar{a};\bar{d}b}{}^{(t)}\propto t^{\frac{1}{2}\{|\Lambda(c\bar{a})|+|\Lambda(\bar{d}b)|\}}, \tag{65}$$

and putting this result into (17) and expanding each of the d functions in the crossing matrix as in (61), and using (64), we see that

$$f_{cd;ab}{}^{(s)}\propto t^{\frac{1}{2}\{|\Lambda(ac)|+|\Lambda(bd)|\}} \quad \text{(bosonic poles)} \tag{66a}$$

for no conspiracy.

For the fermionic poles we get

$$f_{cd;ab}{}^{(s)}\propto t^{\frac{1}{2}\{|\Lambda(ac)|+|\Lambda(bd)|-1\}} \quad \text{(fermionic poles).} \tag{66b}$$

[Equation (66) is true only for the leading term in s as $s\to\infty$.]

Thus the Regge-pole theory leads to a highly restricted spin structure as $t\to 0$ in processes dominated by bosonic or fermionic poles, and this results in many interesting experimental consequences.[17] The theory also demands the existence of parity doublets for fermionic poles and this too should manifest itself in remarkable experimental consequences.

Although the simple solution (54) for the residue behavior eliminates the necessity for conspiracy in the case of bosonic Regge poles, it is possible to find less simple factorizable behaviors for the residues which do require conspiracy. For example, we could take for the residues of each of two conspiring poles of opposite (p)

$$\begin{aligned} &r_{c\bar{a};\bar{d}b}\propto t^{\frac{1}{2}\{|\Lambda(c\bar{a})|+|\Lambda(\bar{d}b)|-2M-1-2\alpha\}} \\ &\qquad \text{for} \quad |\Lambda(c\bar{a})| \quad \text{and} \quad |\Lambda(\bar{d}b)|\geq M, \\ &r_{c\bar{a};\bar{d}b}\propto t^{\frac{1}{2}\{|\Lambda(c\bar{a})|-1-2\alpha\}} \quad \text{for} \quad |\Lambda(c\bar{a})|\geq M>|\Lambda(\bar{d}b)|, \\ &r_{c\bar{a};\bar{d}b}\propto t^{\frac{1}{2}\{|\Lambda(\bar{d}b)|-1-2\alpha\}} \quad \text{for} \quad |\Lambda(\bar{d}b)|\geq M>|\Lambda(c\bar{a})|, \\ &r_{c\bar{a};\bar{d}b}\propto t^{\frac{1}{2}(2M-1-2\alpha)} \\ &\qquad \text{for} \quad |\Lambda(\bar{d}b)| \quad \text{and} \quad |\Lambda(c\bar{a})|<M, \end{aligned} \tag{67}$$

where M is a positive integer, and where the residues must satisfy (49).

This leads to the following behavior for the leading term as $s\to\infty$ of the contribution of a pair of conspiring Regge poles:

Let $\lambda_{\max}=\max\{|\Lambda(ac)|;\ |\Lambda(db)|\}$ and

$$\lambda_{\min}=\min\{|\Lambda(ac)|;\ |\Lambda(db)|\}.$$

Then

$$\begin{aligned} &f_{cd;ab}{}^{(s)}\propto t^{\frac{1}{2}|\Lambda(ac)+\Lambda(db)|+(\lambda_{\min}-M)}, \quad \text{for} \quad \lambda_{\min}\geq M \qquad (68) \\ &f_{cd;ab}{}^{(s)}\propto t^{\frac{1}{2}|\Lambda(ac)+\Lambda(db)|+\frac{1}{2}\lambda_{\min}}, \quad \text{for} \quad \lambda_{\max}\geq M>\lambda_{\min} \\ &f_{cd;ab}{}^{(s)}\propto t^{\frac{1}{2}|\Lambda(ac)+\Lambda(db)|+M-\frac{1}{2}(\lambda_{\max}-\lambda_{\min})}, \quad \text{for} \quad \lambda_{\max}<M. \end{aligned}$$

This has the remarkable property that the *non-spin-flip* amplitude is suppressed by a factor t^M, whereas the amplitudes with $\lambda_{\min}=M$ or 0 and $\lambda_{\max}\geq M$ are allowed to have their normal behavior.

This should be contrasted with (66a) where only the non-spin-flip amplitude avoids suppression. The case $M=1$ has been applied to several processes in Ref. 17.

IV. PROCESSES WITH $m_A=m_C$; $m_B=m_D$

The t-channel process is now of the type (equal-mass pair) → (equal-mass pair) and we denote it as an EE process.

From Eqs. (11) and (12) we note that as $t\to 0$

$$\sin\theta_s\propto t^{1/2}, \tag{69}$$

and it then follows from Eq. (31) that

$$f_{cd;ab}{}^{(s)}\propto t^{\frac{1}{2}|\Lambda(ab)-\Lambda(cd)|} \quad \text{as} \quad t\to 0. \tag{70}$$

The behavior implied by (69) is of fundamental significance. It simply expresses the fact that in the forward direction ($\theta_s=0$ implies $t=0$ for EE processes) no net helicity flip is allowed if angular momentum is to be conserved. It therefore arises purely as a result of the rotational invariance of the theory.

Since each amplitude $f^{(s)}$ is expressed in terms of the $f^{(t)}$ via the crossing relation (17) we see that for every s-channel amplitude with nonzero net helicity flip there will exist a linear combination of t-channel amplitudes which has to vanish as $t\to 0$ at a prescribed rate. This will then imply constraints and relationships among the t-channel amplitudes as $t\to 0$ and will, upon

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[17] T. W. Rogers and G. C. Fox (to be published).

Reggeization, lead to relationships among the trajectories and residues of the participating Regge poles.

In order to study these conditions quantitatively, it will prove convenient to deal directly with the *unmodified* helicity amplitudes $f_{cd;ab}^{(s)}$ and $f_{c\bar{a};\bar{d}b}^{(t)}$ and to use Eq. (17) itself rather than its inverse which we have been using up to now.

From (69) and (17) we thus have that whenever $|\Lambda(ab)-\Lambda(cd)|\neq 0$,

$$\sum_{\bar{a}'b',c'\bar{d}'} M_{cabd}{}^{c'\bar{a}';\bar{d}'b'} f_{c'\bar{a}';\bar{d}'b'}{}^{(t)} \propto t^{\frac{1}{2}|\Lambda(ab)-\Lambda(cd)|} \quad \text{as} \quad t\longrightarrow 0. \tag{71}$$

We shall refer to these as *equations of constraint.*

The equations of constraint are obviously not all independent since parity conservation, time-reversal invariance, etc., reduce the number of independent s-channel amplitudes, and constraint equations arising from related s-channel amplitudes will clearly not give independent information. Therefore, one would conclude that there are as many constraint equations as there are independent s-channel amplitudes. Strictly speaking, in order to get the exact behavior of the t-channel amplitudes one should then solve the whole set of simultaneous equation (71) in the region $t\approx 0$. However, it will prove more convenient to introduce first an approximate type of k.n.b. for the t-channel amplitudes which gives their correct dependence on t as $t\longrightarrow 0$, but which does not guarantee that the Eq. (71) are satisfied. It is thus a necessary but not sufficient specification of the behavior as $t\longrightarrow 0$.

We obtain this behavior by studying the inverse of (17) and picking out in the expression for $f_{c\bar{a};\bar{d}b}^{(t)}$ the most singular term as $t\longrightarrow 0$. We shall not go into the details here since this is essentially the procedure adopted by Wang[11] in studying the behavior of $f^{(t)}$ at $t\approx 0$. However, it was not stressed by Wang that her conditions, while necessary, are insufficient to guarantee accord with the fundamental requirements of (70) and (71). As before, we shall refer to this approximate, and most singular behavior as the kinematically normal behavior.

The k.n.b. may be specified as follows: Let

$$\eta^{c\bar{a}\bar{d}b}=(\eta_C\eta_D/\eta_A\eta_B)(-1)^{s_A+s_B+s_C+s_D}(-1)^{c+\bar{a}+\bar{d}+b}. \tag{72}$$

Then the k.n.b. is given by

$$\begin{aligned} &f_{c\bar{a};\bar{d}b}^{(t)}\longrightarrow \text{const} \quad \text{as } t\longrightarrow 0,\ \text{if } \eta^{c\bar{a}\bar{d}b}=+1,\\ &f_{c\bar{a};\bar{d}b}^{(t)}\propto t^{1/2} \quad \text{as } t\longrightarrow 0,\ \text{if } \eta^{c\bar{a}\bar{d}b}=-1. \end{aligned} \tag{73}$$

In studying the constraint equation (71) it will sometimes happen that the k.n.b. specified in (73) is sufficient to satisfy one or more of the constraint equations. If this occurs, the relevant constraint equations do not carry any further information.

There is another mechanism which complicates the question as to which and how many of the equations are information carrying, and this is connected with parity conservation. To see this we rewrite the left-hand side of (71) as

$$\begin{aligned} L\equiv &\sum_{\bar{a}'b',c'\bar{d}'} M_{cadb}{}^{c'\bar{a}';\bar{d}'b'} f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}\\ &=\tfrac{1}{2}\sum_{\bar{a}'b',c'd'}\{M_{cadb}{}^{c'\bar{a}';\bar{d}'b'} f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}\\ &\qquad +M_{cadb}{}^{-c'-\bar{a}';-\bar{d}'-b'} f_{-c'-\bar{a}';-\bar{d}'-b'}{}^{(t)}\}. \end{aligned} \tag{74}$$

Using parity conservation,

$$f_{-c'-\bar{a}';-\bar{d}'-b}=\eta_g(-1)^{\Lambda(\bar{d}'b')-\Lambda(c'\bar{a}')} f_{c'\bar{a}';\bar{d}'b'}, \tag{75}$$

where

$$\eta_g=\frac{\eta_{\bar{A}}\eta_C}{\eta_B\eta_{\bar{D}}}(-1)^{s_C+s_A-s_D-s_B},$$

Eq. (74) becomes

$$\begin{aligned} L=\tfrac{1}{2}\sum_{\bar{a}'b',c'\bar{d}'}&\{M_{cadb}{}^{c'\bar{a}';\bar{d}'b'}+\eta_g(-1)^{\Lambda(\bar{d}'b')-\Lambda(c'\bar{a}')}\\ &\times M_{cadb}{}^{-c'-\bar{a}';-\bar{d}'-b'}\} f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}. \end{aligned} \tag{76}$$

Also since

$$d_{-\lambda,\mu}{}^{s}(\chi)=(-1)^{s-\mu}d_{\lambda,\mu}{}^{s}(\pi-\chi), \tag{77}$$

we have from (18) that

$$\begin{aligned} M_{cadb}{}^{-c'-\bar{a}';-\bar{d}'-b'}=&(-1)^{s_A+s_B+s_C+s_D-a-b-c-d}\\ &\times d_{\bar{a}'a}{}^{s_A}(\pi-\chi_A)d_{b'b}{}^{s_B}(\pi-\chi_B)d_{c'c}{}^{s_C}(\pi-\chi_C)\\ &\times d_{\bar{d}'\bar{d}}{}^{s_D}(\pi-\chi_D). \end{aligned} \tag{78}$$

Now from (19) as $t\longrightarrow 0$ all the $\chi\longrightarrow \frac{1}{2}\pi$. More precisely for small t one has

$$\chi_i=\tfrac{1}{2}\pi-v_it^{1/2}, \tag{79}$$

with

$$\begin{aligned} &v_A=-v_C=\mu_{AB}(1+v_{AB}t+\cdots)\\ \text{and}\\ &v_B=-v_D=-\mu_{BA}(1+v_{BA}t+\cdots), \end{aligned} \tag{80}$$

where

$$\begin{aligned} &\mu_{AB}=(m_B{}^2-m_A{}^2-s)/2im_A\,\mathcal{S}_{AB}\\ \text{and}\\ &v_{AB}=\tfrac{1}{12}[(1/m_A{}^2)-(2s/\,\mathcal{S}_{AB}{}^2)]. \end{aligned} \tag{81}$$

Thus from (78) and (79)

$$\begin{aligned} \lim_{t\to 0} M_{cadb}{}^{-c'-\bar{a}';-\bar{d}'-b'}=&(-1)^{s_A+s_B+s_C+s_D-a-b-c-d}\\ &\times\lim_{t\to 0} M_{cadb}{}^{c'\bar{a}';\bar{d}'b'}, \end{aligned} \tag{82}$$

and depending on the helicities involved, the two terms in the parentheses of (76) can either reinforce each other or cancel out. If they cancel then the leading term for small t goes at least like $t^{1/2}$ and this is sometimes sufficient to satisfy the constraint equation. (See the examples $\pi N\longrightarrow \pi N$ and $NN\longrightarrow NN$ below.)

Expanding the terms in parentheses in (76) about $t=0$, and collecting together the results of (76)–(82), we arrive at the final form of the constraint equations:

$$\sum_{\bar{a}'b'c'}\sum_{\bar{d}'\leq 0}\frac{1}{1+\delta_{\bar{d}'0}}\mathfrak{M}_{cadb}{}^{c'\bar{a}';\bar{d}'b'} f_{c'\bar{a}';\bar{d}'b'}{}^{(t)} \propto t^{\frac{1}{2}|\Lambda(ab)-\Lambda(cd)|}, \tag{83}$$

with

$$\mathfrak{M}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}=\theta(\eta_{cadb}{}^{c'\bar{a}'\bar{d}'b'})M^{(+)}{}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}$$
$$-t^{1/2}\theta(-\eta_{cadb}{}^{c'\bar{a}'\bar{d}'b'})M^{(-)}{}_{cadb}{}^{c'\bar{a}';\bar{d}'b'},\quad (84)$$

where

$$M^{(+)}{}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}=d_{\bar{a}'a}{}^{sA}d_{b'b}{}^{sB}d_{c'c}{}^{sC}d_{\bar{d}'d}{}^{sD}$$
$$+tv_A[v_B\Delta_{\bar{a}'a}{}^{sA}d_{c'c}{}^{sC}(\Delta_{b'b}{}^{sB}d_{\bar{d}'d}{}^{sD}$$
$$-\Delta_{\bar{d}'d}{}^{sD}d_{b'b}{}^{sB})+v_B\Delta_{c'c}{}^{sC}d_{\bar{a}'a}{}^{sA}$$
$$\times(\Delta_{\bar{d}'d}{}^{sD}d_{b'b}{}^{sB}-\Delta_{b'b}{}^{sB}d_{\bar{d}'b}{}^{sD})$$
$$-v_A(\Delta_{\bar{a}'a}{}^{sA}\Delta_{c'c}{}^{sC}d_{b'b}{}^{sB}d_{\bar{d}'d}{}^{sD}$$
$$+\Delta_{b'b}{}^{sB}\Delta_{\bar{d}'d}{}^{sD}d_{\bar{a}'a}{}^{sA}d_{c'c}{}^{sC})]+t^2v_A{}^2v_B{}^2$$
$$\times\Delta_{\bar{a}'a}{}^{sA}\Delta_{b'b}{}^{sB}\Delta_{c'c}{}^{sC}\Delta_{\bar{d}'d}{}^{sD}+\cdots;\quad (85)$$

$$M^{(-)}{}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}=v_Ad_{b'b}{}^{sB}d_{\bar{d}'d}{}^{sD}(\Delta_{\bar{a}'a}{}^{sA}d_{c'c}{}^{sC}$$
$$-\Delta_{c'c}{}^{sC}d_{\bar{a}'a}{}^{sA})+v_Bd_{\bar{a}'a}{}^{sA}d_{c'c}{}^{sC}$$
$$\times(\Delta_{b'b}{}^{sB}d_{\bar{d}'d}{}^{sD}-\Delta_{\bar{d}'d}{}^{sD}d_{b'b}{}^{sB})$$
$$+tv_Av_B[\Delta_{c'c}{}^{sC}\Delta_{\bar{d}'d}{}^{sD}(v_A\Delta_{\bar{a}'a}{}^{sA}d_{b'b}{}^{sB}$$
$$+v_B\Delta_{b'b}{}^{sB}d_{\bar{a}'a}{}^{sA})-\Delta_{\bar{a}'a}{}^{sA}\Delta_{b'b}{}^{sB}$$
$$\times(v_A\Delta_{c'c}{}^{sC}d_{\bar{d}'d}{}^{sD}$$
$$+v_B\Delta_{\bar{d}'d}{}^{sD}d_{c'c}{}^{sC})]+\cdots;\quad (86)$$

and where we have used the shortened notation

$$d_{\lambda\mu}{}^J=d_{\lambda\mu}{}^J(\theta=\tfrac{1}{2}\pi)\quad (87)$$

and

$$\Delta_{\lambda\mu}{}^J=\frac{d}{d\theta}d_{\lambda\mu}{}^J(\theta)\Big|_{\theta=\pi/2}.$$

Tables of these functions are given in the Appendix I for several spin values.

Finally,

$$\eta_{cadb}{}^{c'\bar{a}'\bar{d}'b'}=(\eta_{\bar{A}}\eta_C/\eta_B\eta_{\bar{D}})$$
$$\times(-1)^{2(s_A+s_B)-a-b-c-d+\Lambda(\bar{d}'b')-\Lambda(c'\bar{a}')}.\quad (88)$$

In practice, if we have $m_A=m_C$ and $m_B=m_D$ we inevitably have that $A=C$ or $A=\bar{C}$ and $B=D$ or $B=\bar{D}$, and thus $s_A=s_C$, $s_B=s_D$. In this case many terms in the sum in (83) are related to each other by parity conservation, conservation of symmetry, or G-parity conservation. The result, although it looks extremely forbidding, is much simpler to use in practical calculation than (83).

One gets

$$\left\{\sum_{\bar{a}'\geq c'\geq 0}\sum_{b'}\sum_{\bar{d}'\geq 0}\frac{1}{1+\delta_{\bar{d}'0}}\frac{1}{1+\delta_{\bar{a}'0}}\frac{1}{1+\delta_{c'\bar{a}'}}+\sum_{\bar{a}'\geq -c'>0}\sum_{b'}\sum_{\bar{d}'>0}\frac{1}{1+\delta_{\bar{a}'-c'}}\right\}\{\mathfrak{M}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}$$
$$+\mathfrak{M}_{cadb}{}^{-c'-\bar{a}';\bar{d}'b'}f_{-c'-\bar{a}';\bar{d}'b'}{}^{(t)}+\mathfrak{M}_{cadb}{}^{\bar{a}'c';\bar{d}'b'}f_{\bar{a}'c';\bar{d}'b'}{}^{(t)}+\mathfrak{M}_{cadb}{}^{-\bar{a}-c';\bar{d}'b'}f_{-\bar{a}'-c';\bar{d}'b'}{}^{(t)}\}$$
$$+\frac{1}{2}\left\{\sum_{\bar{a}'\geq 0}\sum_{b'}\sum_{c'<0}\sum_{\bar{d}'=0}\frac{1}{1+\delta_{\bar{a}'0}}+\sum_{\bar{a}'=0}\sum_{b'}\sum_{c'<0}\sum_{\bar{d}'=0}\right\}\{\mathfrak{M}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}$$
$$+\mathfrak{M}_{cadb}{}^{-c'-\bar{a}';\bar{d}'b'}f_{-c'-\bar{a}';\bar{d}'b'}{}^{(t)}\}\propto t^{\frac{1}{2}|\Lambda(ab)-\Lambda(cd)|}.\quad (89)$$

If the t channel is elastic scattering, then the number of terms in the sum of (89) is further reduced by time-reversal invariance, but the writing down of the summation is so complicated that it is not worth reproducing. It is simpler just to remember the rule

$$f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}=(-1)^{\Lambda(c'\bar{a}')-\Lambda(\bar{d}'b')}f_{\bar{d}'b';c'\bar{a}'}{}^{(t)}\quad \text{(elastic } t \text{ channel only)}.\quad (90)$$

Equations (89) or (83) give the constraints in terms of the full helicity amplitudes $f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}$. For the study of the constraints on the Regge parameters it is convenient to rewrite (89) in terms of the parity-symmetry-conserving partial-wave amplitudes $T^J(p,\rho)$. Using (28) and (89), we get

$$\sum_{p,\rho,J}(2J+1)\sum_{b'}\{\sum_{\bar{a}'\geq c'\geq 0}\sum_{\bar{d}'\geq 0}[(1+\delta_{\bar{d}'0})(1+\delta_{\bar{a}'0})(1+\delta_{c'\bar{a}'})]^{-1}(W_{cadb}{}^{c'\bar{a}';\bar{d}'b'}+\rho n_{CA}W_{cadb}{}^{\bar{a}'c';\bar{d}'b'})T_{c'\bar{a}';\bar{d}'b'}{}^J(p,\rho)$$
$$+\sum_{\bar{a}'\geq -c'>0}\sum_{\bar{d}'\geq 0}[(1+\delta_{\bar{d}'0})(1+\delta_{\bar{a}'-c'})]^{-1}(W_{cadb}{}^{c'\bar{a}';\bar{d}'b'}+p\rho W_{cadb}{}^{-\bar{a}'-c';\bar{d}'b'})T_{c'\bar{a}';\bar{d}'b'}{}^J(p,\rho)$$
$$+\tfrac{1}{2}\sum_{c'<0}\sum_{\bar{d}'\geq 0}(1+\delta_{\bar{d}'0})^{-1}W_{cadb}{}^{c'0;\bar{d}'b'}T_{c'0;\bar{d}'b'}{}^J(p,\rho)\}\propto t^{\frac{1}{2}\{|\Lambda(ab)-\Lambda(cd)|-1\}},\quad (91)$$

where

$$W_{cadb}{}^{c'\bar{a}';\bar{d}'b'}(\cos\theta_t)=[(1+\delta_{\bar{d}'b'})(1+\delta_{\bar{d}'-b'})(1+\delta_{\bar{a}'c'})(1+\delta_{\bar{a}'-c'})]^{1/2}[\mathfrak{M}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}d_{\Lambda(\bar{d}'b'),\Lambda(c'\bar{a}')}{}^J(\cos\theta_t)$$
$$+pn_{C\bar{A}}\mathfrak{M}_{cadb}{}^{-c'-\bar{a}';\bar{d}'b'}d_{\Lambda(\bar{d}'b'),-\Lambda(c'\bar{a}')}{}^J(\cos\theta_t)]\quad (92)$$

and

$$n_{C\bar{A}}=\eta_C\eta_{\bar{A}}(-1)^{2s_A}$$
$$=+1\quad\text{for}\quad BB\text{ or }F\bar{F}\quad (93)$$
$$=-1\quad\text{for}\quad FF.$$

There are two ways of studying the effect of (91) on the Regge parameters. The best method is to do an angular integration with a suitably chosen projection function and to obtain constraint equations for the $T^J(p,\rho)$ directly, and these can then be analytically continued to complex J. However we have been unable to carry out this projection in the *general case*, although it has been possible for the examples below. The next best thing to do is to effectively apply the Sommerfeld-Watson transformation to (91), which amounts to replacing the $f^{(t)}$ in (89) by their asymptotic form for large s. In doing this one may lose a certain amount of information if too few terms are taken in the asymptotic expansion. One gets then relations among the trajectories and residues of the form

$$\sum_{p,\rho,\tau}[2\alpha(p,\rho,\tau)+1]\zeta_\tau\sum_{b'}\Big\{\sum_{\bar{a}'\geq c'\geq 0}\sum_{\bar{d}'\geq 0}[(1+\delta_{\bar{d}'0})(1+\delta_{\bar{a}'0})(1+\delta_{c'\bar{a}'})]^{-1}[\mathcal{W}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}+\rho n_{C\bar{A}}\mathcal{W}_{cadb}{}^{\bar{a}'c';\bar{d}'b'}]$$
$$\times r\tau_{\bar{c}'\alpha';\bar{d}'b';\bar{d}'b'}(p,\rho,\tau)+\sum_{\bar{a}'\geq -c'\geq 0}\sum_{\bar{d}'\geq 0}(1+\delta_{\bar{a}',-c'})^{-1}[\mathcal{W}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}+p\rho\mathcal{W}_{cadb}{}^{-\bar{a}'-c';\bar{d}'b'}]r_{c'\bar{a}';\bar{d}'b'}(p,\rho,\tau)$$
$$+\tfrac{1}{2}\sum_{c'<0}\sum_{\bar{d}'\geq 0}(1+\delta_{\bar{d}'0})^{-1}\mathcal{W}_{cadb}{}^{c'0;\bar{d}'b'}r_{c'0;\bar{d}'b'}(p,\rho,\tau)\Big\}\propto t^{\frac{1}{2}\{|\Lambda(ab)-\Lambda(cd)|-1\}}\,, \quad (94)$$

where

$$\mathcal{W}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}$$
$$=[(1+\delta_{\bar{d}'b'})(1+\delta_{\bar{d}'-b'})(1+\delta_{\bar{a}'c'})(1+\delta_{\bar{a}'-c'})]^{-1}$$
$$\times[\mathfrak{M}_{cadb}{}^{c'\bar{a}';\bar{d}'b'}d_{\Lambda(\bar{d}'b'),\,-\Lambda(c'\bar{a}')}{}^{\alpha(p,\rho,\tau)}(-\cos\theta_t)$$
$$+pn_{C\bar{A}}\mathfrak{M}_{cadb}{}^{-c'-\bar{a}';\bar{d}'b'}d_{\Lambda(\bar{d}'b')\Lambda(c'\bar{a}')}{}^{\alpha(p,\rho,\tau)}(-\cos\theta_t)]. \quad (95)$$

When only two of the particles have nonzero spin (say B and D) the above equations simplify enormously. Only the quantum numbers $p=\rho=+1$ are allowed, and one gets for (88),

$$\sum_J(2J+1)\sum_{\bar{d}'\geq 0}\frac{[(1+\delta_{\bar{d}'b'})(1+\delta_{\bar{d}'-b'})]^{1/2}}{1+\delta_{\bar{d}'0}}\mathfrak{M}_{db}{}^{\bar{d}'b'}T_{\bar{d}'b'}{}^J$$
$$\times d_{\Lambda(\bar{d}'b'),0}{}^J(\cos\theta_t)\propto t^{\frac{1}{2}\{|d-b|-1\}}\,, \quad (96)$$

and a similar simplification for (94).

In order to study the consequences of the constraint equations for the Regge-pole parameters we first consider the effect of the factorization theorem. If we focus our attention on a given *EE* process, we see that the k.n.b. of the residues implied by (73) is not compatible with the factorization theorem. For from (73) we would conclude that

$$r_{c\bar{a};\bar{d}b}\propto t^{-1/2} \quad \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1\,,$$
$$r_{c\bar{a};\bar{d}b}\rightarrow \text{const} \quad \text{if} \quad \eta^{c\bar{a}\bar{d}b}=-1\,, \quad (97)$$

which is not generally compatible with the factorization requirement (51).

We wish now to study what effect the factorization requirement has on the residues, and to see how this influences the behavior of the s-channel amplitudes.

It will be instructive to tackle this problem from two different angles. Firstly we shall study the constraint equations in the *asymptotic limit* and derive the necessary and sufficient behavior of the residues and s-channel amplitudes. We shall then approach the problem using a partial-wave expansion and show that this behavior is no longer sufficient to satisfy the constraint equations if conspiracies and daughters are excluded. It will then be shown that there exists a purely evasive solution satisfying the *exact* constraint equations in which the behavior of the residues is drastically different from the behavior arrived at by studying only the asymptotic limit.

A. Existence of an Evasive Solution in the Asymptotic Limit

Let $f_{c'\bar{a}';\bar{d}'b'}{}^{(t)n}$ be the *leading term* as s (or z_t)$\rightarrow\infty$ in $f_{c'\bar{a}';\bar{d}'b'}{}^{(t)}$ arising from the nth Regge pole. From (29) and the properties of $d_{\lambda\mu}{}^J$ functions, one has that

$$f_{c'\bar{a}';\bar{d}'b'}{}^{(t)n}=\frac{-\pi^2\sqrt{t}}{4(p_{\bar{A}C}p_{\bar{D}B})^{1/2}}(-1)^{\Lambda(\bar{d}'b')}$$
$$\times[(1+\delta_{\bar{a}'c'})(1+\delta_{\bar{a}'-c'})(1+\delta_{\bar{d}'b'})(1+\delta_{\bar{d}'-b'})]^{1/2}$$
$$\times(2\alpha_n+1)\zeta_n r_{c'\bar{a}';\bar{d}'b'}{}^{(n)}$$
$$\times\mathfrak{A}_{\Lambda(\bar{d}'b')}{}^{\alpha n}(z_t)\,\mathfrak{A}_{\Lambda(c'\bar{a}')}{}^{\alpha n}(z_t)\,, \quad (98)$$

where

$$\mathfrak{A}_{\Lambda(\bar{d}'b')}{}^J(z_t)=\Bigg[(-1)^{\Lambda(\bar{d}'b')}\left(\frac{z_t}{2}\right)^J$$
$$\times\frac{(2J)!}{(J+|\Lambda(\bar{d}'b')|)!(J-|\Lambda(\bar{d}'b')|)!}\Bigg]^{1/2} \quad (99)$$

is the function introduced by Fox and Leader[18] which expresses the factorizability of the $d_{\lambda\mu}{}^J$ functions in the asymptotic limit.

Since the residues satisfy the factorization theorem we may put

$$r_{c'\bar{a}';\bar{d}'b'}{}^{(n)}=b_{c'\bar{a}'}{}^{(n)}(t)b_{\bar{d}'b'}{}^{(n)}(t) \quad (100)$$

and then write

$$f_{c'\bar{a}';\bar{d}'b'}{}^{(t)n}=-g_{c'\bar{a}'}{}^{(t)n}h_{\bar{d}'b'}{}^{(t)n}\,, \quad (101)$$

where

$$g_{c'\bar{a}'}{}^{(t)n}=\frac{\pi t^{1/4}}{2\sqrt{p_{\bar{A}C}}}[(1+\delta_{\bar{a}'c'})(1+\delta_{\bar{a}'-c'})(2\alpha_n+1)\zeta_n]^{1/2}$$
$$\times b_{c'\bar{a}'}{}^{(n)}\mathfrak{A}_{\Lambda(c'\bar{a}')}{}^{\alpha n}(z_t) \quad (102)$$

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[18] G. C. Fox and E. Leader, Phys. Rev. Letters **18**, 628 (1967).

and

$$h_{\bar{d}'b'}{}^{(t)n}=\frac{\pi t^{1/4}}{2\sqrt{p_{\bar{D}B}}}(-1)^{\Lambda(\bar{d}'b')}[(1+\delta_{\bar{d}'b'})(1+\delta_{\bar{d}'-b'})$$

$$\times(2\alpha_n+1)\zeta_n]^{1/2}b_{\bar{d}'b'}{}^{(n)}\mathcal{A}_{\Lambda(\bar{d}'b')}{}^{\alpha n}(z_t)\,. \quad (103)$$

If we put

$$f_{cd;ab}{}^{(s)}=\sum_n f_{cd;ab}{}^{(s)n}\,, \quad (104)$$

and make use of the fact that the crossing matrix is a *product* of functions, e.g.,

$$M_{cadb}{}^{c'\bar{a}';\bar{d}'b'}=M_{ca}{}^{c'\bar{a}'}M_{db}{}^{\bar{d}'b'}\,, \quad (105)$$

we can put

$$f_{cd;ab}{}^{(s)n}=-g_{ca}{}^{(s)n}h_{db}{}^{(s)n}\,, \quad (106)$$

where

$$g_{ca}{}^{(s)n}=M_{ca}{}^{c'\bar{a}'}g_{c'\bar{a}'}{}^{(t)n}$$

and

$$h_{db}{}^{(s)n}=M_{db}{}^{\bar{d}'b'}h_{\bar{d}'b'}{}^{(t)n}. \quad (107)$$

In what follows we shall use $g_{\lambda\mu}{}^{(s)}$ and $g_{\lambda'\mu'}{}^{(t)}$ generically for the g and h functions of (107). Now the minimum modification we can make to the behavior (97) in order to satisfy the factorization theorem, is to take[19]

$$\begin{array}{ll} b_{\lambda\mu}{}^{(n)}\propto t^{-1/4} & \text{if} \quad \eta^{\lambda\mu}=+1\,, \\ b_{\lambda\mu}{}^{(n)}\propto t^{1/4} & \text{if} \quad \eta^{\lambda\mu}=-1\,. \end{array} \quad (108)$$

This then implies that

$$\begin{array}{lll} r_{c\bar{a};\bar{d}b}\propto t^{-1/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{and} \quad \eta^{c\bar{a}}=+1 \\ \quad\rightarrow \text{const} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=-1 & \\ \quad\propto t^{1/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{but} \quad \eta^{c\bar{a}}=-1\,, \end{array} \quad (109)$$

and, of course, factorization is satisfied. [We shall show later that actually (109) applies only to poles of the $(+,+)$ variety.] Consider now the generic equation

$$g_{\lambda\mu}{}^{(s)(n)}=M_{\lambda\mu}{}^{\lambda'\mu'}g_{\lambda'\mu'}{}^{(t)n}. \quad (110)$$

Provided that n is a pole of the $(+,+)$ variety, the *structure* of (110) is identical with the constraint equations which would arise in the scattering process

$$L+\pi\rightarrow L+\pi\,,$$

where the L particles have spin s_L and helicities λ, μ, except that $b_{\lambda'\mu'}{}^{(n)}$ would be replaced by $b_{00}{}^{\pi\pi(n)}b_{\lambda'\mu'}{}^{(n)}$. But

$$b_{00}{}^{\pi\pi(n)}b_{\lambda'\mu'}{}^{(n)}=t^{-1/4}b_{\lambda'\mu'}{}^{(n)} \quad (111)$$

by (108). Thus we may associate

$$g_{\lambda\mu}{}^{(s)n}\leftrightarrow f_{\lambda 0;\mu 0}{}^{(s)n}(L\pi\rightarrow L\pi)$$

and

$$g_{\lambda'\mu'}{}^{(t)n}\leftrightarrow f_{00;\lambda'\mu'}{}^{(t)n}(L\bar{L}\rightarrow \pi\pi)\,.$$

Moreover, the behavior

$$\begin{array}{ll} r_{00;\lambda'\mu'}{}^{(n)}\propto t^{-1/2} & \text{if} \quad \eta^{\lambda'\mu'}=+1 \\ \quad\rightarrow \text{const} & \text{if} \quad \eta^{\lambda'\mu'}=-1\,, \end{array}$$

demanded by (97) is, in this process, consistent with (109).

Thus by the original argument which led to (97) we get that

$$g_{\lambda\mu}{}^{(s)n}\propto t^{\frac{1}{2}|\Lambda(\lambda\mu)|}\,, \quad (112)$$

when the $b_{\lambda'\mu'}{}^{(n)}$ are given by (108).

From (106) it now follows that

$$f_{cd;ab}{}^{(s)n}\propto t^{\frac{1}{2}\{|\Lambda(ac)|+|\Lambda(db)|\}}\,; \quad n=(+,+) \quad (113)$$

when the residues $r_{c\bar{a};\bar{d}b}(+,+)$ have the behavior given in (109). Equation (113) should be contrasted with the original behavior (70):

$$f_{cd;ab}{}^{(s)}\propto t^{\frac{1}{2}|\Lambda(ab)-\Lambda(cd)|}=t^{\frac{1}{2}|\Lambda(ac)+\Lambda(db)|}\,,$$

enforced by conservation of angular momentum. We see, therefore, that the factorization theorem leads to the existence of an evasive solution (at least in the asymptotic limit) in which the spin dependence in the forward direction is drastically modified whenever both particles in the reaction have nonzero spin.

The above argument breaks down when the pole n is not of the $(+,+)$ type, since then the Eq. (110) is analogous to fictitious processes, for example, like

$$L+\pi\rightarrow L+\pi'\,,$$

where π' has the same mass as but opposite parity to the π. As listed in Table I, some helicity states are now forbidden. Nevertheless the arguments leading to (97) require only minor modification and lead to the following results:

For poles of the $(-,-)$ variety

$$\begin{array}{lll} r_{c\bar{a};\bar{d}b}(-,-)\propto t^{1/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{and} \quad \eta^{c\bar{a}}=+1 \\ \quad\rightarrow \text{const} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=-1 & \\ \quad\propto t^{-1/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{but} \quad \eta^{c\bar{a}}=-1\,. \end{array} \quad (114)$$

Equation (113) is again valid.

For poles of the $(+,-)$ type

$$\begin{array}{lll} r_{c\bar{a};\bar{d}b}(+,-)\propto t^{3/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{and} \quad \eta^{c\bar{a}}=+1 \\ \quad\propto t & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=-1 & \\ \quad\propto t^{1/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{but} \quad \eta^{c\bar{a}}=-1\,, \end{array} \quad (115)$$

and (113) holds.

Finally for poles of the $(-,+)$ variety

$$\begin{array}{lll} r_{c\bar{a};\bar{d}b}(-,+)\propto t^{1/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{and} \quad \eta^{c\bar{a}}=+1 \\ \quad\propto t & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=-1 & \\ \quad\propto t^{3/2} & \text{if} \quad \eta^{c\bar{a}\bar{d}b}=+1 & \text{but} \quad \eta^{c\bar{a}}=-1\,, \end{array} \quad (116)$$

and again (113) holds. It is worth noting that the leading term in the contributions to the s-channel amplitudes coming from poles of a *given type* possess special symmetry properties in the asymptotic limit. Thus,

$$f_{-cd;-ab}{}^{(s)(p,\rho)}=p\eta_C\eta_{\bar{A}}(-1)^{2s_A+a-c}f_{cd;ab}{}^{(s)(p,\rho)}\,, \quad (117)$$

$$f_{ad;cb}{}^{(s)(p,\rho)}=p\rho(-1)^{a-c}f_{cd;ab}{}^{(s)(p,\rho)}\,, \quad (118)$$

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[19] I am indebted to G. C. Fox for enlightenment on this point.

and

$$f_{-ad;-cb}{}^{(s)(p,\rho)}=\rho\eta_C\eta_{\overline{A}}(-1)^{2s_A}f_{cd;ab}{}^{(s)(p,\rho)}, \qquad (119)$$

and similar results when b and d are manipulated. Equation (118) is actually true also in the nonasymptotic region.

B. Existence of an Evasive Solution in the Exact Case

We shall now show that the above behavior is quite insufficient to satisfy the constraint equations in the nonasymptotic limit. To do this we shall study several examples, and we shall learn from these that an evasive solution can still be found, but with a more drastic behavior for the residues than (109), (114)–(116). We shall then give a proof of the existence of an evasive solution in the general case.

1. Examples

The calculations are facilitated by having a list of the independent amplitudes for an arbitrary process, and this is provided in Appendix II.

(a) $\pi\pi\rightarrow\pi\pi$. There is only one residue denoted by $r^{\pi\pi;\pi\pi}(+,+)$ and it has the behavior

$$r^{\pi\pi;\pi\pi}(+,+)\propto t^{-1/2}. \qquad (120)$$

There are, of course, no constraint equations.

(b) $\pi\rho\rightarrow\pi\rho$. There are two constraint equations arising from the amplitudes $f_{0;1}{}^{(s)}$ and $f_{-1;1}{}^{(s)}$. The former carries no information, and the latter leads, using (96), to the equation

$$\sum(2J+1)\{[T_{11}{}^J-\sqrt{2}T_{00}{}^J]P_J(\theta_t) + T_{1-1}{}^J d_{20}{}^J(\theta_t)\}\propto t^{1/2}, \qquad (121)$$

where

$$T_{\lambda\mu}{}^J=\langle\pi\pi|T^J|\rho\rho;\lambda,\mu\rangle.$$

Projecting with $P_{J+1}(\theta_t)-P_{J-1}(\theta_t)$, we get after some manipulation

$$T_{11}{}^{J+1}-\sqrt{2}T_{00}{}^{J+1}-\left[\frac{(J+2)(J+3)}{J(J+1)}\right]^{1/2}T_{1,-1}{}^{J+1} - \left\{T_{11}{}^{J-1}-\sqrt{2}T_{00}{}^{J-1}-\left[\frac{(J-1)(J-2)}{J(J+1)}\right]^{1/2}T_{1-1}{}^{J-1}\right\} \propto t^{1/2}. \qquad (122)$$

We continue this equation to complex J and assume that there is a pole at $J=\alpha$. (It has of course, $p=\rho=+$.)

Taking $J=\alpha-1$ in (122) we get for the residues

$$r_{11}{}^{\pi\pi;\rho\rho}(+,+)-\sqrt{2}r_{00}{}^{\pi\pi;\rho\rho}(+,+) -\left[\frac{(\alpha+1)(\alpha+2)}{\alpha(\alpha-1)}\right]^{1/2}r_{1-1}{}^{\pi\pi;\rho\rho}(+,+)\propto t^{1/2}. \qquad (123)$$

Taking $J=\alpha+1$, however, we get

$$r_{11}{}^{\pi\pi;\rho\rho}(+,+)-\sqrt{2}r_{00}{}^{\pi\pi;\rho\rho}(+,+) -\left[\frac{\alpha(\alpha-1)}{(\alpha+1)(\alpha+2)}\right]^{1/2}r_{1-1}{}^{\pi\pi;\rho\rho}(+,+)\propto t^{1/2}. \qquad (124)$$

Thus we must have, if we avoid daughterlike sequences,

$$r_{1-1}{}^{\pi\pi;\rho\rho}(+,+)\propto t^{1/2},$$

and

$$\begin{aligned} r_{00}{}^{\pi\pi;\rho\rho}(+,+)&=a_{00}t^{-1/2}+O(t^{1/2}),\\ r_{11}{}^{\pi\pi;\rho\rho}(+,+)&=\sqrt{2}a_{00}t^{-1/2}+O(t^{1/2}). \end{aligned} \qquad (125)$$

(c) $\pi N^*\rightarrow\pi N^*$ (*N^* with spin* $\frac{3}{2}$). There are four constraint equations arising from the amplitudes $f_{\frac{3}{2};\frac{1}{2}}{}^{(s)}$, $f_{-\frac{1}{2}\frac{1}{2}}{}^{(s)}$, $f_{-\frac{1}{2}\frac{3}{2}}{}^{(s)}$, and $f_{-\frac{3}{2}\frac{3}{2}}{}^{(s)}$. Of these the first two do not carry information. The others lead to the equations

$$\sum(2J+1)\{[3(T_{\frac{3}{2}\frac{3}{2}}{}^J-T_{\frac{1}{2}\frac{1}{2}}{}^J)P_J(\theta_t)+(\sqrt{\tfrac{3}{2}})T_{\frac{3}{2}-\frac{1}{2}}{}^J d_{20}{}^J(\theta_t)] \times v_B t^{1/2}+[(\sqrt{\tfrac{3}{2}})T_{\frac{3}{2}\frac{1}{2}}{}^J+3T_{\frac{1}{2}-\frac{1}{2}}{}^J]d_{1,0}{}^J(\theta_t) - T_{\frac{3}{2}-\frac{3}{2}}{}^J d_{30}{}^J(\theta_t)\}\propto t \qquad (126)$$

and

$$\sum(2J+1)[(\sqrt{6})(T_{\frac{3}{2}\frac{3}{2}}{}^J-T_{\frac{1}{2}\frac{1}{2}}{}^J)P_J(\theta_t) - T_{\frac{3}{2}-\frac{1}{2}}{}^J d_{20}{}^J(\theta_t)]\propto t^{1/2}. \qquad (127)$$

Equation (127) has precisely the same structure as (121) and leads to the requirements

$$\begin{aligned} r_{\frac{3}{2}\frac{3}{2}}{}^{\pi\pi;N^*\overline{N}^*}(+,+)&=a_{\frac{3}{2}\frac{3}{2}}t^{-1/2}+O(t^{1/2}),\\ r_{\frac{1}{2}\frac{1}{2}}{}^{\pi\pi;N^*\overline{N}^*}(+,+)&=a_{\frac{3}{2}\frac{3}{2}}t^{-1/2}+O(t^{1/2}),\\ r_{\frac{3}{2}-\frac{1}{2}}{}^{\pi\pi;N^*\overline{N}^*}(+,+)&\propto t^{1/2}. \end{aligned} \qquad (128)$$

Using (127) and (128) we see that (126) requires

$$\sum(2J+1)\{[(\sqrt{\tfrac{3}{2}})T_{\frac{3}{2}\frac{1}{2}}{}^J+3T_{\frac{1}{2}-\frac{1}{2}}{}^J]d_{10}{}^J(\theta_t) - T_{\frac{3}{2}-\frac{3}{2}}{}^J d_{30}{}^J(\theta_t)\}\propto t. \qquad (129)$$

Projecting with

$$d_{30}{}^{J+1}(\theta_t)-\left[\frac{(J-2)(J-3)}{(J+3)(J+4)}\right]^{1/2}d_{30}{}^{J-1}(\theta_t),$$

we get

$$\left[\frac{(J-1)J(J+1)(J+2)}{(J+3)(J+4)}\right]^{1/2}[(\sqrt{\tfrac{3}{2}})(T_{\frac{3}{2}\frac{1}{2}}{}^{J+1}-T_{\frac{3}{2}\frac{1}{2}}{}^{J-1}) +3(T_{\frac{1}{2}-\frac{1}{2}}{}^{J+1}-T_{\frac{1}{2}-\frac{1}{2}}{}^{J-1})]+T_{\frac{3}{2}-\frac{3}{2}}{}^{J+1}-T_{\frac{3}{2}-\frac{3}{2}}{}^{J-1}\propto t. \qquad (130)$$

By similar methods to those used following (122) we deduce that

$$r_{\frac{3}{2}-\frac{3}{2}}{}^{\pi\pi;N^*\overline{N}^*}(+,+)\propto t,$$

$$r_{\frac{3}{2}\frac{1}{2}}{}^{\pi\pi;N^*\overline{N}^*}(+,+)=a_{\frac{3}{2}\frac{1}{2}}+O(t), \qquad (131)$$

and

$$r_{\frac{1}{2}-\frac{1}{2}}{}^{\pi\pi;N^*\overline{N}^*}(+,+)=\tfrac{1}{6}\sqrt{6}a_{\frac{3}{2}\frac{1}{2}}+O(t).$$

In all the above examples, the t channel involved coupling to the $\pi\pi$ system so that only one type of Regge pole, with $p=\rho=+1$, was allowed. Conspiracy is therefore not really a possibility, and the above exercise simply shows that an evasive solution as against a daughterlike solution, exists.

We now look at more complicated examples, involving a mixture of Regge poles.

(d) $NN \rightarrow NN$. There are two amplitudes $f_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}{}^{(s)}$ and $f_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}{}^{(s)}$ which could give rise to constraint equations. The first of these turns out not to carry information and the second leads to the requirement

$$\begin{aligned}\sum(2J+1)\{2T_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}{}^{J}(-+)P_J(\theta_t)+T_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}{}^{J}(++)\\ \times[d_{1-1}{}^{J}(\theta_t)-d_{11}{}^{J}(\theta_t)]+T_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}-\frac{1}{2}}{}^{J}(--)\\ \times[d_{1-1}{}^{J}(\theta_t)+d_{11}{}^{J}(\theta_t)]\}\propto t^{1/2}.\end{aligned} \quad (132)$$

Note that the connection between our general notation and the more usual one for the NN problem is as follows:

$$\begin{aligned}T_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}{}^{J}(-,+)&=f_0{}^J,\\ T_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}{}^{J}(-,-)&=f_1{}^J,\\ T_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}{}^{J}(+,+)&=f_{11}{}^J,\\ T_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}{}^{J}(+,+)&=f_{22}{}^J,\\ T_{-\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}{}^{J}(+,+)&=f_{12}{}^J.\end{aligned} \quad (133)$$

Projecting with $P_{J+1}(\theta_t)-P_{J-1}(\theta_t)$ yields

$$\frac{J+2}{J+1}f_{22}{}^{J+1}-\frac{J-1}{J}f_{22}{}^{J-1}-f_0{}^{J+1}+f_0{}^{J-1}-\frac{2J+1}{J(J+1)}f_1{}^J\propto t^{1/2}. \quad (134)$$

For an evasive solution the terms belonging to each type of Regge pole must separately satisfy (134). Thus we get

$$\begin{aligned}r_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}{}^{N\bar{N};N\bar{N}}(+,+)&\propto t^{1/2},\\ r_{-\frac{1}{2}\frac{1}{2};-\frac{1}{2}\frac{1}{2}}{}^{N\bar{N};N\bar{N}}(-,-)&\propto t^{1/2},\end{aligned} \quad (135)$$

and

$$r_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}{}^{N\bar{N};N\bar{N}}(-,+)\propto t^{1/2}.$$

(e) $\rho\rho\rightarrow\rho\rho$. In this example we shall see the vital role played by the factorization theorem. The process is described by seventeen independent amplitudes of which ten give rise to constraint equations. Of these, four yield no information and we are left with the constraints arising from $f_{11;1-1}{}^{(s)}$, $f_{00;1-1}{}^{(s)}$, $f_{10;01}{}^{(s)}$, $f_{10;-10}{}^{(s)}$, $f_{01;1-1}{}^{(s)}$, and $f_{-11;1-1}{}^{(s)}$. They are, in the above order:

$$\begin{aligned}\sum(2J+1)\{[2T_{11;11}{}^{J}(+,+)-4T_{00;00}{}^{J}(+,+)]P_J(\theta_t)+4T_{11;1-1}{}^{J}(+,+)d_{2,0}{}^{J}(\theta_t)-T_{10;10}{}^{J}(+,+)\\ \times[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]+T_{-11;1-1}{}^{J}(+,+)[d_{2-2}{}^{J}(\theta_t)+d_{22}{}^{J}(\theta_t)]-\tfrac{5}{2}T_{10;10}{}^{J}(-,-)[d_{11}{}^{J}(\theta_t)-d_{1-1}{}^{J}(\theta_t)]\\ +2T_{-11;1-1}{}^{J}(-,-)[d_{2-2}{}^{J}(\theta_t)-d_{22}{}^{J}(\theta_t)]-2T_{10;10}{}^{J}(-,+)[d_{11}{}^{J}(\theta_t)-d_{1-1}{}^{J}(\theta_t)]\\ -\tfrac{1}{2}T_{10;10}{}^{J}(+,-)[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]\}\propto t^{1/2};\end{aligned} \quad (136)$$

$$\begin{aligned}\sum(2J+1)\{T_{-11;1-1}{}^{J}(+,+)[d_{22}{}^{J}(\theta_t)-d_{2-2}{}^{J}(\theta_t)]-T_{-11;1-1}{}^{J}(-,-)[d_{22}{}^{J}(\theta_t)+d_{2-2}{}^{J}(\theta_t)]\\ -[T_{10;10}{}^{J}(-,-)+T_{10;10}{}^{J}(-,+)][d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]+2T_{11;11}{}^{J}(-,+)P_J(\theta_t)\}\propto t^{1/2};\end{aligned} \quad (137)$$

$$\begin{aligned}\sum(2J+1)\{[T_{10;10}{}^{J}(+,+)-T_{10;10}{}^{J}(+,-)][d_{11}{}^{J}(\theta_t)-d_{1-1}{}^{J}(\theta_t)]+[T_{10;10}{}^{J}(-,+)-T_{10;10}{}^{J}(-,-)]\\ \times[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]-T_{-11;1-1}{}^{J}(+,+)[d_{22}{}^{J}(\theta_t)-d_{2-2}{}^{J}(\theta_t)]+T_{-11;1-1}{}^{J}(-,-)\\ \times[d_{22}{}^{J}(\theta_t)+d_{2-2}{}^{J}(\theta_t)]+2T_{11;11}{}^{J}(-,+)P_J(\theta_t)\}\propto t^{1/2};\end{aligned} \quad (138)$$

$$\begin{aligned}\sum(2J+1)\{2[T_{11;11}{}^{J}(+,+)-\sqrt{2}T_{11;00}{}^{J}(+,+)]P_J(\theta_t)+2\sqrt{2}T_{-11;00}{}^{J}(+,+)d_{20}{}^{J}(\theta_t)-T_{-11;1-1}{}^{J}(+,+)\\ \times[d_{22}{}^{J}(\theta_t)+d_{2-2}{}^{J}(\theta_t)]+T_{-11;1-1}{}^{J}(-,-)[d_{22}{}^{J}(\theta_t)-d_{2-2}{}^{J}(\theta_t)]\}\propto t^{1/2};\end{aligned} \quad (139)$$

$$\begin{aligned}\sum(2J+1)\{v_At^{1/2}[4\sqrt{2}(2T_{00;00}{}^{J}(+,+)-T_{11;11}{}^{J}(+,+)+2T_{11;11}(-,+))P_J(\theta_t)-16T_{-11;00}{}^{J}(+,+)d_{20}{}^{J}(\theta_t)\\ +2\sqrt{2}(2T_{10;10}{}^{J}(+,+)+3T_{10;10}{}^{J}(-,+))(d_{1-1}{}^{J}(\theta_t)-d_{11}{}^{J}(\theta_t))-2\sqrt{2}(2T_{10;10}{}^{J}(-,-)+3T_{10;10}{}^{J}(+,-))\\ \times(d_{1-1}{}^{J}(\theta_t)+d_{11}{}^{J}(\theta_t))+2\sqrt{2}T_{-11;1-1}{}^{J}(+,+)d_{22}{}^{J}(\theta_t)-2\sqrt{2}T_{-11;1-1}{}^{J}(-,-)(d_{22}{}^{J}(\theta_t)-d_{2-2}{}^{J}(\theta_t))]\\ +2(\sqrt{2}T_{11;10}{}^{J}(+,+)+3T_{11;10}{}^{J}(-,+)-2T_{01;00}{}^{J}(+,+))d_{01}{}^{J}(\theta_t)+\sqrt{2}T_{01;1-1}{}^{J}(+,+)(d_{21}{}^{J}(\theta_t)-d_{2-1}{}^{J}(\theta_t))\\ -\sqrt{2}T_{01;1-1}{}^{J}(-,-)(d_{21}{}^{J}(\theta_t)+d_{2-1}{}^{J}(\theta_t))\}\propto t;\end{aligned} \quad (140)$$

and

$$\begin{aligned}\sum(2J+1)\{2[2T_{00;00}{}^{J}(+,+)-2\sqrt{2}T_{11;00}{}^{J}(+,+)+T_{11;11}{}^{J}(+,+)]P_J(\theta_t)+4[T_{11;1-1}{}^{J}(+,+)-\sqrt{2}T_{00;1-1}{}^{J}(+,+)]\\ \times d_{20}{}^{J}(\theta_t)+T_{-11;1-1}{}^{J}(+,+)[d_{22}{}^{J}(\theta_t)+d_{2-2}{}^{J}(\theta_t)]-T_{-11;1-1}{}^{J}(-,-)[d_{22}{}^{J}(\theta_t)-d_{2-2}{}^{J}(\theta_t)]-5T_{10;10}{}^{J}(+,-)\\ \times[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]-6T_{10;10}{}^{J}(-,+)[d_{11}{}^{J}(\theta_t)-d_{1-1}{}^{J}(\theta_t)]-v_At^{1/2}\sqrt{2}[(2T_{01;00}{}^{J}(+,+)-\sqrt{2}T_{11;10}{}^{J}(+,+)\\ +2\sqrt{2}T_{11;10}{}^{J}(-,+))d_{01}{}^{J}(\theta_t)+\tfrac{1}{2}\sqrt{2}T_{01;1-1}{}^{J}(+,+)(d_{2-1}{}^{J}(\theta_t)-d_{21}{}^{J}(\theta_t))+\tfrac{1}{2}\sqrt{2}T_{01;1-1}{}^{J}(-,-)\\ \times(d_{2-1}{}^{J}(\theta_t)+d_{21}{}^{J}(\theta_t))]\}\propto t^{3/2}.\end{aligned} \quad (141)$$

Since, for the moment, we are looking for an evasive type solution, the above equations have to be satisfied separately for each type of Regge pole.

Consider first the $(+,-)$ type. From Eq. (141) we see that

$$\sum(2J+1)T_{10;10}{}^{J}(+,-)\times[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]\propto t^{3/2}. \tag{142}$$

Thus we must have

$$r_{10;10}{}^{\rho\rho;\rho\rho}(+,-)\propto t^{3/2}. \tag{143}$$

All the other equations are then automatically satisfied or oversatisfied and this is the only $(+,-)$ residue appearing in the problem.

Let us turn now to the $(-,+)$ type. From (138) we have

$$\sum(2J+1)\{2T_{11;11}{}^{J}(-,+)P_J(\theta_t)+T_{10;10}{}^{J}(-,+)\times[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]\}\propto t^{1/2}, \tag{144}$$

and from (137)

$$\sum(2J+1)\{2T_{11;11}{}^{J}(-,+)P_J(\theta_t)-T_{10;10}{}^{J}(-,+)\times[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]\}\propto t^{1/2}. \tag{145}$$

Adding and subtracting (144) and (145), we get that

$$r_{11;11}{}^{\rho\rho;\rho\rho}(-,+)\propto t^{1/2}$$

and

$$r_{10;10}{}^{\rho\rho;\rho\rho}(-,+)\propto t^{1/2}. \tag{146}$$

However, from (129),

$$\sum(2J+1)\{3T_{10;10}{}^{J}(-,+)[d_{11}{}^{J}(\theta_t)-d_{1-1}{}^{J}(\theta_t)]+2v_At^{1/2}T_{11;10}{}^{J}(-,+)d_{01}{}^{J}(\theta_t)\}\propto t^{3/2}, \tag{147}$$

so that $r_{11;10}{}^{\rho\rho;\rho\rho}(-,+)$ can only go as constant or t as $t\rightarrow 0$. But by the factorization theorem

$$r_{11;11}r_{10;10}=(r_{11;10})^2,$$

so that the above behavior for $r_{10;10}$ is not acceptable and we must have

$$r_{10;10}{}^{\rho\rho;\rho\rho}(-,+)\propto t^{3/2} \tag{148}$$

and

$$r_{11;10}{}^{\rho\rho;\rho\rho}(-,+)\propto t. \tag{149}$$

All the other equations are now satisfied.

Consider now Regge poles of type $(-,-)$. From (138),

$$\sum(2J+1)\{T_{-11;1-1}{}^{J}(-,-)[d_{22}{}^{J}(\theta_t)+d_{2-2}{}^{J}(\theta_t)]-T_{10;10}{}^{J}(-,-)[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]\}\propto t^{1/2}, \tag{150}$$

and from (137),

$$\sum(2J+1)\{T_{-11;1-1}{}^{J}(-,-)[d_{22}{}^{J}(\theta_t)+d_{2-2}{}^{J}(\theta_t)]-T_{10;10}{}^{J}(-,-)[d_{11}{}^{J}(\theta_t)+d_{1-1}{}^{J}(\theta_t)]\propto t^{1/2}, \tag{151}$$

from which we deduce that

$$r_{10;10}{}^{\rho\rho;\rho\rho}(-,-)\propto t^{1/2}$$

and

$$r_{-11;1-1}{}^{\rho\rho;\rho\rho}(-,-)\propto t^{1/2}. \tag{152}$$

However, from (141),

$$\sum(2J+1)\{T_{-11;1-1}{}^{J}(-,-)[d_{22}{}^{J}(\theta_t)-d_{2-2}{}^{J}(\theta_t)]+v_At^{1/2}T_{01;1-1}{}^{J}(-,-)\times[d_{2-1}{}^{J}(\theta_t)+d_{21}{}^{J}(\theta_t)]\}\propto t^{3/2}, \tag{153}$$

so that $r_{01;1-1}(-,-)$ must go as constant or t as $t\rightarrow 0$. By the factorization theorem

$$r_{10;10}r_{-11;1-1}=-(r_{10;1-1})^2,$$

so that the above behavior of $r_{-11;1-1}$ is not acceptable and we get instead

$$r_{-11;1-1}{}^{\rho\rho;\rho\rho}(-,-)\propto t^{3/2} \tag{154}$$

and

$$r_{10;1-1}{}^{\rho\rho;\rho\rho}(-,-)\propto t. \tag{155}$$

It is easily seen that the other equations are now satisfied.

Lastly, we look at the more involved case where the Regge pole is of type $(+,+)$. Since the factorization theorem connects the processes $\pi\pi\rightarrow\pi\pi$, $\pi\rho\rightarrow\pi\rho$, and $\rho\rho\rightarrow\rho\rho$ we can use Eqs. (120) and (125) to get immediately that

$$r_{11;11}{}^{\rho\rho;\rho\rho}(+,+)=2a_{00}{}^{2}t^{-1/2}[1+O(t)], \tag{156}$$

$$r_{00;00}{}^{\rho\rho;\rho\rho}(+,+)=a_{00}{}^{2}t^{-1/2}[1+O(t)], \tag{157}$$

and

$$r_{-11;1-1}{}^{\rho\rho;\rho\rho}(+,+)\propto t^{3/2}. \tag{158}$$

Equation (137) is now oversatisfied. Using (158) in (138) leaves the requirement that

$$\sum(2J+1)T_{10;10}{}^{J}(+,+)\times[d_{11}{}^{J}(\theta_t)-d_{1-1}{}^{J}(\theta_t)]\propto t^{1/2}, \tag{159}$$

implying

$$r_{10;10}{}^{\rho\rho;\rho\rho}(+,+)\propto t^{1/2}. \tag{160}$$

Looking at the role of $r_{01;1-1}$ in (141), and using (158) and the factorization theorem, we conclude that

$$r_{01;1-1}{}^{\rho\rho;\rho\rho}(+,+)\propto t. \tag{161}$$

Similarly, we deduce that

$$r_{01;00}{}^{\rho\rho;\rho\rho}(+,+)=a_{00}+O(t) \tag{162}$$

and

$$r_{01;11}{}^{\rho\rho;\rho\rho}(+,+)=\sqrt{2}a_{00}+O(t). \tag{163}$$

Moreover from (156), (157), and (158) we get that

$$r_{-11;00}{}^{\rho\rho;\rho\rho}(+,+)=a_{00}t^{1/2}+O(t^{3/2}) \tag{164}$$

and

$$r_{11;1-1}{}^{\rho\rho;\rho\rho}(+,+)=\sqrt{2}a_{00}t^{1/2}+O(t^{3/2}). \tag{165}$$

Using the results (156)–(165), we see that Eqs. (136) and (140) are satisfied.

Finally, using (156) and (157) in the factorization theorem we get that

$$r_{11;00}{}^{\rho\rho;\rho\rho}(+,+)=\sqrt{2}a_{00}{}^{2}t^{-1/2}+O(t^{1/2}), \tag{166}$$

and now Eq. (139) is also satisfied. The cancellation in (141) is a more subtle matter. All the terms automatically satisfy the equation except for the coefficient of $P_J(\theta_t)$. We must therefore show that

$$\sum(2J+1)[2T_{00;00}{}^{J}(+,+)-2\sqrt{2}T_{11;00}{}^{J}(+,+) \\ +T_{11;11}{}^{J}(+,+)]P_J(\theta_t)\propto t^{3/2}. \tag{167}$$

Using the factorization theorem *without* approximation, the term in square brackets becomes

$$\begin{aligned}&[T_{11;11}{}^{J}(+,+)]^{-1}[2(T_{11;00}{}^{J}(+,+))^2\\ &\quad-2\sqrt{2}T_{11;00}{}^{J}(+,+)T_{11;11}{}^{J}(+,+)\\ &\quad+(T_{11;11}{}^{J}(+,+))^2]=[T_{11;11}{}^{J}(+,+)]^{-1}\\ &\quad\times[\sqrt{2}T_{11;00}{}^{J}(+,+)-T_{11;11}{}^{J}(+,+)]^2\\ &\qquad\qquad\propto t^{1/2}\times t=t^{3/2},\end{aligned}$$

using (156) and (166). Thus all the constraint equations are satisfied.

Looking at the results of these examples we see emerging a definite pattern of behavior for the residues in an evasive solution. Namely, that

$$r_{a\bar{a};\bar{b}b}(p,\rho)\propto t^{\frac{1}{2}\{|\Lambda(a\bar{a})|+|\Lambda(\bar{b}b)|-1\}} \quad \text{for} \quad p\rho=+1 \tag{168}$$

and

$$r_{a\bar{a};\bar{b}b}(p,\rho)\propto t^{\frac{1}{2}\{|\Lambda(a\bar{a})|+|\Lambda(\bar{b}b)|+1\}} \quad \text{for} \quad p\rho=-1. \tag{169}$$

This behavior is clearly compatible with the factorization theorem.

2. *General Case*

We shall now show in the general case that there exists an evasive solution with behavior given by (168) and (169).

From (18), (78), and (80), we have that

$$M_{\lambda\mu}{}^{\lambda'\mu'}=d_{\lambda'\lambda}{}^{s}(\tfrac{1}{2}\pi-\epsilon)d_{\mu'\mu}{}^{s}(\tfrac{1}{2}\pi+\epsilon), \tag{170}$$

where

$$\epsilon=t^{1/2}v(t) \tag{171}$$

and

$$v(t)=v_0+v_1t+v_2t^2+\cdots. \tag{172}$$

We shall need the following lemmas.

Lemma 1:

$$\sum_{\lambda'}\frac{d^n}{d\epsilon^n}M(\epsilon)_{\lambda\mu}{}^{\lambda'\lambda'}|_{\epsilon=0}=0 \quad \text{if} \quad |\lambda-\mu|>n.$$

Proof: By definition

$$d_{\lambda'\lambda}{}^{s}(\theta)=\langle s\lambda'|e^{-i\theta J_y}|s\lambda\rangle.$$

Therefore,

$$\begin{aligned}\sum_{\lambda'}\frac{d^n}{d\epsilon^n}M(\epsilon)_{\lambda\mu}{}^{\lambda'\lambda'}&=\frac{d^n}{d\epsilon^n}\sum_{\lambda'}\langle\lambda'|e^{-i(\pi/2-\epsilon)J_y}|\lambda\rangle\\ &\qquad\times\langle\lambda'|e^{-i(\pi/2+\epsilon)J_y}|\mu\rangle\\ &=\frac{d^n}{d\epsilon^n}\sum_{\lambda'}\langle\lambda|e^{+i(\pi/2-\epsilon)J_y}|\lambda'\rangle\\ &\qquad\times\langle\lambda'|e^{-i(\pi/2+\epsilon)J_y}|\mu\rangle\\ &=\frac{d^n}{d\epsilon^n}\langle\lambda|e^{-2i\epsilon J_y}|\mu\rangle\\ &=(-2i)^n\langle\lambda|J_y{}^ne^{-2i\epsilon J_y}|\mu\rangle.\end{aligned}$$

Thus

$$\sum_{\lambda'}\frac{d^n}{d\epsilon^n}M(\epsilon)_{\lambda\mu}{}^{\lambda'\lambda'}|_{\epsilon=0}=(-2i)^n\langle\lambda|J_y{}^n|\mu\rangle \\ =\langle\lambda|(J_--J_+)^n|\mu\rangle, \tag{173}$$

which is clearly zero if $|\lambda-\mu|>n$.

Lemma 2: Let

$$A(s,\lambda)=[(s+\lambda+1)(s-\lambda)]^{1/2},$$

then

$$\sum_{\lambda'}A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-1) \\ \times M_{\lambda\mu}{}^{\lambda'+n,\lambda'}(\epsilon=0)=0 \quad \text{if} \quad |\lambda-\mu|>n.$$

(We take $n\geq 0$. If $n<0$ a similar result follows.)

Proof: Since

$$J_+|s,\lambda\rangle=A(s,\lambda)|s,\lambda+1\rangle,$$

we have

$$\begin{aligned}&\sum_{\lambda'}A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-1)M_{\lambda\mu}{}^{\lambda'+n,\lambda'}(\epsilon)\\ &\quad=\sum_{\lambda'}A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-1)\\ &\qquad\qquad\times\langle\lambda|e^{+i(\pi/2-\epsilon)J_y}|\lambda'+n\rangle\langle\lambda'|e^{-i(\pi/2+\epsilon)J_y}|\mu\rangle\\ &\quad=\sum_{\lambda'}A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-2)\\ &\qquad\times\langle\lambda|e^{+i(\pi/2-\epsilon)J_y}J_+|\lambda'+n-1\rangle\langle\lambda'|e^{-i(\pi/2+\epsilon)J_y}|\mu\rangle,\end{aligned}$$

and repeating this process,

$$=\sum_{\lambda'}\langle\lambda|e^{+i(\pi/2-\epsilon)J_y}J_+{}^n|\lambda'\rangle\langle\lambda'|e^{-i(\pi/2+\epsilon)J_y}|\mu\rangle \\ =\langle\lambda|e^{+i(\pi/2-\epsilon)J_y}J_+{}^ne^{-i(\pi/2+\epsilon)J_y}|\mu\rangle. \tag{174}$$

Thus at $\epsilon=0$,

$$\sum_{\lambda'}A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-1)M_{\lambda\mu}{}^{\lambda'+n,\lambda'}(\epsilon=0) \\ =\langle\lambda|e^{+i(\pi/2)J_y}J_+{}^ne^{-i(\pi/2)J_y}|\mu\rangle.$$

Now,

$$\begin{aligned}e^{i(\pi/2)J_y}J_+e^{-i(\pi/2)J_y}&=J_z+iJ_y\\ &=J_z+\tfrac{1}{2}(J_+-J_-).\end{aligned}$$

Thus

$$\sum_{\lambda'} A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-1)M_{\lambda\mu}{}^{\lambda'+n,\lambda'}(\epsilon=0)$$
$$=\langle\lambda|[J_z+\tfrac{1}{2}(J_+-J_-)]^n|\mu\rangle, \quad (175)$$

which is clearly zero if $|\lambda-\mu|>n$.

Lemma 3:

$$\sum_{\lambda'} \frac{d^r}{d\epsilon^r} A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-1)$$
$$\times M_{\lambda\mu}{}^{\lambda'+n,\lambda'}(\epsilon)|_{\epsilon=0}=0 \quad \text{if} \quad |\lambda-\mu|>n+r. \quad (176)$$

From (174) the left-hand side of (176) at any ϵ is

$$=\frac{d^r}{d\epsilon^r}\langle\lambda|e^{i(\pi/2-\epsilon)J_y}J_+{}^n e^{-i(\pi/2+\epsilon)J_y}|\mu\rangle$$

$$=\sum_{m=0}^{r}\binom{r}{m}\langle\lambda|(-iJ_y)^m e^{i(\pi/2-\epsilon)J_y}J_+{}^n(-iJ_y)^{r-m}$$
$$\times e^{-i(\pi/2+\epsilon)J_y}|\mu\rangle.$$

Thus at $\epsilon=0$,

$$\sum_{\lambda'} \frac{d^r}{d\epsilon^r} A(s,\lambda')A(s,\lambda'+1)\cdots$$
$$A(s,\lambda'+n-1)M_{\lambda\mu}{}^{\lambda'+n,\lambda'}(\epsilon)|_{\epsilon=0}$$
$$=\sum_{m=0}^{r}\binom{r}{m}\langle\lambda|(-iJ_y)^m[J_z+iJ_y]^n(-iJ_y)^{r-m}|\mu\rangle, \quad (177)$$

which is clearly zero if $|\lambda-\mu|>n+r$.

Lemma 4:

$$\sum_{\lambda'}(\lambda'+\tfrac{1}{2}n)A(s,\lambda')A(s,\lambda'+1)\cdots A(s,\lambda'+n-1)\frac{d^r}{d\epsilon^r}$$
$$\times M_{\lambda\mu}{}^{\lambda'+n,\lambda'}(\epsilon)|_{\epsilon=0}=0 \quad \text{if} \quad |\lambda-\mu|>n+r+1. \quad (178)$$

This follows by similar methods upon replacing $\lambda'+\frac{1}{2}n$ by $(J_z-\frac{1}{2}n)$ when acting on the state $|\lambda'+n\rangle$.

The contribution of a single Regge pole to $f^{(t)}$ is given by (29) as

$$f_{c'\,c'+n;\,b'+m\,b'}{}^{(t)(p,\rho,\tau)}$$

$$=(-1)^{1+m}\frac{\pi^2\sqrt{t}}{4(p_{C\bar{A}}p_{\bar{D}B})^{1/2}}[(1+\delta_{n0})(1+\delta_{n\,-2c'})(1+\delta_{m0})$$
$$\times(1+\delta_{m\,-2b'})]^{1/2}[2\alpha(p,\rho,\tau)+1]\zeta_\tau r_{c'\,c'+n;\,b'+m\,b'}(p,\rho,\tau)$$
$$\times d_{mn}{}^{\alpha(p,\rho,\tau)}(-z_t)$$

$$=(\sqrt{t})r_{c'\,c'+n;\,b'+m\,b'}(p,\rho,\tau)[(1+\delta_{n0})(1+\delta_{n\,-2c'})$$
$$\times(1+\delta_{m0})(1+\delta_{m\,-2b'})]^{1/2}L_{mn}(t,s), \quad (179)$$

say, where $L_{mn}(t,s)$ is finite at $t=0$.

Consider first the case $p\rho=+1$. We put

$$[(1+\delta_{n0})(1+\delta_{m0})(1+\delta_{n\,-2c'})$$
$$\times(1+\delta_{m\,-2b'})]^{1/2}r_{c'\,c'+n;\,b'+m\,b'}{}^{(p\rho=+1)}$$
$$=t^{\frac{1}{2}\{|n|+|m|-1\}}R_{c'\,c'+n;\,b'+m\,b'}{}^{p\rho=+1}(t), \quad (180)$$

with

$$R(t)=R^{(0)}+tR^{(1)}+t^2R^{(2)}+\cdots; \quad (181)$$

and we expand the crossing matrices about $t=0$ in the form

$$M_{\lambda\mu}{}^{\lambda'\mu'}(\epsilon)=M_{\lambda\mu}{}^{(0)\lambda'\mu'}+\epsilon M_{\lambda\mu}{}^{(1)\lambda'\mu'}$$
$$+\epsilon^2M_{\lambda\mu}{}^{(2)\lambda'\mu'}+\cdots, \quad (182)$$

where, clearly,

$$M_{\lambda\mu}{}^{(r)\lambda'\mu'}=(1/r!)(d^r/d\epsilon^r)M_{\lambda\mu}{}^{\lambda'\mu'}(\epsilon)|_{\epsilon=0}. \quad (183)$$

Thus (176) becomes

$$f_{cd:ab}{}^{(s)p\rho=+1}=\sum_{n=-2s_A}^{2s_A}\;\sum_{m=-2s_B}^{2s_B}\;\sum_{c',b'}\;\sum_{r,r'}\epsilon_A{}^r\epsilon_B{}^{r'}$$
$$\times M_{ca}{}^{(r)c'\,c'+n}M_{db}{}^{(r')b'+m\,b'}f_{c'\,c'+n;\,b'+m\,b'}{}^{(t)}$$
$$=\sum_{n=-2s_A}^{2s_A}\;\sum_{m=-2s_B}^{2s_B}t^{\frac{1}{2}\{|n|+|m|\}}L_{mn}(t,s)$$
$$\times\sum_{r,r'=0}^{\infty}t^{\frac{1}{2}(r+r')}v_A{}^r(t)v_B{}^{r'}(t)$$
$$\times\sum_{c',b'}M_{ca}{}^{(r)c'\,c'+n}M_{db}{}^{(r')b'+m\,b'}$$
$$\times R_{c'\,c'+n;\,b'+m\,b'}{}^{p\rho=+1}(t). \quad (184)$$

If we now take

$$R_{c'\bar{a}';\bar{d}'b'}{}^{p\rho=+1}(t)=A(s_A;y_{c'\bar{a}'})$$
$$\times A(s_A;y_{c'\bar{a}'}+1)\cdots A(s_A;x_{c'\bar{a}'}-1)$$
$$\times A(s_B;y_{\bar{d}'b'})A(s_B;y_{\bar{d}'b'}+1)\cdots A(s_B;x_{\bar{d}'b'}-1)$$
$$\times C(t;p,\rho;\Lambda(c'\bar{a}'))C(t;p,\rho;\Lambda(\bar{d}'b')), \quad (185)$$

where

$$y_{c'\bar{a}'}=\min\{c';\bar{a}'\},$$
$$x_{c'\bar{a}'}=\max\{c';\bar{a}'\}, \text{ etc.} \quad (186)$$

and

$$\lim_{t\to0}C(t;p,\rho;\Lambda(c',\bar{a}'))=C(p,\rho;\Lambda(c'\bar{a}')) \quad (187)$$

exists and is finite, and where

$$C(p,\rho;\Lambda(\bar{a}'c'))=\rho n_{C\bar{A}}C(p,\rho;\Lambda(c'\bar{a}')), \quad (188)$$

and use the results of the Lemmas, we see that the sum over b' and c' in (184) is zero if $|c-a|>|n|+r$ and/or $|d-b|>|m|+r'$.

Thus every term on the right-hand side of (184) of order $t^{\frac{1}{2}\{|n|+|m|+r+r'\}}$ contributes only to s-channel amplitudes with $|c-a|+|d-b|\leq|n|+|m|+r+r'$. Alternatively, $f_{cd;ab}{}^{(s)}$ has in it only powers of t greater than or equal to $|c-a|+|d-b|$, i.e.,

$$f_{cd;ab}{}^{(s)(p\rho=+1)}\propto t^{\frac{1}{2}\{|\Lambda(ca)|+|\Lambda(db)|\}}. \quad (189)$$

Thus we have the result that (185) and (180) provide an evasive solution to the constraint equations, for which the s-channel amplitudes possess a *factorizable t dependence*, even though the amplitudes themselves are not actually factorizable in the nonasymptotic limit.

From (177), (180), and (185) the behavior of the actual residues is

$$\begin{aligned}r_{c'\bar{a}';\bar{d}'b'}(p\rho=+1)\propto t^{\frac{1}{2}\{|c'-\bar{a}'|+|\bar{d}'-b'|-1\}}[(1+\delta_{c'\bar{a}'})\\ \times(1+\delta_{c'-\bar{a}'})(1+\delta_{\bar{d}'b'})(1+\delta_{\bar{d}'-b'})]^{-1/2}\times C(p,\rho;\Lambda(c'\bar{a}'))\\ \times C(p,\rho;\Lambda(\bar{d}'h'))A(s_A;y_{c'\bar{a}'})\cdots A(s_A;x_{c'\bar{a}'}-1)\\ \times A(s_B;y_{\bar{d}'b'})\cdots A(s_B;x_{\bar{d}'b'}-1).\quad(190)\end{aligned}$$

For the case $p\rho=-1$, the above solution is unacceptable since one must have

$$f_{c\bar{a};\bar{d}b}^{(t)}(p,\rho)=p\rho f_{-\bar{a}-c;\bar{d}b}^{(t)}(p,\rho),\quad(191)$$

which for $p\rho=-1$ is not satisfied by (185).

Using Lemma 4 we construct a solution with the correct symmetry (191) by taking

$$\begin{aligned}[(1+\delta_{n0})(1+\delta_{m0})(1+\delta_{n-2c'})\\ \times(1+\delta_{m-2b'})]^{1/2}r_{c'\ c'+n;b'+m\ b'}(p\rho=-1)\\ =t^{\frac{1}{2}\{|n|+|m|+1\}}R_{c'\ c'+n;b'+m\ b'}{}^{p\rho=-1}(t),\quad(192)\end{aligned}$$

and then choosing

$$\begin{aligned}R_{c'\bar{a}';\bar{d}'b'}{}^{p\rho=-1}(t)=(c'+\bar{a}')(\bar{d}'+b')\\ \times A(s_A;y_{c'\bar{a}'})\cdots A(s_A;x_{c'\bar{a}'}-1)\\ \times A(s_B;y_{\bar{d}'b'})\cdots A(s_B;x_{\bar{d}'b'}-1)\\ \times C(t;p,\rho;\Lambda(c'\bar{a}'))C(t;p,\rho;\Lambda(\bar{d}'b')).\quad(193)\end{aligned}$$

Applying the same arguments as used in the $p\rho=+1$ case, we see that a term on the right-hand side of (184) with given value of $|n|+|m|+r+r'$ can in this case contribute to $|c-a|+|d-b|\leq|n|+|m|+r+r'+2$. Hence the necessity of the extra power of t in (192). Thus (192) and (193) provide an evasive solution for which the s-channel helicity amplitudes have the behavior

$$f_{cd;ab}^{(s)p\rho=-1}\propto t^{\frac{1}{2}\{|\Lambda(ca)|+|\Lambda(db)|\}},\quad(194)$$

with, as required by (118),

$$f_{ad;ab}^{(s)p\rho=-1}\equiv0.\quad(195)$$

The residues themselves then behave as

$$\begin{aligned}r_{c'\bar{a}';\bar{d}'b'}(p\rho=-1)\propto t^{\frac{1}{2}\{|c'-\bar{a}'|+|\bar{d}'-b'|+1\}}[(1+\delta_{c'\bar{a}'})\\ \times(1+\delta_{c'-\bar{a}'})(1+\delta_{\bar{d}'b'})(1+\delta_{\bar{d}'-b'})]^{-1/2}\\ \times C(p,\rho;\Lambda(c'\bar{a}'))C(p,\rho;\Lambda(\bar{d}'b'))(c'+\bar{a}')\\ \times(\bar{d}'+b')A(s_A;y_{c'\bar{a}'})\cdots A(s_A;x_{c'\bar{a}'}-1)\\ \times A(s_B;y_{\bar{d}'b'})\cdots A(s_B;x_{\bar{d}'b'}-1).\quad(196)\end{aligned}$$

Thus the behavior conjectured on the basis of the examples is shown to be true in general, i.e., there does exist an evasive solution and it has the remarkable property that it leads to s-channel amplitudes whose main t behavior, as $t\rightarrow0$, is factorizable.

It is worth noting that the helicity dependence of the residue behavior (186) deduced in the EE case by satisfying the equal-mass constraint equations to *all orders* in s, coincides with the behavior (54) obtained by satisfying the unequal-unequal (UU) constraints to *leading order* only in s. The reason for this is discussed in Sec. VI.

As was done in the UU case we can also consider *conspiratorial* solutions involving poles of opposite p. We then get for the leading term in s the same behavior as given in Eq. (68). The residues behave as in (67) but there is no longer a simple condition like (49) relating the residues of opposite p. Instead one has

$$f^{(s)}=f^{(s)(1)}+f^{(s)(2)},$$

say, with

$$f_{cd;ab}^{(s)(1)}=\hat{s}f_{cd;ab}^{(s)(2)}[1+O(t^{\lambda_{\min}})],$$

where $\hat{s}=-\operatorname{sgn}(\Lambda(ac)\Lambda(db))$ and

$$f_{cd;ab}^{(s)(1)}\propto t^{\frac{1}{2}(\lambda_{\max}-\lambda_{\min})}.$$

V. PROCESSES WITH $m_A\neq m_C$; $m_B=m_D$

The t-channel process is of the type (unequal-mass pair) → (equal-mass pair) and we denote it as a UE process.[20]

As in Sec. III, we study the behavior of $\bar{f}^{(t)}(t,s)$ as $t\rightarrow0$ using Eq. (32).

It is seen from (13) and (16) that

$$\cos\theta_t\propto t^{1/2},\quad\sin\theta_t\rightarrow1\quad\text{as}\quad t\rightarrow0,\quad(197)$$

so that the factors involving $\cos\frac{1}{2}\theta_t$ and $\sin\frac{1}{2}\theta_t$ are innocuous in (32).

However, the rotation angles χ_A and χ_C for the equal-mass pair are very sensitive to the order in which the limits $s\rightarrow\infty$, $t\rightarrow0$ are taken. Thus

$$\begin{aligned}\cos\chi_A=-\frac{1}{S_{AB}t^{1/2}(t-4m_A{}^2)^{1/2}}[t(s+m_A{}^2-m_B{}^2)\\ +2m_A{}^2(m_B{}^2-m_D{}^2)],\quad(198)\end{aligned}$$

and if we fix s and let $t\rightarrow0$ we get

$$\cos\chi_A\propto t^{-1/2},\quad(199)$$

and, similarly,

$$\cos\chi_C\propto t^{-1/2}.\quad(200)$$

The limits of the other angles are unexceptional, so that using the property

$$d_{\lambda\mu}{}^J(\chi)\propto(\cos\chi)^J\quad(201)$$

as $\cos\chi\rightarrow\infty$, we are led in (32) to the result

$$\bar{f}_{c\bar{a};\bar{d}b}^{(t)}\propto t^{-\frac{1}{2}(s_A+s_C)},\quad(202)$$

independently of the helicities. One then finds[11] from

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[20] Much of the argument in this section is due to T. W. Rogers.

(26) that

$$\bar{f}_{c\bar{a};\bar{d}b}{}^{(t)}(p,\rho) \propto t^{-\frac{1}{2}(s_A+s_C)} \quad \text{for} \quad p = e_{CA}(-1)^{\Lambda/2}$$
$$\propto t^{-\frac{1}{2}(s_A+s_C-1)} \text{for} \quad p = -e_{CA}(-1)^{\Lambda/2}, \tag{203}$$

where

$$e_{CA} = +\eta_A/\eta_C \quad \text{if } A \text{ and } C \text{ are bosons}$$
$$= -\eta_A/\eta_C \quad \text{if } A \text{ and } C \text{ are fermions},$$

and therefore, naïvely, that the residues ought to have k.n.b.

$$r_{c\bar{a};\bar{d}b}(p,\rho) \propto t^{\frac{1}{2}[\lambda_m - \alpha(p,\rho)]} t^{-\frac{1}{2}(s_A+s_C+1)}$$

or

$$\propto t^{\frac{1}{2}[\lambda_m - \alpha(p,\rho)]} t^{-\frac{1}{2}(s_A+s_C)} \tag{204}$$

for

$$p = \pm e_{CA}(-1)^{\Lambda/2},$$

respectively.

But this behavior is totally at variance with the behavior deduced for EE and UU processes, since if we consider four processes, say

$$\begin{aligned} A+B &\rightarrow A+B, \\ G+G &\rightarrow H+H, \\ A+G &\rightarrow A+H, \\ G+B &\rightarrow H+B, \end{aligned} \tag{205}$$

where $m_G \neq m_H$, so that the processes are, respectively, of the type EE, UU, UE, and UE, then the residues must satisfy the factorization requirement

$$r_{a'\bar{a};\bar{b}b'}{}^{(\mathrm{EE})} r_{h'\bar{g};\bar{h}g'}{}^{(\mathrm{UU})} = r_{a'\bar{a};\bar{h}g'}{}^{(\mathrm{UE})} r_{h'\bar{g};\bar{b}b'}{}^{(\mathrm{UE})}, \tag{206}$$

and (204) is quite incompatible with (206), (54), and (109) or (185).

To get the correct behavior we consider only the *leading term for large s* in (198) then

$$\cos\chi_A \propto t^{1/2} \quad \text{as} \quad t \rightarrow 0, \tag{207}$$

and, similarly,

$$\cos\chi_C \propto t^{1/2}. \tag{208}$$

In fact, the behavior of χ_A and χ_C in this case is exactly the same as in the EE case, (79) and (80), provided we take only the leading term in s in the latter equation. Moreover, for the *leading term in s* in χ_B and χ_D we have from (19) precisely the same behavior as in the UU case (58).

Thus so long as we deal with only the leading term in s, the equal-mass pair and the unequal-mass pair in the UE process behave exactly as they would in EE- and UU-type processes, respectively. It is now clear that the behavior of the leading term in the s-channel amplitude can be deduced directly from the result (106) since in the proof of (106) no reference was made to the mass relations and since (106) was deduced for arbitrary t; i.e., corresponding to taking limits in the order s large then t small.

Thus we have for the leading term, in the absence of conspiracy,

$$f_{cd;ab}{}^{(s)} \propto t^{\frac{1}{2}\{|\Lambda(ac)|+|\Lambda(bd)|\}}, \tag{209}$$

and the residues can be evaluated from

$$r_{a\bar{a};\bar{d}b}{}^{(\mathrm{UE})} = [r_{a\bar{a};\bar{a}a}{}^{(\mathrm{EE})} r_{b\bar{d};\bar{d}b}{}^{(\mathrm{UU})}]^{1/2}, \tag{210}$$

using (54) with either (109), (114), (115), and (116) or (190) and (196), according to whether the coupling to the equal-mass pair is evasive or not.

It is well known that the limit $t \rightarrow 0$ is subtle when unequal masses are present.[21] The correct way to approach $t=0$ is probably through a fixed-s dispersion relation and this would correspond to the order of limits used in deriving (209) and (210). Again in this case there can be conspiratorial solutions involving poles of opposite p and the behavior (209) is then replaced by that given in (68).

VI. GROUP THEORY AT $t=0$

A. Group-Theory Approach

Consider the s-channel process

$$A+B \rightarrow C+D$$

and let

$$K = p_A - p_C, \quad Q = p_A - p_D \tag{211}$$

be the momentum-transfer four-vectors, so that from Eq. (4)

$$t = K^2. \tag{212}$$

Take an arbitrary Lorentz frame in which

$$p_A = ((\mathbf{p}^2 + m_A{}^2)^{1/2}, \mathbf{p})$$

and

$$p_C = ((\mathbf{p}'^2 + m_C{}^2)^{1/2}, \mathbf{p}'). \tag{213}$$

Then if $m_A = m_C$, it is easy to show that at $t=0$, in every Lorentz frame, K is a null vector, i.e.,

$$K = (0,0,0,0). \tag{214}$$

That is, $t=0$ implies forward scattering in all reference systems.

However, if $m_A \neq m_C$ and $t=0$ is a physical point for the process, then if we go to the s-channel c.m. frame with the z axis along, or antiparallel, to $\mathbf{p}-\mathbf{p}'$, then at $t=0$

$$K = \frac{m_A{}^2 - m_C{}^2 + m_D{}^2 - m_B{}^2}{\sqrt{s}}(1,0,0,1), \tag{215}$$

i.e., K is a lightlike four-vector. It follows that K is a lightlike vector in all frames which can be reached from the s-channel c.m. frame by arbitrary Lorentz transformations.

Let us now go to the c.m. frame of the t channel. Put

$$p_A = (-(\mathbf{p}^2 + m_A{}^2)^{1/2}, \mathbf{p}),$$

and

$$p_C = ((\mathbf{p}^2 + m_C{}^2)^{1/2}, \mathbf{p}), \tag{216}$$

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[21] D. Z. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967).

so that

$$K=(-(\mathbf{p}^2+m_A)^{1/2}-(\mathbf{p}^2+m_C^2)^{1/2},0,0,0)$$
$$=(-\sqrt{t},0,0,0). \qquad (217)$$

Then in the limit $t\rightarrow 0$, K becomes the null vector $K=(0,0,0,0)$ *independently* of any relation amongst the masses.

Comparing (214), (215), and (217) it is clear that in going to the limit $t=0$ while in the t-channel c.m. frame we must be doing something quite drastic. This is also shown by the fact that at $t=0$, $p\equiv|\mathbf{p}|=im$ in the case $m_A=m_C\equiv m$, whereas we need $p=i\infty$ if $m_A\neq m_C$.

From (214) and (215) one would argue that at $t=0$ the relevant symmetry groups of the physical scattering amplitudes are $O(3,1)$ in the case $m_A=m_C$, $m_B=m_D$ and a group G isomorphic to $T_2\times O(2)$ (the group of translations and rotations in 2 dimensions) for the case of unequal masses. On the other hand, from (217) it is argued[9] that the relevant group is $O(3,1)$ [or $O(4)$ if s is below threshold] in all cases, independently of the masses; and this leads to a classification of Regge poles in terms of "Lorentz" poles. Since Regge poles presumably have an independent existence it seems most unlikely that there is any fundamental preference for $O(3,1)$, since if one treats the general mass case, steering clear of $t=0$ in the t-channel c.m. frame, one is simply never led to $O(3,1)$ at finite s.

This difficulty at $t=0$ in the t-channel c.m. frame is intimately bound up with the troubles which were discovered even in the spinless case.[1,21] In the usual discussion of these difficulties one is always thinking of $s\rightarrow\infty$. We wish to stress that the difficulties at $t=0$ have little to do with $s\rightarrow\infty$.

Consider for simplicity a spinless process. Let $A(s,t)$ be the invariant scattering amplitude. Let s lie in a closed domain $\mathfrak{D}$.

Then certainly the limit $t\rightarrow 0$ exists and

$$\lim_{t\rightarrow 0} A(s,t)=H(s)\,, \quad \text{say.} \qquad (218)$$

Also we have

$$A(s,t)=f^{(s)}(s,z_s)=f^{(t)}(t,z_t) \qquad (219)$$

and, therefore,

$$\lim_{t\rightarrow 0} f^{(s)}(s,z_s)=\lim_{t\rightarrow 0} f^{(t)}(t,z_t)=H(s)\,. \qquad (220)$$

However, we have, for the unequal-mass case, that

$$\lim_{t\rightarrow 0} z_t=1 \quad \text{for all} \quad s\in\mathfrak{D}\,,$$

and therefore, *naïvely*,

$$\lim_{t\rightarrow 0} f^{(t)}(t,z_t)=f^{(t)}(0,1) \qquad (221)$$

contradicting (220). The fallacy, of course, lies in the assumption that if

$$f(x)=\hat{f}(\phi(x))$$

then

$$\lim_{x\rightarrow 0} \hat{f}(\phi(x))=\hat{f}(\lim_{x\rightarrow 0} \phi(x))\,,$$

a result which is only true if the mapping $x\rightarrow\phi(x)$ is nonsingular at $x=0$, which is not true for the mapping

$$(s,t)\rightarrow(t,z_t) \quad \text{at} \quad t=0\,.$$

Thus irrespective of whether s is large or not, one *cannot* take the limit $t\rightarrow 0$ if $f^{(t)}$ regarded as a function of t and z_t. Or, if one does take the limit, it may have nothing to do with the physics at $t=0$. Now we claim that the fallacy which leads to the spurious results, that $O(3,1)$ holds independently of the masses, is precisely of this nature. For in order to deduce the symmetry it is necessary to regard $f_{\lambda\mu}{}^{(t)}(s,t)$ as a function of the vectors K and $p=p_A+p_C$, $p'=p_B+p_D$, i.e.,

$$f_{\lambda\mu}{}^{(t)}(s,t)=\hat{f}_{\lambda\mu}(K,p,p')\,,$$

and it is not true, in the unequal-mass case, that

$$\lim_{t\rightarrow 0} f_{\lambda\mu}{}^{(t)}(s,t)=\hat{f}_{\lambda\mu}(\lim_{t\rightarrow 0} K,\ \lim_{t\rightarrow 0} p,\ \lim_{t\rightarrow 0} p')\,,$$

since the mapping

$$(s,t)\rightarrow(K,p,p')$$

is singular at $t=0$. Thus the group $O(3,1)$ is not relevant for the unequal-mass case in general. However, let us now see what happens as $s\rightarrow\infty$. We *assume* that for the unequal-mass case there exists an asymptotic expansion of the form

$$f_{\mu\lambda}{}^{(s)}(P,K,Q)=s^{\alpha(t)}G_{\mu\lambda}(P,K,Q)\,, \qquad (222)$$

where

$$\lim_{s\rightarrow\infty} G_{\mu\lambda}(P,K,Q)$$

exists and is equal to

$$G_{\mu\lambda}(\lim_{s\rightarrow\infty} P,\ \lim_{s\rightarrow\infty} K,\ \lim_{s\rightarrow\infty} Q)$$

and

$$G_{\mu\lambda}(P,K,Q)=G_{\mu\lambda}(P,K,Q)\big|_{s=\infty}$$
$$+\frac{1}{s}\left[\frac{d}{d(1/s)}G_{\mu\lambda}(P,K,Q)\right]_{s=\infty}+\cdots. \qquad (223)$$

Then the term $\lim_{s\rightarrow\infty}G_{\mu\lambda}(P,K,Q)$ has in it $\lim_{s\rightarrow\infty}K=(0,0,0,0)$, and therefore, for the *leading term*, and only for the leading term, the relevant symmetry group becomes $O(3,1)$. Thus, in summary, we expect that

$$f_{\mu\lambda}{}^{(s)}\sim\beta_{\mu\lambda}(t)s^{\alpha(t)}+B_{\mu\lambda}(s,t)\cdots, \qquad (224)$$

where $B_{\mu\lambda}(s,t)=O(s^{\alpha(t)-1})$ and where the relevant groups would be $O(3,1)$ for $\beta_{\mu\lambda}(t)$ and $T_2\times O(2)$ for $B_{\mu\lambda}(s,t)$. This result explains why we found the same residue behavior when treating the EE case to all orders in s

and the UU case only to leading order in s. (See end of Sec. IV.) It also suggests that the role of daughters in the UU case is simply to eliminate a spurious $O(3,1)$ symmetry from terms of order $s^{\alpha(t)-1}$ and lower, that develops as a result of our present methods of Reggeization.

B. Uniqueness of the Lorentz-Pole Hypothesis

It has been claimed[10] that it is not possible for a number of Lorentz poles to conspire in such a way as to give a result equivalent to having one single nonconspiring Regge pole. The proof given results from a study of $\pi\rho\to\pi\rho$. We wish to show that this claim is not justified.

In the first place, our general evasive solution is a counter-example to this claim. The evasive solution is eliminated in Ref. 10 by the implicit assumption that $r_{1,-1}^{\pi\pi;\rho\rho}$ does not vanish at $t=0$ [see Eq. (125)].

In the second place, even assuming that $r_{1,-1}\neq 0$ at $t=0$, a proof of the existence of an infinite sequence of Regge poles using only the process $\pi\rho\to\pi\rho$ certainly does not say anything about the question of conspiracy, since in this process true conspiracy is in any case impossible since only one type of Regge pole can contribute to it. If one assumes $r_{1,-1}\neq 0$ in the constraint equation (122), then one is certainly forced into an infinite sequence of Regge poles. On the other hand, in the case $NN\to NN$, we know that the constraint equation can be satisfied by a *finite* conspiring sequence of Regge poles.[2] But, as we mentioned in Sec. I, this finite sequence might well be incompatible in other processes.

Thus we believe that the question of group-theoretic solutions to the constraint equations is still open. It may turn out that only two extreme alternatives are possible: evasion or $O(3,1)$ conspiracy, but the proof is lacking.

VII. DISCUSSION AND CONCLUSION

We have seen above that the constraint equations impose very serious restrictions on the trajectories and residues of Regge poles in the neighborhood of $t=0$. We have analyzed some possible methods of satisfying these constraints in the general case of the scattering of particles of arbitrary spin and in several examples, in particular the construction of an *evasive solution*, and have indicated the experimental consequences. It is clear, however, that these conditions, fundamentally, have little to do with the Regge model itself. Any model which uses as input the helicity amplitudes in the crossed channel will run into the same difficulties.[22] The reason for this can be seen group-theoretically. The reduction of Poincaré group down to the little group proceeds quite differently according to whether the total four-momentum is a timelike, spacelike, or null vector, i.e., according as $t\gtrless 0$ or $t=0$. The crossing matrix, which relates the regions $t>0$ and $t<0$, has a simple structure because the "amount of symmetry" in the two regions is the same and therefore the number of *independent* helicity amplitudes needed in each region is the same. On the other hand, at $t=0$ there is greater symmetry, hence fewer independent amplitudes are required to describe the s-matrix, and therefore the constraints. The attempts to study the situation at $t=0$ by group-theoretic methods are fraught with difficulty. The relevant groups for processes with equal-mass particles or unequal-mass particles are quite different. The symmetries hold only at one point $t=0$. The $O(3,1)$ or $O(4)$ partial-wave amplitudes are diagonal only *at* $t=0$, so away from $t=0$ it might be necessary to have new poles appearing in the nondiagonal amplitudes.

It is instructive to consider the situation at $t=0$ in some less complicated theories than the Regge model. For example, in single elementary-particle exchange models, one always describes the exchanged particle as a representation of the Lorentz group. Thus a ρ meson is treated as a four-vector, an A_1 meson as an axial four-vector, and so on. This is so in both the Feynman-diagram approach and in dispersion theory. Alternatively, one could imagine describing ρ or A_1 meson exchange by means of a Breit-Wigner resonance in the relevant crossed-channel partial-wave helicity amplitude, and these two methods are not at all equivalent, except in special circumstances.

Consider, for example, nucleon-nucleon scattering. In the latter approach the A_1 would appear as a resonance in the t-channel amplitude $\bar{f}_1{}^J(t)$ at $J=1$, $t=m_{A_1}{}^2$. It would thus contribute only to the amplitude $\bar{f}_3$ in the notation of Ref. (3). On the other hand, A_1 exchange calculated using dispersion-theoretic methods leads to a contribution to F_A, the invariant function which is the coefficient of the axial coupling $(\gamma_5\gamma_\mu)\cdot(\gamma_5\gamma_\mu)$, and from this one finds a contribution to both $\bar{f}_3$ and $\bar{f}_1$. In fact one gets

$$\bar{f}_1=-(p^2/m^2)\bar{f}_3\,, \tag{225}$$

which is just right to satisfy (1) as $t\to 0$ (which implies $p^2\to -m^2$). The reason for this discrepancy is clear. The axial four-vector behaves like an axial three-vector at, and only at the point $t=m_{A_1}{}^2$. Everywhere else it behaves like a mixture of an axial three-vector and a pseudoscalar. Only the axial three-vector part resonates and, close to $t=m_{A_1}{}^2$, one could neglect the pseudoscalar part. However at $t=0$ it is essential to have both parts in precisely the right proportions.

For ρ-meson exchange there is no discrepancy. One would expect contributions to $\bar{f}_2$, $\bar{f}_4$, and $\bar{f}_5$ by the partial-wave approach, and this is what one finds on treating the ρ as a four-vector with just electric coupling. Moreover, the contribution to $\bar{f}_4$ has an explicit factor of t in it guaranteeing that (1) is satisfied. The difference between the ρ and A_1 cases lies in the fact that the ρ

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[22] It is possible that Weinberg's approach to Feynman diagrams [Phys. Rev. **133**, B1318 (1964)] may also suffer from this difficulty.

is coupled to a conserved current whereas the A_1 is not.[23]

Another example is nucleon exchange in the direct channel of $\pi N\rightarrow\pi N$. Naïvely one might have tried to describe the nucleon as a pole in the p-wave amplitude with zero contribution to the s-wave. It is well known that nucleon exchange as normally treated contributes to both s and p waves and it is essential to keep both contributions as $s\rightarrow 0$.

In all these cases, in which the elementary-particle exchange is fed into the invariant amplitudes of the problem, there is never any trouble at $t=0$. This is because the decomposition into invariants does not utilize the little group and is therefore a global process. Thus $t=0$ is simply not a special point. Thus parametrizing a theory which is to be useful globally, i.e., for all t, in terms of the amplitudes of a single channel, does not seem to be a very transparent or logical scheme. Indeed Regge theory seems to be the only case where this is done, and this, presumably, because of too naïve a generalization of the potential-theory results to the relativistic situation.

All this leads one to suspect, therefore, that it ought to be possible to formulate Regge-pole theory in a covariant form which would be a more natural extension of the potential situation to the relativistic one, and in which the question of constraints at $t=0$ would not arise. It would, of course, be possible to reexpress the content of such a theory in terms of Regge poles as presently used, presumably in a unique way, and this would then be equivalent to giving a unique prescription for the solution of the constraint equations .

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APPENDIX I

For convenience we list here some relevant values of the functions

$$d_{\lambda\mu}{}^J\equiv d_{\lambda\mu}{}^J(\theta=\pi/2)$$

and

$$\Delta_{\lambda\mu}{}^J=\frac{d}{d\theta}d_{\lambda\mu}{}^J(\theta)\,|_{\theta=\pi/2}\,.$$

$d_{\lambda\mu}{}^J(\theta=\pi/2)$.

$J=\frac{1}{2}$:

$$\frac{1}{\sqrt{2}}\times\ \begin{array}{c|cc} \downarrow\lambda\quad\mu\rightarrow & \frac{1}{2} & -\frac{1}{2} \\ \hline \frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & 1 & 1 \end{array}\,.$$

$J=1$:

$$\frac{1}{2}\times\ \begin{array}{c|ccc} \downarrow\lambda\quad\mu\rightarrow & 1 & 0 & -1 \\ \hline 1 & 1 & -\sqrt{2} & 1 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} \\ -1 & 1 & \sqrt{2} & 1 \end{array}\,.$$

$J=\frac{3}{2}$:

$$\frac{1}{2\sqrt{2}}\times\ \begin{array}{c|cccc} \downarrow\lambda\quad\mu\rightarrow & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \hline \frac{3}{2} & 1 & -\sqrt{3} & \sqrt{3} & -1 \\ \frac{1}{2} & \sqrt{3} & -1 & -1 & \sqrt{3} \\ -\frac{1}{2} & \sqrt{3} & 1 & -1 & -\sqrt{3} \\ -\frac{3}{2} & 1 & \sqrt{3} & \sqrt{3} & 1 \end{array}\,.$$

$J=2$:

$$\frac{1}{4}\times\ \begin{array}{c|ccccc} \downarrow\lambda\quad\mu\rightarrow & 2 & 1 & 0 & -1 & -2 \\ \hline 2 & 1 & -2 & \sqrt{6} & -2 & 1 \\ 1 & 2 & -2 & 0 & 2 & -2 \\ 0 & \sqrt{6} & 0 & -2 & 0 & \sqrt{6} \\ -1 & 2 & 2 & 0 & -2 & -2 \\ -2 & 1 & 2 & \sqrt{6} & 2 & 1 \end{array}\,.$$

$$\frac{d}{d\theta}d_{\lambda\mu}{}^J(\theta)\,|_{\theta=\pi/2}\,.$$

$J=\frac{3}{2}$:

$$-\frac{1}{2\sqrt{2}}\times\ \begin{array}{c|cccc} \downarrow\lambda\quad\mu\rightarrow & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \hline \frac{3}{2} & \frac{3}{2} & -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\sqrt{3} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & \frac{5}{2} & -\frac{5}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & \frac{5}{2} & \frac{5}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{3}{2} & -\frac{3}{2} & -\frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} & \frac{3}{2} \end{array}\,.$$

$J=1$:

$$\frac{1}{2}\times\ \begin{array}{c|ccc} \downarrow\lambda\quad\mu\rightarrow & 1 & 0 & -1 \\ \hline 1 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ -1 & 1 & 0 & -1 \end{array}\,.$$

APPENDIX II

We list here an enumeration of the independent amplitudes for some processes with arbitrary spin.

(i) Elastic fermion-fermion scattering:

$$F_1+F_2\rightarrow F_1+F_2$$

with helicity transitions $\langle f_1'f_2'|f_1f_2\rangle$. Independent elements:

$f_2'>0$: (a) $f_2=f_2'$; $f_1'\geq f_1$,

(b) $f_2'=-f_2$; $f_1\leq -f_1'$,

(c) $f_2'>f_2>0$; all f_1, f_1',

(d) $f_2'>-f_2>0$; all f_1, f_1'.

Number of elements$=\frac{1}{4}M(M+2)$, where

$$M=(2s_1+1)(2s_2+1)\,.$$

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[23] These matters have been discussed in more detail by L. Durand, III, (see Ref. 5) and J. C. Taylor, Clarendon Laboratory Report, 1967 (unpublished).

(ii) Elastic scattering of identical fermions:

$$F+F\to F+F$$

with helicity transitions $\langle f_1'f_2'|f_1f_2\rangle$. Independent elements:

$f_2'>0$: (a) $f_2'=f_2$ with $-f_2'\le(f_1=f_1')\le f_2$ and with $f_1'>-f_1>0$,

(b) $f_2'=-f_2$ with $-f_2'\le(f_1=-f_1')\le f_2'$,

(c) $f_2'>f_2>0$ with $f_1=\pm f_1'$ and with $f_1'\neq f_1$, f_1' and f_1 of same sign, and $-f_2'\le f_1'\le f_2'$,

(d) $f_2'>-f_2>0$; all f_1, f_1'.

Number of elements $=\frac{1}{3}R[8(R^3+1)-R(2R-1)]$, where $R=s+\frac{1}{2}$.

(iii) Elastic fermion-boson scattering:

$$F+B\to F+B$$

with helicity transitions $\langle b'f'|bf\rangle$. Independent elements

$f'>0$: (a) $b'=0$; all b and f,

(b) b', $b\neq 0$ and same elements as for $F_1+F_2\to F_1+F_2$.

Number of elements $=\frac{1}{4}M(M+2)$, where

$$M=(2s_F+1)(2s_B+1).$$

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Regge-Pole Couplings to Nucleons in a Field-Theory Model

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Perturbation theory is used to consider the coupling of both normal- and abnormal-parity Regge poles to nucleons. The kinematic dependence of the residue functions on the trajectory function is obtained. All spin-flip amplitudes have fixed poles in both the angular momentum and energy variables. One of the abnormal parity trajectories satisfies a conspiracy condition corresponding to Freedman and Wang's class II. The daughter pole in this case moves exactly parallel to the leading trajectory, unlike the daughter arising from unequal-mass kinematics.

I. INTRODUCTION

PERTURBATION-THEORY models of Regge poles have been useful as a technique for testing various conjectured properties of Reggeized scattering amplitudes.[1] On the other hand, all of the work on summing infinite classes of perturbation-theory diagrams has involved spinless particles; while the results so obtained are interesting, they are not particularly useful for the phenomenologist who attempts to fit experimentally observed high-energy cross sections with Regge poles. Particles with spin are always involved. The problem of summing diagrams with internal particles having spin has not been solved.[1,2] The difficulties due to the necessity of including more than just simple ladder diagrams have made the analysis of eighth-order diagrams prohibitive; summation of higher-order diagrams is out of the question. In this paper we consider the more modest problem of coupling previously developed Regge poles to external states involving particles with spin. Only the external states and at most one internal line involve spin. The motion of the Regge poles in the complex l plane is assumed to be determined entirely by the coupling to the lowest-mass intermediate states which are composed of spinless scalar and pseudoscalar particles. If real Regge trajectories are dominated by nearby singularities, the model is a quite reasonable one for determining the coupling to higher spin states.

We analyze the coupling of boson trajectories of both parities to nucleon-antinucleon states. The method is applicable to states containing particles of any spin, but the nucleon system is the most interesting one from the experimental point of view. We obtain expressions for the asymptotic form of the t-channel invariant helicity amplitudes for the processes $MM'\to MM'$, $N\bar{N}\to MM'$, and $N\bar{N}\to\bar{N}N$; M denotes a spinless meson whose parity depends on whether we are considering normal or abnormal parity trajectories. The kinematic dependence of the Regge residues on the trajectory function $\alpha(t)$ is such that the coefficient of the leading power of $(-s/s_0)$ is $\Gamma(-\alpha)$ for every amplitude con-

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[1] For a complete discussion of the techniques and justification of high-energy perturbation theory, as well as references to the problems to which it has been applied, see R. Eden, P. Landshoff, D. Olive, and J. C. Polkinghorne, *The Analytic S Matrix* (Cambridge University Press, New York, 1966), Chap. 3.

[2] J. C. Polkinghorne, J. Math. Phys. 5, 1491 (1964); J. V. Greenman, *ibid.* 7, 1782 (1966); 8, 26 (1967).