can be taken as m_ρ^{-2}, the three ρ-exchange graphs contribute an effective 4π interaction

$$4m_\rho^2 g_\rho^{-2}\lambda^4(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})^2.$$

But this is exactly cancelled by the $(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})^2$ term arising directly from (7.10). Thus, if we wish to add ρ-exchange terms to the results of current algebra for low-energy π-π scattering, we must also add compensating terms whose effect is to convert the propagator $(q^2+m_\rho^2)^{-1}$ into $-(q^2/m_\rho^2)(q^2+m_\rho^2)^{-1}$. The total contribution made by ρ exchange plus compensating terms to low-energy π-π scattering is of fourth order in m_π, and hence negligible.

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Test of Time-Reversal Invariance in the Decay Process $K^+ \rightarrow \mu^+ + \nu + e^+ + e^-$†

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CP violation occurring in the $K_L \rightarrow 2\pi$ decay is assumed to take place when the electromagnetic and weak interactions occur simultaneously. An analysis of the $K^+ \rightarrow \mu^+ + \nu + e^+ + e^-$ decay is presented which shows that the measurement of the muon polarization in this decay is a practical way of determining this effect. The reduction of the principal background from $K_{\mu 3}$ is discussed. Such an experiment would also provide valuable information on structure radiation.

I. INTRODUCTION

Among the many theories and models that have been proposed since the discovery of the decay mode $K_L{}^0 \rightarrow \pi^+\pi^-$,[1] those that invoke the electromagnetic interaction[2] are of particular interest. These hypotheses that the electromagnetic interaction is related to CP violation, can explain "naturally" the factor

$$|\eta_{+-}| = [P(K_L{}^0 \rightarrow \pi^+\pi^-)/\Gamma(K_S{}^0 \rightarrow \pi^+\pi^-)]^{1/2} \approx \alpha/\pi$$

and the large admixture of the $\Delta I = \frac{3}{2}$ amplitude in the process $K_L{}^0 \rightarrow \pi\pi$.[3] The results of the recent experiments on weak decay processes $K_L{}^0 \rightarrow \pi\mu\nu$,[4] $\Lambda \rightarrow p\pi^-$,[5] and Ne^{19} β decay[6] showing no evidence of CP violation for these decay modes may serve as indirect support for these hypotheses.

On the other hand, the experiments with relatively high statistics, such as the upper limit[7] for $\eta \rightarrow \pi^0 e^+ e^-$ and the charge asymmetry[8] between the two charged pions in $\eta \rightarrow \pi^+\pi^-\pi^0$, show that the present experimental status is at least consistent with the conventional C- and P-conserving electromagnetic theory. Thus, even though a definite conclusion should not be drawn until further experimental information on Compton scattering[9] and other processes becomes available, we are tempted to think that the ordinary C- and P-conserving electromagnetic interaction can describe nature fairly well to the order of $g_{st}e$, where g_{st} is the strong-interaction coupling constant.

However, we are aware that there is no experiment which checks definitely the possibility of CP-violating electromagnetic interaction accompanied by weak inter-

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[1] $K_L{}^0 \rightarrow \pi^+ + \pi^-$ was first observed by J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters **13**, 138 (1964). For a summary of the experimental situation, see V. L. Fitch, Rapporteur's Talk, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, Calif., 1966* (University of California Press, Berkeley, 1967), p. 63.

[2] Possibility of the connection between CP violation and electromagnetic interaction is discussed by many authors; see, for example, J. Bernstein, G. Feinberg, and T. D. Lee, Phys. Rev. **139**, B1650 (1965).

[3] J. M. Gaillard, F. Krienen, W. Galbraith, A. Hussri, M. R. Jane, N. H. Lipman, G. Manning, T. Rutcliffe, P. Day, A. G. Parham, B. T. Payne, A. C. Sherwood, H. Faissner, and H. Reithler, Phys. Rev. Letters **18**, 20 (1967); J. W. Cronin, P. F. Kunz, W. S. Risk, and P. C. Wheeler, *ibid.* **18**, 25 (1967).

[4] For the most recent experimental result, see K. K. Young, M. J. Longo, and J. A. Helland, Phys. Rev. Letters **18**, 806 (1967).

[5] O. E. Overseth and R. F. Roth, Phys. Rev. Letters **19**, 391 (1967).

[6] F. P. Calaprice, E. D. Commins, H. M. Gibbs, and G. L. Wick, Phys. Rev. Letters **18**, 918 (1967).

[7] D. Berley, E. L. Hart, D. C. Rahm, D. L. Stonehill, B. Thevenet, W. J. Willis, and S. S. Yamamoto, Phys. Rev. **142**, 893 (1966); see also C. Baglin, A. Bezaguet, B. Degrange, F. Jacquet, P. Musset, U. Nguyen-Khac, and G. Nihoul-Boutang, Phys. Letters **22**, 219 (1966).

[8] A. M. Cnops, G. Finocchiaro, J. C. Lassale, P. Mittner, P. Zanella, J. P. Dufey, B. Gobbi, M. A. Pouchon, and A. Muller, Phys. Letters **22**, 546 (1966).

[9] Generally it is difficult to detect C- and T-violating but P-conserving electromagnetic effects among hadrons. See, for example, K. Hiida and T. Ebata, Phys. Rev. **154**, 1337 (1967).

action.[10] We recall that for the explanation of $K_L \to 2\pi$ as a radiative correction, such an interaction is indispensible.[11] Many experiments have been proposed to check the possibility.[12,13] The purpose and essential idea of the present study on $K^+ \to \mu^+\nu e^+e^-$ is the same as those mentioned in Refs. 12 and 13. We, however, pay specific attention to the feasibility of the proposal as a practical experiment. An advantage in studying the CP-noninvariant properties of the radiative leptonic decay such as $K^+ \to \mu^+\nu\gamma(e^+e^-)$ is that it can provide a "yes or no" experiment free from ambiguities due to the strong final-state interactions.

Another aspect of the investigation of the decay mode $K^+ \to \mu^+\nu e^+e^-$ would reveal the contributions of the "structure" radiation in a strangeness-changing interaction.

II. OBSERVABLES OF THE DECAY PROCESSES

In this section, expressions of the decay rates and muon polarizations for the processes

$$K^+(p) \to \mu^+(l)+\nu(q)+\gamma(k) \tag{2.1}$$

and

$$K^+(p) \to \mu^+(l)+\nu(q)+e^-(s_1)+e^+(s_2) \tag{2.2}$$

are developed. The letters in parentheses represent the four-momenta of the respective particles. The amplitude for both processes (2.1) and (2.2) is

$$S=-i(2\pi)^4\delta(p-k-q-l)NG_K e\bar{v}(l)f_\mu{}^K\epsilon_\mu u(q)\,, \tag{2.3}$$

where

$$f_\mu{}^K=\left\{m\left[\frac{p_\mu}{p\cdot k-\frac{1}{2}k^2}-\frac{2l_\mu}{2k\cdot l+k^2}+\frac{(\gamma\cdot k)\gamma_\mu}{2k\cdot l+k^2}\right]\right.$$
$$+\frac{A_K}{M^2}[p_\mu(\gamma\cdot k)-\gamma_\mu(p\cdot k)]+\frac{iB_K}{M^2}\epsilon_{\alpha\beta\sigma\mu}p_\alpha\gamma_\beta k_\sigma$$
$$\left.+\frac{C_K}{M^2}k^2\gamma_\mu\right\}\tfrac{1}{2}(1-i\gamma_5)\,. \tag{2.4}$$

The expressions (2.3) and (2.4) are written for both processes (2.1) and (2.2), and for the latter process the electron-positron pair is produced by a virtual photon with four-momentum k, $k^2\neq 0$. In addition, ϵ_μ is the photon polarization for process (2.1) and $\epsilon_\mu=(e/k^2)\times\bar{v}(s_2)\gamma_\mu u(s_1)$ for process (2.2). The normalization factor N for these processes is given by

$$N=(2\pi)^{-6}\left(\frac{m}{4p_0k_0E}\right)^{1/2} \quad \text{for decay mode (2.1)}$$

and

$$N=(2\pi)^{-15/2}\left(\frac{mm_e{}^2}{2p_0EE_1E_2}\right)^{1/2} \quad \text{for decay mode (2.2)}.$$

M is the mass of the kaon and the muon has a mass m and an energy E. The energies of the electron and positron are E_1 and E_2, respectively, and their mass is m_e. The spinors of the various particles can be identified by the indicated momentum parameters.

The first term in Eq. (2.4) is the "inner bremsstrahlung" term associated with the various direct couplings of the photon to a charged particle line in the Feynman graphs [Figs. 1(a) and 1(b)]. The other terms represent three different possible modes of "structure" radiation where the photon is emitted at the weak vertex[10] [Fig. 1(c)]. The dimensionless coupling constants A, B, and C are complex if CP invariance is not assumed. The phenomenological coupling constant G_K of Eq. (2.3) is determined by the rate for $K^+\to\mu^+\nu$, given by

$$\Gamma(K^+\to\mu^+\nu)=\frac{G_K{}^2}{4\pi}\tfrac{1}{4}Mm^2\left(1-\frac{m^2}{M^2}\right)^2. \tag{2.5}$$

The system of units where $\hbar=c=1$ is used, so that $e^2/4\pi=1/137$. For convenience, the density matrix is defined by

$$\rho_{\mu\nu}\equiv\frac{\gamma\cdot l+m}{2m}f_\mu\tfrac{1}{2}\gamma\cdot qf_\nu{}'\frac{\gamma\cdot l+m}{2m}\,, \tag{2.6}$$

where $f_\nu{}'=\gamma_0 f_\nu{}^\dagger\gamma_0$, and $f_\nu{}^\dagger$ is the Hermitian conjugate of f_ν. Thus the observable momentum spectrum and polarization of the muon are related to the quantities $\mathrm{Tr}(\rho_{\mu\nu})$ and $\mathrm{Tr}(i\gamma_5\gamma_\alpha\rho_{\mu\nu})$, respectively. The explicit expressions for these quantities are given in Appendix A.

The decay distributions for the two decay modes (2.1) and (2.2) are given. With the above definition of $\rho_{\mu\nu}$, the decay distribution for $K\to\mu\nu\gamma$ in the rest frame of K^+ is

$$\Gamma(K\to\mu\nu\gamma)=\frac{G_K{}^2e^2}{(2\pi)^3}\frac{m}{2M}\int[\mathrm{Tr}(\rho_{\mu\nu})S_{\mu\nu}]$$
$$+\delta(\cos\theta-\cos\theta')dEdkd\cos\theta\,, \tag{2.7}$$

where θ is the angle between the muon and photon momenta,

$$\cos\theta'=[(M-E)^2-2k(M-E)-|\mathbf{l}|^2]/2k|\mathbf{l}| \tag{2.8}$$

and

$$S_{\mu\nu}=g_{\mu\nu}-(n_\mu k_\nu+n_\nu k_\mu)/(n\cdot k),\quad n_\mu n_\nu=1 \tag{2.9}$$

where the second term is included to make the longitudinal and timelike polarizations of the real photon cancel. This term must be included because $\mathrm{Tr}(\rho_{\mu\nu})$ is

[10] We classify the electromagnetic interaction of the hypothetical intermediate bosons into the catagory of "electromagnetic interaction accompanied by weak interaction," or "structure" radiation.

[11] Note that most of the radiative correction diagrams, except those in which virtual photons are emitted from the weak vertex, are either CP invariant by definition or CP invariant in the $SU(3)$ limit. For the last statement, see N. Cabibbo, Phys. Rev. Letters **14**, 965 (1965).

[12] J. L. Gervais, J. Iliopoulos, and J. M. Kaplan, Phys. Letters **20**, 432 (1966).

[13] S. W. MacDowell, Phys. Rev. Letters **17**, 1116 (1966); **18**, 227(E) (1967).

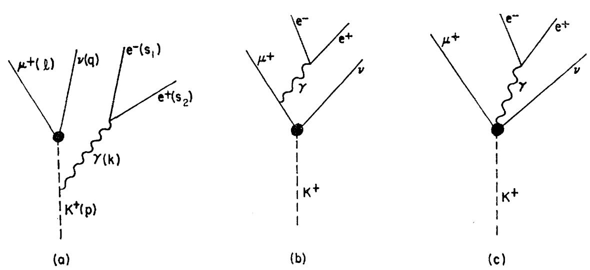

FIG. 1. Feynman diagrams for the process $K^+ \to \mu^+\nu e^+e^-$.

not explicitly gauge invariant as noted in Appendix A due to the dropping of terms proportional to k_μ or k_ν in $\mathrm{Tr}(\rho_{\mu\nu})$. The component of the muon polarization in the direction α, $\mathbf{P}_\alpha$ is given by

$$\mathbf{P}_\alpha = -\mathrm{Tr}(i\gamma_5\gamma_\alpha\rho_{\mu\nu}S_{\mu\nu})/\mathrm{Tr}(\rho_{\mu\nu}S_{\mu\nu}). \quad (2.10)$$

The polarization in the rest frame of the muon is obtained by applying the appropriate Lorentz transformation so that the timelike polarization of the muon vanishes. The corresponding quantities for the $K^+ \to \mu^+\nu e^-e^+$ (2.2) mode with the electron mass neglected are given by

$$\Gamma(K \to \mu\nu ee) = \frac{e^4}{(2\pi)^6}\frac{m}{4M}G_K^2 \int dE d\cos\theta |\mathbf{k}|^2 d|\mathbf{k}|$$

$$\times \int \frac{|\mathbf{s}_1| d\Omega_s}{q_0(k_0 - |\mathbf{k}|\cos\Theta)}[\mathrm{Tr}(\rho_{\mu\nu})S_{\mu\nu}], \quad (2.11)$$

where θ is the angle between the muon momentum and the virtual photon momentum, $\mathbf{k} \equiv \mathbf{s}_1 + \mathbf{s}_2$, and the angle Θ is the angle between the momenta of the electron and virtual photon, $\mathbf{s}_1$ and $\mathbf{k}$. The angular integration $d\Omega_s$ is over $\mathbf{s}_1$, with respect to fixed $\mathbf{k}$ and $\mathbf{l}$. The component of muon polarization in the direction α, $\mathbf{P}_\alpha$ for the process (2.2) is

$$\mathbf{P}_\alpha = -[\mathrm{Tr}(i\gamma_5\gamma_\alpha\rho_{\mu\nu})S_{\mu\nu}]/[\mathrm{Tr}(\rho_{\mu\nu})S_{\mu\nu}]. \quad (2.12)$$

The quantity $S_{\mu\nu}$ is given by

$$S_{\mu\nu} = -[2k_\mu k_\nu - 2r_\mu r_\nu - g_{\mu\nu}(k^2 - r^2)]/(k^2)^2, \quad (2.13)$$

where $r_\mu \equiv (s_1 - s_2)_\mu$. To obtain the muon polarization in its rest frame in Sec. IV, the Lorentz transformation is applied on $\mathbf{P}_\alpha$, as described under Eq. (2.10). The resulting expressions are lengthy and not very illuminating so they will not be reproduced here. Quantitative results of numerical calculations for these polarizations are presented in Sec. IV.

The polarization of the muon can be expressed as

$$\mathbf{P} = L\mathbf{l} + K\mathbf{k} + R\mathbf{r} + N_1[\mathbf{k}\times\mathbf{l}] + N_2[\mathbf{r}\times\mathbf{l}] + N_3[\mathbf{k}\times\mathbf{r}], \quad (2.14)$$

where L, K, R, N_1, N_2, and N_3 are complicated functions of kinematical variables, which are derived from the expressions (2.12) by making use of the results of Appendix A. Contributions to L, K, and R are predominantly from CP-conserving terms. There are small contributions from the CP-conserving amplitudes to the N_i's, but, as we shall see in Sec. IV, these contributions are practically zero. The physical explanation for this is the following. Those CP-conserving contributions to N_i's are always accompanied by the factor[14] $\mathbf{l}\cdot[\mathbf{k}\times\mathbf{r}]$, and since $\mathbf{r}$ is almost parallel to $\mathbf{k}$ this factor is very small. Therefore, detection of the polarization due to N_i's leads to the unambiguous conclusion of CP violation.

Identical expressions occur for the pion decays $\pi \to \mu\nu\gamma$ and $\pi \to \mu\nu ee$ with appropriate changes of coupling constants and meson masses. The contributions from the structure terms are less than 1% and studies of CP-violating effects in these decay modes seem unfeasible experimentally at the present time.

III. ESTIMATE OF A AND B

Even though no reliable methods are available for calculating the magnitudes of the coupling constants A, B, and C introduced in the previous section it is desirable to have an estimate of them to aid in planning an experiment. In order for these estimates to provide information on the CP-violating effects to be expected in an experiment, we assume that CP is maximally violated as defined by

$$|\mathrm{Re}\,A| = |\mathrm{Im}\,A|, \quad |\mathrm{Re}\,B| = |\mathrm{Im}\,B|. \quad (3.1)$$

This assumption is supported by the fact that both Im A and Im B cannot be small if the photon is related to the process $K_L \to 2\pi$ in an essential way.

To estimate A (i.e., A_K), the analogous decay $\pi \to e\nu\gamma$ is used to obtain a value for A_π which is then related to A_K by the Cabibbo theory. For estimating the magnitude of A_π, an "axial-vector current" for the process $\rho^0 \to \pi + l + \nu$ is first constructed. Since this current should satisfy the requirement that the ρ is coupled to the conserved current and the hypothesis of the partially conserved axial-vector current (PCAC),[15] it can be written as

$$\langle \pi^\alpha(p) | J_\mu{}^{\beta(A)}(q) | \rho_\nu{}^\gamma(k) \rangle = G_A \epsilon_{\alpha\beta\gamma} \frac{4g_{\rho\pi}}{m_\rho{}^2} \frac{2M_N}{g_{\pi N}} \times \left[(p\cdot k)\delta_{\mu\nu}\epsilon_\nu - (p\cdot\epsilon)k_\mu - \frac{k^2(p\cdot\epsilon)}{q^2 - \mu^2} q_\mu \right]. \quad (3.2)$$

In this expression, $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric tensor for charge indices, and $g_{\rho\pi}$ and $g_{\pi N}$ are the $\rho\pi\pi$ and πNN coupling constants, respectively. ϵ_ν is the ρ-meson polarization, and m_ρ and M_N are the masses of the ρ meson and the nucleon, respectively. Equation (3.2), together with the ρ-meson–isovector-photon Lagrangian:

$$\mathfrak{L}(\gamma,\rho) = (em_\rho{}^2/2\gamma_\rho)A_\mu\rho_\mu{}^3, \quad (3.3)$$

[14] This is what is expected from the results of Refs. 12 and 13.
[15] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

where the ρ-meson dominance in the electromagnetic form factor means

$$g_{\rho\pi}/\gamma_\rho=1\,, \tag{3.4}$$

gives the following expression for $\pi\to l\nu\gamma$:

$$\begin{aligned} f_\mu\epsilon_\mu &= eG_Am_\rho^{-2}(4M_N/g_{\pi N})\epsilon_{\alpha\beta 3} \\ &\quad\times[(p\cdot k)(j\cdot\epsilon)-(p\cdot\epsilon)(k\cdot j)] \\ &= eG_\pi\mu^{-2}(2\mu^2/m_\rho^2)\epsilon_{\alpha\beta 3} \\ &\quad\times[(p\cdot k)(j\cdot\epsilon)-(p\cdot\epsilon)(k\cdot j)]. \end{aligned} \tag{3.5}$$

In the above, j is the leptonic weak current, and PCAC has been used to obtain the last line. Now, comparison of Eq. (3.5) to Eq. (2.4) yields

$$A_\pi=2\mu^2/m_\rho^2=0.068\,. \tag{3.6}$$

Thus, if CP is violated to the order of the weak times the electromagnetic interaction even for the strangeness-conserving decay $\pi\to e\nu\gamma$, then we should have

$$|\mathrm{Re}\,A_\pi|=|\mathrm{Im}\,A_\pi|\simeq 0.068\,. \tag{3.7}$$

Since an experiment[16] analyzed under the assumption of CP conservation gives

$$|A_\pi^{\mathrm{exp}}|=0.01 \quad\text{or}\quad 0.05\,, \tag{3.8}$$

our number (3.7) is quite compatible with experiment. Additional information is required to determine whether A_π^{exp} is real or complex.

The use of Cabibbo's theory[17] allows a straightforward estimate of A_K. Since the Cabibbo angle is included in the definition of G_K ($G_K\approx\sin\theta G_\pi$), we obtain from (3.6)

$$|\mathrm{Re}\,A_K|=|\mathrm{Im}\,A_K|\simeq 2M_K^2/m_\rho^2=0.84\,. \tag{3.9}$$

An estimate for the magnitude of the second-structure radiation-mode coupling constant B already exists. Gervais, Iliopoulos, and Kaplan[12] give in our notation

$$|\mathrm{Re}\,B_K|=|\mathrm{Im}\,B_K|\simeq 5\,. \tag{3.10}$$

Another estimate can be obtained by using Cabibbo's theory to transform the value

$$|B_\pi|\simeq 2.5\times 10^{-2}\,, \tag{3.11}$$

which is derived from the $\pi^0\to 2\gamma$ decay. The result is

$$\begin{aligned} |\mathrm{Re}\,B_K|=|\mathrm{Im}\,B_K|&=|B_\pi|(M_K^2/\mu^2) \\ &\quad\times(g_{K^*VK}/g_{\rho\omega\pi})\simeq 0.3\,. \end{aligned} \tag{3.12}$$

In this expression, V stands for ρ and ω, and ω is the isoscalar member of the vector-meson octet.

We have no way to estimate the third constant C. This term, however, does not contribute to the real-photon mode (2.1), and its contribution to the electron-pair mode is small. (See the numerical results presented in Sec. IV.)

IV. NUMERICAL RESULTS

The differential decay rates and the muon polarizations in $K^+\to\mu^+\nu\gamma$ (2.1) and $K^+\to\mu^+\nu e^-e^+$ (2.2) are computed numerically. These observables, as discussed in the Sec. II, e.g., Eqs. (2.7) and (2.11), are evaluated through a Monte Carlo method using an electronic computer. A brief description of the process employed is found in Appendix B.

Many authors have studied the process $K^+\to\mu^+\nu\gamma$, and we present only the pertinent parts of the results to compare with the decay mode $K^+\to\mu^+\nu e^-e^+$. The differential decay spectrum for $K^+\to\mu^+\nu\gamma$ is dominated by the inner bremsstrahlung term in Eq. (2.4). Since the bremsstrahlung term is infrared-divergent, this process is greatly enhanced with soft photons (see Fig. 2). This decay rate of the process is very sensitive to the cutoff energy of the photon. The partial branching ratio for the process when the photon energy $E_\gamma\geq 10$ MeV and $A=B=0$ is found to be

$$R(K^+\to\mu^+\nu\gamma)=4.0\times 10^{-3}\,. \tag{4.1}$$

When we include the structure terms, and vary their magnitudes and relative phases, we find that the branching ratio is modified up to approximately 10% of the above value (4.1) for the range $|A|$ and $|B|\lesssim 1$.

If a possibility of CP noninvariance for this process is allowed, A and B are complex relative to the bremsstrahlung term. One of the effects of the CP noninvariance is the muon polarization normal to the decay plane. The possibilities of large normal polarization of the muon have been shown.[12,13] This normal polarization results from the interference between the bremsstrahlung term and the structure terms in Eq. (2.4). Since

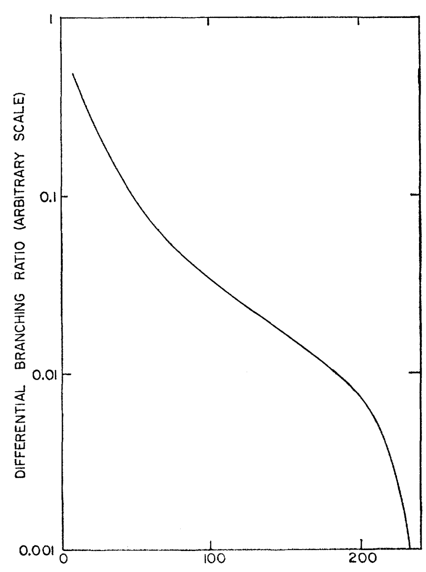

FIG. 2. Differential branching ratio versus photon energy for $K^+\to\mu^+\nu\gamma$.

[16] P. Deppomier, J. Heintje, C. Rubbia, and V. Soergel, Phys. Letters **7**, 285 (1963).

[17] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

only the former term is infrared-divergent and the latter terms are not, an appreciable normal polarization of the muon occurs accompanied with photons at the high-energy end of the spectrum. Indeed a normal polarization of muon up to 57% at photon energy 165 MeV was shown in Ref. 12. As mentioned above, the photon spectrum is dominated by the soft photons, so that the partial rate for those decaying with large normal polarization of muons is lowered appreciably from the value of (4.1). In practice, the observation of the effect is further obscured by the large background from $K^+ \to \pi^0\mu^+\nu$.

For the decay process $K^+ \to \mu^+\nu e^- e^+$ (2.2), we immediately see from Eq. (2.4) that the process is dominated by small k^2 (Fig. 3). Compared to the case of the process $K^+ \to \mu^+\nu\gamma$ [Eq. (2.1)], however, the magnitude of the bremsstrahlung is relatively suppressed. The

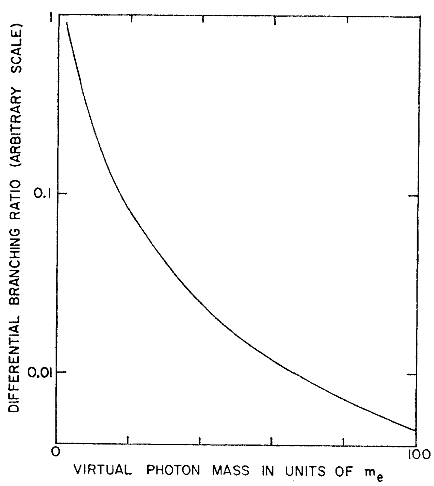

FIG. 3. Differential branching ratio versus virtual photon mass for $K^+ \to \mu^+\nu e^+e^-$ when $A=B=C=0$.

mass of the virtual photon λ is defined by $\lambda^2=k^2=k_0^2-|\mathbf{k}|^2$, and for the analysis below, it is crucial to observe that a small λ could be associated with a large $\mathbf{k}$ and thereby an energetic electron-positron pair.

(i) We find the branching ratio when both electron and positron energies are larger than 10 MeV to be

$$R(K^+ \to \mu^+\nu e^+e^-)=5.2\times10^{-6} \tag{4.2}$$

if $A=B=C=0$. For the range of $|A|$ and $|B|\lesssim 1$, this value is modified by a factor of approximately 2 [see Fig. 4(a)].

(ii) When the cutoff energies of both electron and positron are increased to higher values, 40 and 80 MeV, the muon spectrum is modified mostly at the higher momentum end [Fig. 4(b)]. This means that the decay rate in the interesting muon momentum range, $30 \leq l \leq 150$ MeV/c (see below), remains relatively constant for higher cutoff energies of the electron pair. For the

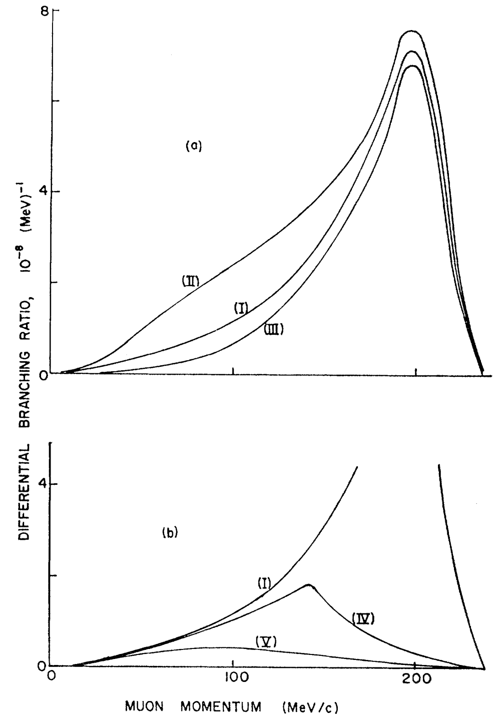

FIG. 4. Muon spectra for $K^+ \to \mu^+\nu e^+e^-$. (a) Cutoff energies of both electron and positron are 10 MeV. (I) Bremsstrahlung term only. $A=0$ and $B=0$. (II) $A=-1$ and $B=+1$. (III) $A=+1$ and $B=-1$. (b) Bremsstrahlung term only for various cutoff energies E for both electron and positron. (I) $E=10$ MeV, (IV) $E=40$ MeV, and (V) $E=80$ MeV.

purpose of a practical experiment we observe the following.

(iii) The opening angle of the electron pair is less than 10° for 95% of the events. In addition, the direction of the vector $\mathbf{k}\equiv\mathbf{s}_1+\mathbf{s}_2$ can be approximated by the bisector of $\mathbf{s}_1$ and $\mathbf{s}_2$ to within 1° accuracy. For our process we define a "decay plane" by the vectors $\mathbf{l}$ and the bisector.

(iv) The angular distribution of the pair relative to the muon momentum, $\hat{l}\cdot\hat{k}$ is almost uniform.

The muon polarization is given by Eq. (2.14):

$$\mathbf{P}=L\mathbf{l}+K\mathbf{k}+R\mathbf{r}+N_1[\mathbf{k}\times\mathbf{l}]+N_2[\mathbf{r}\times\mathbf{l}]+N_3[\mathbf{k}\times\mathbf{r}].$$

As noted in the previous section, contributions to L, K, and R are from the CP-conserving terms, and there are negligible contributions from the CP-conserving amplitude to the parameters N_1, N_2, and N_3, which are adding up incoherently. In Fig. 5 we present some of our results. To study the effects of CP violation, we let $|A|=|B|=1$ and vary the relative phases of these parameters, neglecting the contributions from the parameter C.

(v) For CP violation in a coordinate system where the z axis is chosen along the muon momentum $\mathbf{l}$, the transverse component of the muon polarization perpendicular to the "decay plane," P_x, is comparable to the muon polarization in the other two directions for the

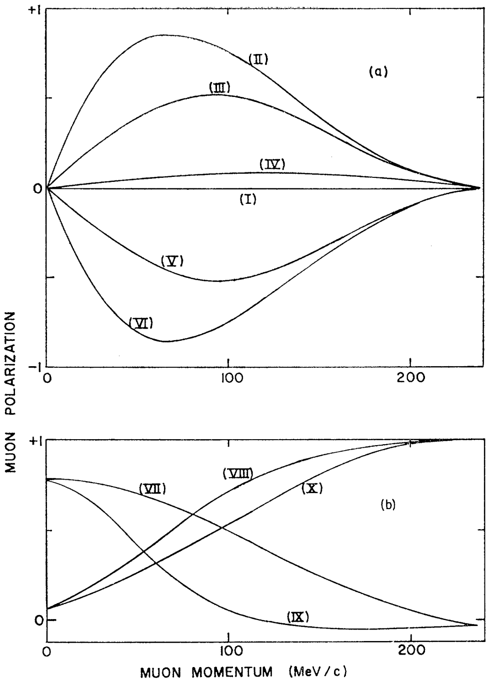

FIG. 5. Muon polarization for $K^+ \to \mu^+\nu e^+e^-$, where the cutoff energy of the pair, $E=10$ MeV. (a) P_x (for the definition of axis, see text). (I) $\mathrm{Im}A=\mathrm{Im}B=0$. (II) $A=(1+i)/\sqrt{2}$, $B=(-1-i)/\sqrt{2}$. (III) $A=(1+i)/\sqrt{2}$, $B=(1-i)/\sqrt{2}$. (IV) $A=(1-i)/\sqrt{2}$, $B=(1-i)/\sqrt{2}$. (V) $A=(1-i)/\sqrt{2}$, $B=(1+i)/\sqrt{2}$. (VI) $A=(1-i)/\sqrt{2}$, $B=(-1+i)/\sqrt{2}$. (b) Polarization in the "decay plane," P_y and P_z. (VII) and (VIII): P_y and P_z for bremsstrahlung term only. (IX) and (X): P_y and P_z for $A=(1+i)/\sqrt{2}$ and $B=(-1-i)/\sqrt{2}$.

range of l, $30 \lesssim l \lesssim 150$ MeV/c. For example, when the cutoff energies of both electron and positron are 10 MeV/c, and when $A=(1+i)/\sqrt{2}$ and $B=(-1-i)/\sqrt{2}$, we find $(P_x,P_y,P_z)=(0.84,\ 0.24,\ 0.33)$ around $l\approx 70$ MeV/c.

(vi) For the higher cutoff energies of the pair, P_x is enhanced because of the fact that the "uninteresting" bremsstrahlung term is suppressed. For the same parameters as the above example, we find $(P_x,P_y,P_z)=(0.89,\ 0.19,\ 0.39)$ and $(0.92,\ 0.02,\ 0.42)$ for cutoff energies of the pair 40 and 80 MeV, respectively.

(vii) Furthermore, the transverse polarization of the muon in the "decay plane" P_y is relatively sensitive to the signs of Re(A) and Re(B). The values of P_y together with those of P_z for typical sets of A and B are shown in Fig. 6.

V. DISCUSSION OF $K_{\mu 3}$ BACKGROUND

In this section we discuss methods of separating $K^+ \to \mu^+\nu e^+e^-$ events from the principal background $K_{\mu 3}$, $K^+ \to \mu^+\nu\pi^0$ followed by π^0 decay through a Dalitz pair. Since the muon polarization in the $K_{\mu 3}$ decay is known to lie in the decay plane,[18] this background causes only dilution of statistics in the determination of CP-violating polarization P_x. Consequently a ratio of a number of events over background $R\approx 1/1$ or better will suffice to detect P_x in $K^+ \to \mu^+\nu e^+e^-$ decay. For the detection of the structure terms which require the measurements of P_y and P_z, one would need more precise knowledge of the muon polarization in $K_{\mu 3}$ in the decay plane[18] than those available at the present time. Aside from this fact, a similar argument can be made for the determination of P_y as in the case of P_x above. We focus our attention on a region where P_x is relatively large, i.e., the "interesting region" of $30\leq l \leq 150$ MeV/c and virtual photon energy $k_0 \geq 50$ MeV. Within this region, the branching ratio of $K^+ \to \mu^+\nu e^+e^-$ decay is found to be 2.4×10^{-6}. Using Monte Carlo generated events for $K^+ \to \mu^+\nu\pi^0$ followed by $\pi^0 \to e^+e^-\gamma$,[19] we calculate that the rate of the background is 3.2×10^{-4} in a region of $30\leq l\leq 150$ MeV/c and the energy of the Dalitz pair $k_0\geq 50$ MeV. Two independent calculations were performed, with these $K_{\mu 3}$ events interpreted as $K^+ \to \mu^+\nu e^+e^-$ decays where we assumed that the odd photons escape detection. The observables required in these calculations were the muon momentum l, the energy of the electron-positron pair k_0, and the angle between the muon direction and the bisector of the pair. The first calculation was the mass of the virtual photon λ, assuming that the only missing particle was a neutrino, and the second was the missing mass of the "neutrino" q^2, assuming that λ is small ($\lambda<10$ MeV). A plot of λ^2 versus q^2 is shown in Fig. 7. Since the predominant contribution to $K^+ \to \mu^+\nu e^+e^-$ decay comes from small λ^2 as discussed in the previous section, most of the genuine events fall on a line segment near the

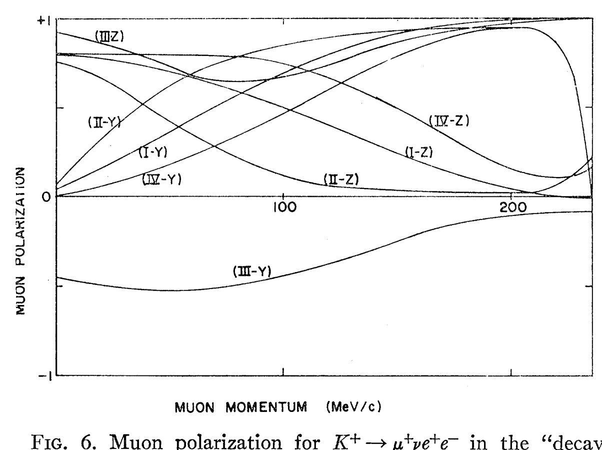

FIG. 6. Muon polarization for $K^+ \to \mu^+\nu e^+e^-$ in the "decay plane" when A and B are real. (I) $A=B=0$. (II) $A=1$, $B=0$ (III) $A=1$, $B=-1$. (IV) $A=0$, $B=1$. Y and Z indicate the y and z components of the muon polarization, P_y and P_y, respectively.

[18] For a recent summary of experimental situation, see L. B. Auerbach, A. K. Mann, W. K. McFarlane, and F. J. Sciulli, Phys. Rev. Letters **19**, 464 (1967).

[19] The $K_{\mu 3}$ spectrum was computed with pure vector coupling and $\xi=f_-/f_+=0$. [See R. H. Dalitz, in *Proceedings of the International School of Physics "Enrico Fermi", Course 32* (Academic Press Inc., New York, 1966).] The branching ratio is taken as 3.4% from A. H. Rosenfield, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, P. Soding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. **39**, 1 (1967).

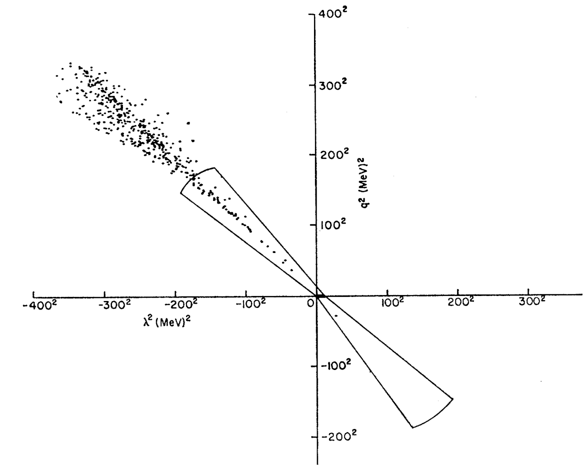

FIG. 7. The distribution of λ^2 and q^2. The dots represent 500 Monte Carlo events for the background of $K_{\mu 3}$. The $K^+\to\mu^+\nu e^+e^-$ events lie on the heavy line segment near the origin. These events with errors mentioned in the text would be distributed in the area inside the solid lines.

origin, $q^2=0$ and $1<\lambda<10$ MeV. When we assign errors expected in typical experiments, e.g., an uncertainty of 30 MeV in the pair energy and 10 MeV/c error in the muon momentum, the $K^+\to\mu^+\nu e^+e^-$ events will be distributed approximately symmetrically in the second and fourth quadrants as shown in the Fig. 7. All the $K_{\mu 3}$ background events fall in the second quadrant and if similar errors are assigned to the observables of the $K_{\mu 3}$ background, we find the event-over-background ratio $R<1/1$ for those events in the fourth quadrant. Additional enhancement of this ratio is obtained by following tests. The opening angle of the Dalitz pair is found to be less acutely distributed when compared to the pair from $K^+\to\mu^+\nu e^+e^-$ events [see (iii), Sec. III]. Without losing any good events, we can reject 45% of the Dalitz pairs if the pair opening angles are confined to 10°. Allowing experimental uncertainties, we take a 15° cutoff on the pair opening angles and obtain a 36% rejection of the background. Secondly, the correlation between the energy of the pair and the angle between the muon momentum and the bisector of the pair enables us to discriminate the additional background. Finally, the conversion of the odd photon in the detector will increase the event-over-background ratio further.

VI. CONCLUSION

We have presented an analysis of the $K^+\to\mu^++\nu+e^++e^-$ decay including CP-violating effects. This analysis shows that if CP violation occurs in a process involving the electromagnetic and weak interactions, it is feasible to detect it experimentally by polarization measurements of these decay products. In addition, valuable information on structure radiation associated with this decay can be obtained.

Although the recently measured upper limits of the electric dipole moment (EDM) of the neutron[20] would seem to eliminate the possibility of significant CP violation occurring in this decay, the following argument shows that no incompatibility yet exists. From the transition moment of $\Sigma^+\to p+\gamma$, which is known to be $\sim 4\times 10^{-21}e$ cm,[21] we can easily estimate the EDM of the neutron. If CP is maximally violated in all radiative weak interactions of hadrons, the EDM of the neutron would be $\sim 10^{-20}e$ cm since the strangeness-nonchanging current could contribute. However, if we assume that only the strangeness-changing current accompanying the electromagnetic interaction is CP-violating, then our prediction for the EDM of the neutron would be (factor)$\times 10^{-22}e$ cm. Thus a smaller upper limit for the EDM of the neutron of, say, $10^{-23}e$ cm (considering possible "dynamical cancellation") is needed to rule out the possible CP-violation effects in the $K^+\to\mu^++\nu+e^++e^-$ decay.

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APPENDIX A

The explicit expressions for $\mathrm{Tr}(\rho_{\mu\nu})$ and $\mathrm{Tr}(i\gamma_5\gamma_\alpha\rho_{\mu\nu})$ are

$$2m\,\mathrm{Tr}(\rho_{\mu\nu})=\Big((q\cdot l)(p_\mu x-2l_\mu y)(p_\nu x-2l_\nu y)+2y(p_\mu x-2l_\mu y)[(k\cdot l)q_\nu-(k\cdot q)l_\nu]$$
$$-y^2\{2(k\cdot l)(k\cdot q)g_{\mu\nu}+k^2[2l_\mu q_\nu-g_{\mu\nu}(q\cdot l)]\}+2\,\mathrm{Re}\left(\frac{A}{M^2}\right)m[(k\cdot q)(p_\mu x-2l_\mu y)p_\nu-(p\cdot k)q_\mu(p_\nu x-2l_\nu y)$$
$$+yk^2p_\mu q_\nu+y(p\cdot k)(k\cdot q)g_{\mu\nu}]-2\,\mathrm{Re}\left(\frac{B}{M^2}\right)myg_{\sigma\sigma'}(\epsilon_{\alpha\beta\sigma\mu}q_\alpha k_\beta)(\epsilon_{\alpha'\beta'\sigma'\nu}p_{\alpha'}k_{\beta'})$$

[20] P. D. Miller, W. B. Dress, J. K. Baird, and N. F. Ramsey, Phys. Rev. Letters **19**, 381 (1967); C. G. Shull and R. Nathans, *ibid.* **19**, 384 (1967).
[21] K. Tanaka, Phys. Rev. **140**, B463 (1965).

$$+2\,\mathrm{Re}\left(\frac{C}{M^2}\right)k^2m[(p_\mu x-2l_\mu y)q_\nu+y(k\cdot q)g_{\mu\nu}]+\left|\frac{A}{M^2}\right|^2$$

$$\times\{[2(k\cdot l)(k\cdot q)-k^2(q\cdot l)]p_\mu p_\nu+(p\cdot k)^2[2l_\mu q_\nu-(q\cdot l)g_{\mu\nu}]-2(p\cdot k)p_\mu[(k\cdot l)q_\nu+(k\cdot q)l_\nu]\}$$

$$+\left|\frac{B}{M^2}\right|^2[-2(\epsilon_{\alpha\beta\sigma\mu}p_\alpha l_\beta k_\sigma)(\epsilon_{\alpha'\beta'\sigma'\nu}p_{\alpha'}l_{\beta'}k_{\sigma'})-(q\cdot l)g_{\sigma\sigma'}(\epsilon_{\alpha\beta\sigma\mu}p_\alpha k_\beta)(\epsilon_{\alpha'\beta'\sigma'\nu}p_{\alpha'}k_{\beta'})]$$

$$+\left|\frac{C}{M^2}\right|^2(k^2)^2[2l_\mu q_\nu-(q\cdot l)g_{\mu\nu}]+2\,\mathrm{Re}\left(\frac{A}{M^2}\frac{B^*}{M^2}\right)[p_\mu g_{\delta\delta'}(\epsilon_{\alpha\beta\sigma\delta}p_\alpha l_\beta k_\sigma)(\epsilon_{\delta'\alpha'\beta'\nu}p_{\alpha'}k_{\beta'})$$

$$+(p\cdot k)g_{\sigma\sigma'}(\epsilon_{\alpha\beta\sigma\nu}q_\alpha l_\beta)(\epsilon_{\sigma'\alpha'\beta'\mu}p_{\alpha'}k_{\beta'})]+2\,\mathrm{Re}\left(\frac{B}{M^2}\frac{C^*}{M^2}\right)k^2g_{\sigma\sigma'}(\epsilon_{\alpha\beta\sigma\mu}q_\alpha l_\beta)(\epsilon_{\alpha'\beta'\sigma'\nu}p_{\alpha'}k_{\beta'})$$

$$+2\,\mathrm{Re}\left(\frac{C}{M^2}\frac{A^*}{M^2}\right)k^2\{(p\cdot k)[2l_\mu q_\nu-(q\cdot l)g_{\mu\nu}]-p_\mu[(k\cdot l)q_\nu+(k\cdot q)l_\nu]\}\Bigg)$$

$$+\Bigg\{-2\,\mathrm{Im}\left(\frac{B}{M^2}\right)m(p_\mu x-2l_\mu y)(\epsilon_{\alpha\beta\sigma\nu}p_\alpha q_\beta k_\sigma)+2\,\mathrm{Im}\left(\frac{A}{M^2}\frac{B^*}{M^2}\right)[p_\mu(k\cdot l)(\epsilon_{\alpha\beta\sigma\nu}p_\alpha q_\beta k_\sigma)$$

$$+p_\mu(k\cdot q)(\epsilon_{\alpha\beta\sigma\nu}p_\alpha l_\beta k_\sigma)-l_\mu(p\cdot k)(\epsilon_{\alpha\beta\sigma\nu}p_\alpha q_\beta k_\sigma)-q_\mu(p\cdot k)(\epsilon_{\alpha\beta\sigma\nu}p_\alpha l_\beta k_\sigma)]$$

$$+2\,\mathrm{Im}\left(\frac{A}{M^2}\frac{C^*}{M^2}\right)k^2p_\mu(\epsilon_{\alpha\beta\sigma\nu}p_\alpha l_\beta k_\sigma)\Bigg\}\,. \quad \text{(A1)}$$

The "polarization Q" can be separated into CP-conserving Q_1, and CP-violating Q_2 as follows:

$$Q_{\alpha,\mu\nu}\equiv\mathrm{Tr}(i\gamma_5\gamma_\alpha\rho_{\mu\nu})=(Q_1)_{\alpha,\mu\nu}+(Q_2)_{\alpha,\mu\nu}\,, \quad \text{(A2)}$$

where

$$2m^2(Q_1)_{\alpha,\mu\nu}=l_\alpha\Bigg(-(p_\mu x-2l_\mu y)(p_\nu x-2l_\nu y)(q\cdot l)-2y(p_\mu x-2l_\mu y)[(k\cdot l)q_\nu-(k\cdot q)l_\nu]$$

$$+y^2[2(k\cdot l)(k\cdot q)g_{\mu\nu}-k^2(q\cdot l)g_{\mu\nu}+2k^2l_\mu q_\nu]+\left|\frac{A}{M^2}\right|^2\{p_\mu p_\nu[2(k\cdot l)(k\cdot q)-(q\cdot l)k^2]$$

$$+(p\cdot k)^2[2l_\mu q_\nu-(q\cdot l)g_{\mu\nu}]-2(p\cdot k)p_\mu[(k\cdot l)q_\nu+l_\nu(k\cdot q)]\}-\left|\frac{B}{M^2}\right|^2[2(\epsilon_{\beta\gamma\delta\mu}p_\beta l_\gamma k_\delta)(\epsilon_{\beta'\gamma'\delta'\nu}p_{\beta'}l_{\gamma'}k_{\delta'})$$

$$+(q\cdot l)g_{\delta\delta'}(\epsilon_{\beta\gamma\delta\mu}p_\beta k_\gamma)(\epsilon_{\beta'\gamma'\delta'\nu}p_{\beta'}k_{\gamma'})]+\left|\frac{C}{M^2}\right|^2(k^2)^2[2l_\mu q_\nu-(q\cdot l)g_{\mu\nu}]$$

$$+2\,\mathrm{Re}\left(\frac{A}{M^2}\frac{B^*}{M^2}\right)[p_\mu(\epsilon_{\beta\gamma\delta\sigma}l_\beta k_\gamma q_\delta)g_{\sigma\sigma'}(\epsilon_{\sigma'\beta'\gamma'\nu}p_{\beta'}k_{\gamma'})-(p\cdot k)(\epsilon_{\beta\gamma\delta\mu}l_\gamma q_\delta)g_{\beta\beta'}(\epsilon_{\beta'\gamma'\delta'\nu}p_{\gamma'}k_{\delta'})]$$

$$-2\,\mathrm{Re}\left(\frac{B}{M^2}\frac{C^*}{M^2}\right)k^2(\epsilon_{\beta\gamma\delta\mu}l_\gamma q_\delta)g_{\beta\beta'}(\epsilon_{\beta'\gamma'\delta'\nu}p_{\gamma'}k_{\delta'})+2\,\mathrm{Re}\left(\frac{C}{M^2}\frac{A^*}{M^2}\right)k^2$$

$$\times\{(p\cdot k)[2l_\mu q_\nu-(q\cdot l)g_{\mu\nu}]-p_\mu[(k\cdot l)q_\nu+l_\nu(k\cdot q)]\}\Bigg)$$

$$+m^2((p_\mu x-2l_\mu y)(p_\nu x-2l_\nu y)q_\alpha+2y(p_\mu x-2l_\mu y)[k_\alpha q_\nu-g_{\alpha\nu}(k\cdot q)]-y^2\{2k_\alpha g_{\mu\nu}(k\cdot q)+k^2[2g_{\alpha\mu}q_\nu-g_{\mu\nu}q_\alpha]\})$$

$$+2m\,\mathrm{Re}\left(\frac{A}{M^2}\right)((p_\mu x-2l_\mu y)\{(p\cdot k)[(q\cdot l)g_{\alpha\nu}-q_\alpha l_\nu]-p_\nu[(q\cdot l)k_\alpha-q_\alpha(k\cdot l)]\}$$

$$+yp_\mu\{2k_\alpha l_\nu(k\cdot q)-2(k\cdot l)g_{\alpha\nu}(k\cdot q)+k^2[g_{\alpha\nu}(q\cdot l)-q_\alpha l_\nu]\}$$

$$-y(p\cdot k)\{k_\alpha[2l_\mu q_\nu-g_{\mu\nu}(q\cdot l)]-(k\cdot l)[2g_{\alpha\mu}q_\nu-g_{\mu\nu}q_\alpha]\})$$

$$-2m\,\mathrm{Re}\left(\frac{B}{M^2}\right)\{g_{\alpha\alpha'}(p_\mu x-2l_\mu y)(\epsilon_{\alpha'\beta'\gamma'\delta'}l_{\beta'}q_{\gamma'})g_{\delta'\beta}(\epsilon_{\beta\gamma\delta\nu}p_\gamma k_\delta)-yg_{\alpha\alpha'}(\epsilon_{\alpha'\beta'\gamma'\mu}p_{\beta'}k_{\gamma'})(\epsilon_{\beta\gamma\delta\nu}k_\beta l_\gamma q_\delta)$$

$$+yq_\alpha g_{\beta\beta'}(\epsilon_{\beta\gamma\delta\mu}p_\gamma k_\delta)(\epsilon_{\beta'\gamma'\delta'\nu}k_{\gamma'}l_{\delta'})+yg_{\alpha\alpha'}[(\epsilon_{\alpha'\beta'\gamma'\mu}k_{\beta'}l_{\gamma'})(\epsilon_{\beta\gamma\delta\nu}p_\beta q_\gamma k_\delta)$$

$$+(\epsilon_{\alpha'\beta'\gamma'\mu}k_{\beta'}q_{\gamma'})(\epsilon_{\beta\gamma\delta\nu}p_\beta l_\gamma k_\delta)-(q\cdot l)g_{\beta\beta'}(\epsilon_{\alpha'\beta'\gamma'\nu}k_{\gamma'})(\epsilon_{\beta\gamma\delta\mu}p_\gamma k_\delta)]\}$$

$$+2m\,\mathrm{Re}\left(\frac{C}{M^2}\right)k^2((p_\mu x-2l_\mu y)[(q\cdot l)g_{\alpha\nu}-q_\alpha l_\nu]-y\{k_\alpha[2l_\mu q_\nu-g_{\mu\nu}(q\cdot l)]-(k\cdot l)[2g_{\alpha\mu}q_\nu-g_{\mu\nu}q_\alpha]\})$$

$$-m^2\left|\frac{A}{M^2}\right|^2\{p_\mu p_\nu[2k_\alpha(k\cdot q)-q_\alpha k^2]+(p\cdot k)^2(2g_{\alpha\mu}q_\nu-q_\alpha g_{\mu\nu})-2(p\cdot k)p_\mu[k_\alpha q_\nu+g_{\alpha\nu}(k\cdot q)]\}$$

$$-m^2\left|\frac{B}{M^2}\right|^2[2g_{\alpha\alpha'}(\epsilon_{\alpha'\beta'\gamma'\mu}p_{\beta'}k_{\gamma'})(\epsilon_{\beta\gamma\delta\nu}p_\beta l_\gamma k_\delta)-q_\alpha(\epsilon_{\beta\gamma\delta\mu}p_\gamma k_\delta)g_{\beta\beta'}(\epsilon_{\beta'\gamma'\delta'\nu}p_{\gamma'}k_{\delta'})]$$

$$-m^2\left|\frac{C}{M^2}\right|^2(k^2)^2[2g_{\alpha\mu}q_\nu-q_\alpha g_{\mu\nu}]+2m^2\,\mathrm{Re}\left(\frac{A}{M^2}\frac{B^*}{M^2}\right)[(p\cdot k)g_{\alpha\alpha'}(\epsilon_{\alpha'\beta'\gamma'\mu}q_{\beta'})g_{\gamma'\gamma}(\epsilon_{\gamma\beta\delta\nu}p_\beta k_\delta)$$

$$-p_\mu g_{\alpha\alpha'}(\epsilon_{\alpha'\beta'\gamma'\delta'}k_{\beta'}q_{\gamma'})g_{\delta\delta'}(\epsilon_{\delta\beta\gamma\nu}p_\beta k_\gamma)]+2m^2\,\mathrm{Re}\left(\frac{B}{M^2}\frac{C^*}{M^2}\right)k^2g_{\alpha\alpha'}(\epsilon_{\alpha'\beta'\gamma'\mu}q_{\beta'})g_{\gamma'\gamma}(\epsilon_{\gamma\alpha\beta\nu}p_\alpha k_\beta)$$

$$+2m^2\,\mathrm{Re}\left(\frac{C}{M^2}\frac{A^*}{M^2}\right)k^2\{p_\mu[k_\alpha q_\nu+g_{\alpha\nu}(k\cdot q)]-(p\cdot k)[2g_{\alpha\mu}q_\nu-q_\alpha g_{\mu\nu}]\}\,, \quad \text{(A3)}$$

and

$$2m^2(Q_2)_{\alpha,\mu\nu}=2l_\alpha\left\{\mathrm{Im}\left(\frac{A}{M^2}\frac{B^*}{M^2}\right)(\epsilon_{\beta\gamma\delta\mu}p_\beta l_\gamma k_\delta)[(p\cdot k)l_\mu-p_\mu(k\cdot l)+p_\mu(k\cdot q)-(k\cdot p)q_\mu]\right.$$

$$\left.-\mathrm{Im}\left(\frac{B}{M^2}\frac{C^*}{M^2}\right)k^2[l_\mu(\epsilon_{\beta\gamma\delta\nu}p_\beta l_\gamma k_\delta)-q_\mu(\epsilon_{\beta\gamma\delta\nu}p_\beta l_\gamma k_\delta)]+\mathrm{Im}\left(\frac{C}{M^2}\frac{A^*}{M^2}\right)k^2p_\mu(\epsilon_{\beta\gamma\delta\nu}l_\beta k_\gamma q_\delta)\right\}$$

$$-2m\,\mathrm{Im}\left(\frac{A}{M^2}\right)\{p_\mu(p_\nu x-2l_\mu y)g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\delta}l_\beta k_\alpha q_\delta)+(p\cdot k)(p_\nu x-2l_\nu y)g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\mu}l_\beta q_\gamma)$$

$$+k^2yp_\mu g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}l_\beta q_\gamma)+(p\cdot k)yg_{\mu\nu}g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\delta}l_\beta q_\gamma k_\delta)$$

$$-2y[p_\mu(k\cdot q)g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}l_\beta k_\gamma)-(p\cdot k)q_\mu g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}l_\beta k_\gamma)]\}$$

$$-2m\,\mathrm{Im}\left(\frac{B}{M^2}\right)((p_\mu x-2l_\mu y)[(q\cdot l)g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}p_\beta k_\gamma)+q_\alpha(\epsilon_{\beta\gamma\delta\nu}p_\beta l_\gamma k_\delta)]$$

$$-y(\epsilon_{\beta\gamma\delta\mu}p_\beta l_\gamma k_\delta)\{k_\alpha(2l_\nu-p_\nu)-g_{\alpha\nu}[(k\cdot l)-(k\cdot q)]\}+yg_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\mu}p_\beta k_\gamma)[q_\nu(k\cdot l)-l_\nu(k\cdot q)])$$

$$-2m\,\mathrm{Im}\left(\frac{C}{M^2}\right)k^2[(p_\mu x-2l_\mu y)g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}l_\beta q_\gamma)+yg_{\mu\nu}g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\delta}l_\beta q_\gamma k_\delta)]$$

$$+2m^2\,\mathrm{Im}\left(\frac{A}{M^2}\frac{B^*}{M^2}\right)\{(\epsilon_{\beta\gamma\delta\nu}p_\beta l_\gamma k_\delta)[p_\mu k_\alpha-g_{\alpha\mu}(k\cdot p)]+g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}p_\beta k_\gamma)[p_\mu(k\cdot q)-q_\mu(k\cdot p)]\}$$

$$+2m^2\,\mathrm{Im}\left(\frac{B}{M^2}\frac{C^*}{M^2}\right)k^2[g_{\alpha\mu}(\epsilon_{\beta\gamma\delta\nu}p_\beta l_\gamma k_\delta)+q_\mu g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}p_\beta k_\gamma)]$$

$$-2m^2\,\mathrm{Im}\left(\frac{C}{M^2}\frac{A^*}{M^2}\right)k^2p_\mu g_{\alpha\alpha'}(\epsilon_{\alpha'\beta\gamma\nu}k_\beta q_\gamma)\,. \quad \text{(A4)}$$

In the above expressions x and y are given by

$$x=m(p\cdot k-\tfrac{1}{2}k^2)^{-1},\quad y=m(2k\cdot l+k^2)^{-1}. \tag{A5}$$

As a matter of course, the expressions given by Eqs. (A1)–(A3) satisfy gauge invariance (we have omitted terms proportional to k_μ or k_ν) and the requirement

$$l_\alpha Q_{\alpha,\mu\nu}=0. \tag{A6}$$

For the radiative decay $K\to\mu\nu\gamma$, the indices μ and ν should be contracted with the tensor

$$S_{\mu\nu}=g_{\mu\nu}-(n_\mu k_\nu+n_\nu k_\mu)/n\cdot k$$

and for the decay $K\to\mu\nu e^+e^-$, the indices μ and ν should be contracted with the symmetric tensor given by

$$S_{\mu\nu}=\frac{1}{(k^2)^2}\operatorname{Tr}[(\gamma\cdot s_1+m_e)\gamma_\mu(\gamma\cdot s_2+m_e)\gamma_\nu]$$

$$=-\frac{1}{(k^2)^2}[2k_\mu k_\nu-2r_\mu r_\nu-g_{\mu\nu}(k^2-r^2)]. \tag{A7}$$

APPENDIX B

Numerical integration was performed through a Monte Carlo method using an IBM 7094 computer. The method used is conventional and will not be described here. A brief description of the method used to choose a set of Monte Carlo events is given below. Approximately 20 000 events for a set were found to give a statistically satisfactory sample.

Equation (2.10) is rewritten as

$$\Gamma(K\to\mu\nu ee)=\Gamma(K\to\mu\nu)\left(\frac{\alpha}{\pi}\right)^2\left(\frac{1}{4\pi m}\right)\left[\frac{(M^2-m^2)}{2M}\right]^2$$

$$\times\int\int\int\frac{|l|d|l|}{E}\frac{kq_0}{k_0}\lambda d\lambda dk$$

$$\times\int\frac{|s_1|}{k_0-k\cos\Theta}J(\Omega_s,\Omega_s{}')d\Omega_s{}'\operatorname{Tr}(\rho), \tag{B1}$$

where $\alpha=1/137$, $\lambda^2=k^2$, and the substitutions

$$\Gamma(K\to\mu\nu)=(G_K{}^2/4\pi)(m^2/M)\{(M^2-m^2)/2M\}^2 \tag{B2}$$

and

$$\partial\cos\theta/\partial\lambda^2=-q_0/2klk_0 \tag{B3}$$

have been made. We also made a change of Ω_s, the angular variables of the electron in the K^+ rest frame, into $\Omega_s{}'$ in the rest frame of the virtual photon with mass λ, moving with velocity β_λ in the K^+ rest frame. Then

$$J(\Omega_s,\Omega_s{}')=\frac{1}{(1-\beta_\lambda{}^2)^{1/2}}\left(\frac{\sin\Theta}{\sin\Theta'}\right)^3\left(1+\frac{\beta_\lambda}{\beta_e{}'}\cos\Theta\right), \tag{B4}$$

where primes denote quantities measured in the rest frame of the virtual photon.

We chose random samples uniformly distributed in a rectangular volume defined by three sides l, k, and λ, with their ranges:

$$0\leq(l \text{ and } k)\leq(M^2-m^2)/2M \tag{B5}$$

and

$$2m_e\leq\lambda\leq(M-m). \tag{B6}$$

In practice, however, we used the following method to facilitate the speed of computation. First we chose l inside the range of Eq. (B5); then for a given value of l, we restricted the domain of k and λ inside the physical region defined by four equations,

$$\lambda=2m_e, \tag{B7a}$$

$$\lambda^2=(M-E-l)^2-2(M-E-l)k, \tag{B7b}$$

$$\lambda^2=(M-E-l)^2+2(M-E-l)k, \tag{B7c}$$

and

$$\lambda^2=(M-E+l)^2-2(M-E+l)k. \tag{B7d}$$

Samples so chosen were carefully weighed according to the ratio of the unphysical area to the physical area. As noted in Sec. IV, most of the important contribution comes from small λ^2 events. It was found to be more practical to compute with a restricted λ: Two cases, $2m_e<\lambda<30m_e$ and $30m_e<\lambda<100m_e$, were studied. The contribution from the latter interval of λ is small compared to the first case. The contribution from samples with $\lambda>100m_e$ is negligible. After l, k, and λ were chosen, we chose electron variables by letting the virtual photon "decay" isotropically in its rest frame into an electron-positron pair.