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Nonlinear Realizations of Chiral Symmetry*

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We explore possible realizations of chiral symmetry, based on isotopic multiplets of fields whose transformation rules involve only isotopic-spin matrices and the pion field. The transformation rules are unique, up to possible redefinitions of the pion field. Chiral-invariant Lagrangians can be constructed by forming isotopic-spin-conserving functions of a covariant pion derivative, plus other fields and their covariant derivatives. The resulting models are essentially equivalent to those that have been derived by treating chirality as an ordinary linear symmetry broken by the vacuum, except that we do not have to commit ourselves as to the grouping of hadrons into chiral multiplets; as a result, the unrenormalized value of g_A/g_V need not be unity. We classify the possible choices of the chiral-symmetry-breaking term in the Lagrangian according to their chiral transformation properties, and give the values of the pion-pion scattering lengths for each choice. If the symmetry-breaking term has the simplest possible transformation properties, then the scattering lengths are those previously derived from current algebra. An alternative method of constructing chiral-invariant Lagrangians, using ρ mesons to form covariant derivatives, is also presented. In this formalism, ρ dominance is automatic, and the current-algebra result from the ρ-meson coupling constant arises from the independent assumption that ρ mesons couple universally to pions and other particles. Including ρ mesons in the Lagrangian has no effect on the π-π scattering lengths, because chiral invariance requires that we also include direct pion self-couplings which cancel the ρ-exchange diagrams for pion energies near threshold.

I. INTRODUCTION

CURRENT algebra is useful because it allows us to obtain physical predictions from chiral symmetry. We have recently noted[1] that for soft-pion processes the same predictions can also be derived by a different method: Just use the lowest-order graphs generated by any chiral-invariant Lagrangian. The Lagrangian method has since been applied to pion production,[2] η decay,[3] K interactions and decay,[4] and, in various extended versions, to meson mass ratios and decay amplitudes,[5] and to the pion electromagnetic mass difference.[6] Opinions differ[7] as to whether any fundamental significance resides in the Lagrangians that have been used, but there is no doubt that they provide both a convenient method of calculation and a valuable heuristic guide to theorems that can be proved with current algebra.

There are two ways of constructing our chiral-invariant Lagrangians, which mirror two different views of the meaning of chiral symmetry. The first, *conventional* method[8] is to construct $\mathcal{L}$ to be manifestly chiral-invariant, as if chirality were an ordinary linear symmetry like isospin. For example, in the σ model[9] the $\boldsymbol{\pi}$ and σ fields form a four-vector coupled to nucleons in the combination $\sigma+i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5$, and the nucleon mass arises from the nonvanishing vacuum expectation value $\langle\sigma\rangle_0=-m_N/G$. In a closely related model[10] the Lagrangian takes the same form, but with σ replaced everywhere with $[(m_N/G)^2-\boldsymbol{\pi}^2]^{1/2}$. Such models suffer from a fundamental disadvantage: They hide the fact that soft pions are emitted in clusters by derivative cou-

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1 S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

2 L.-N. Chang, Phys. Rev. **162**, 1497 (1967); Ph.D. thesis (unpublished).

3 W. A. Bardeen, L. S. Brown, B. W. Lee, and H. T. Nieh, Phys. Rev. Letters **18**, 1170 (1967). Precisely the same calculation was done by S. Shei, but not published, because it appeared that the matrix element was too small by a factor m_π^2/m_η^2 to account for the observed decay. Bardeen *et al.* treat the η-π vertex in what seems to me a dubious manner, and thereby escape this difficulty.

4 B. Zumino (to be published); S. Iwao (to be published).

5 J. Schwinger, Phys. Letters **24B**, 473 (1967); S. Weinberg, Phys. Rev. Letters **18**, 507 (1967) (see footnote 7); S. Glashow, H. Schnitzer, and S. Weinberg, Phys. Rev. Letters **19**, 139 (1967) [Eq. (13) ff]; M. Lévy (to be published); J. W. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967); S. Glashow and S. Weinberg (to be published). The decay amplitudes derived using Lagrangian methods by Schwinger [and then in a somewhat more general form by Wess and Zumino] were subsequently rederived using current algebra by H. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).

6 J. Schwinger (to be published); D. B. Fairlie and K. Yoshida (to be published). The corresponding current-algebra calculation was done by T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters **18**, 759 (1967).

7 In particular, Schwinger has argued that as long as the origin of symmetries remains obscure, the phenomenological Lagrangian provides a suitable arena for their study. [J. Schwinger, Phys. Rev. **152**, 1219 (1966); also Refs. 5 and 6, and private communication.] Others like myself remain uneasy at using a symmetry on the phenomenological level, when it is not clear how *any* fundamental Lagrangian could give rise to the supposed symmetry of phenomena. From this point of view, chirality is in good shape because we have current algebra to underwrite it, but $SU(6)$ remains obscure. Time will tell.

8 J. Schwinger, Ann. Phys. (N. Y.) **2**, 407 (1957); M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); F. Gürsey, *ibid.* **16**, 230 (1960); in *Proceedings of the 1960 Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, (1960), p. 572; Ann. Phys. **12**, 91 (1961); F. Gürsey and B. Zumino, (unpublished); P. Chang and F. Gürsey, Phys. Rev. **164**, 1752 (1967); H. S. Mani, Y. Tomozawa, and Y. P. Yao, Phys. Rev. Letters **18**, 1084 (1967); L. S. Brown, Phys. Rev. **163**, 1802 (1967); and J. A. Cronin, Phys. Rev. **161**, 1483 (1967); and Refs. 1-4.

9 J. Schwinger and M. Gell-Mann and M. Lévy, Ref. 8.

10 F. Gürsey and M. Gell-Mann and M. Lévy, Ref. 8.

plings from external lines.[11] For this reason, it proves necessary to perform a chiral field-dependent rotation which eliminates the nonderivative coupling of σ and $\boldsymbol{\pi}$ and replaces it with a nonlinear derivative coupling of the chiral rotation vector, identified as a new pion field.

In the second, *nonlinear* method,[12] one recognizes from the beginning that chirality is not like other symmetries, because it relates processes involving different numbers of soft pions. Therefore, the Lagrangian is constructed so that it is invariant under chiral transformations expressed, not in terms of isospin matrices and γ_5's, but in terms of isospin matrices and the pion field. In this way we work from the beginning with the nonlinear derivative couplings which emerged only at the end of the first method.

The conventional method has the advantage of expressing the Lagrangian in a manifestly renormalizable form.[9] The nonlinear method has the advantage of putting the Lagrangian in a useful form without our having to work our way through a chiral rotation. One other difference between the two methods is that the second yields *only* those results which can be obtained from current algebra, while the first mixes up these results with others which reflect our prejudices as to how hadrons are grouped into chiral multiplets. For instance, in the σ model one assumes that the nucleon is in a $(\frac{1}{2},0)+(0,\frac{1}{2})$ linear representation of $SU(2)\times SU(2)$, and one finds that the unrenormalized value of the weak-coupling-constant ratio g_A/g_V is unity; in the nonlinear method we never have to ask to what kind of chiral multiplet the nucleon belongs, and the unrenormalized g_A/g_V can be anything we like. It is not clear whether this should be counted as an advantage for the conventional or the nonlinear method.

This article will present a systematic development of the nonlinear approach to chiral invariance. In Sec. II we show that the most general possible nonlinear pion transformation rule is equivalent to

$$[X_a,\pi_b]=-i\lambda^{-1}[\tfrac{1}{2}(1-\lambda^2\boldsymbol{\pi}^2)\delta_{ab}+\lambda^2\pi_a\pi_b], \quad (1.1)$$

where X_a is the chiral generator (a, $b=1$, 2, 3) and λ is a constant. By "equivalent" we mean that any other possible transformation law can be converted to (1.1) by a suitable redefinition of the pion field.[13] In Sec. III we show that the corresponding transformation for a general field ψ is

$$[\mathbf{X},\psi]=\lambda(\mathbf{t}\times\boldsymbol{\pi})\psi, \quad (1.2)$$

where $\mathbf{t}$ is the isospin matrix for ψ. (We do not limit ourselves to the nucleon field here; ψ could be the field of a K meson, a baryon resonance, etc.) In Sec. IV we show that a chiral-invariant Lagrangian can be constructed simply by forming an arbitrary isospin-invariant Lagrangian out of the ψ, their "covariant derivatives,"

$$D_\mu\psi=\partial_\mu\psi+2i\lambda^2(1+\lambda^2\boldsymbol{\pi}^2)^{-1}\mathbf{t}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\psi; \quad (1.3)$$

and a pion covariant derivative,

$$D_\mu\boldsymbol{\pi}=(1+\lambda^2\boldsymbol{\pi}^2)^{-1}\partial_\mu\boldsymbol{\pi}. \quad (1.4)$$

By studying the axial-vector current, we find that

$$\lambda=F_{\pi 0}^{-1}, \quad (1.5)$$

where $F_{\pi 0}$ is the unrenormalized value of the pion decay amplitude. We show in Sec. V that the linearly transforming fields of the conventional approach can be constructed from the $\boldsymbol{\pi}$ and ψ. In Sec. VI we explain what differences may arise among current-algebra results for the pion-pion scattering lengths[14,15] by showing their value is entirely determined by the chiral transformation properties of the symmetry-breaking terms in $\mathcal{L}$ (see Table I). In Sec. VII we discuss the possible role of the ρ meson, and show that the covariant derivative (1.3) can be replaced with

$$\mathfrak{D}_\mu\psi=\partial_\mu\psi-ig_0\mathbf{t}\cdot\boldsymbol{\rho}_\mu\psi, \quad (1.6)$$

provided we use the Yang-Mills Lagrangian for $\boldsymbol{\rho}$, and

TABLE I. Values of the pion-pion scattering lengths (in pion Compton wavelengths) under various assumptions about the chiral transformation properties of the term in the Lagrangian which breaks chiral symmetry. The rows labelled with a value of N represent the cases where this term transforms according to the representation $(N/2,N/2)$, i.e., like a traceless symmetric tensor of rank N. The last two rows represent the possibility that the symmetry-breaking term is simply $-\frac{1}{2}m_\pi^2\boldsymbol{\pi}^2$, with $\boldsymbol{\pi}$ defined to transform according to Eqs. (2.18) or (2.21). (The first, second, and last rows present the results of Refs. 14, 18, and 12, respectively.)

Transformation of symmetry-breaking term	a_0	a_2	a_0/a_2
Four-vector ($N=1$)	0.20	-0.06	$-7/2$
Tensor ($N=2$)	0.35	0	∞
Tensor ($N=3$)	0.55	0.08	95/14
⋯	⋯	⋯	⋯
$\boldsymbol{\pi}^2$; see Eq. (2.18)	0.06	-0.11	$-1/2$
$\boldsymbol{\pi}^2$; see Eq. (2.21)	0.12	-0.09	$-3/2$

[11] S. Weinberg, Phys. Rev. Letters **16**, 879 (1966).

[12] See particularly, J. Schwinger, Ref. 5.

[13] It perhaps should be emphasized that we are free to use any pseudoscalar isovector object as the pion field; different choices give different matrix elements off the mass shell, but they all give the same S matrix.

[14] The pion-pion scattering lengths were calculated using current algebra by S. Weinberg [Phys. Rev. Letters **17**, 616 (1966)] under the assumption that the symmetry-breaking term in the Lagrangian is the 0 component of a chiral four-vector.

[15] We are not concerned here with the validity of the expansion technique used in Ref. 14. It was clear from the beginning that the low-energy π-π interaction might be so strong that current algebra simply could not be used. However, the consensus of those who have studied this problem is that the expansion technique used in Ref. 14 is at least self-consistent; i.e., it yields scattering lengths small enough so that the unitarity corrections are even smaller. See N. N. Khuri, Phys. Rev. **153**, 1477 (1967); F. J. Meiere, *ibid.* **159**, 1462 (1967); J. Sucher and C. H. Woo, Phys. Rev. Letters **18**, 723 (1967); K. Kang and J. Akiba, Phys. Rev. **164**, 1836 (1967). J. Iliopoulos (to be published). In addition, the successful application of current algebra to a wide variety of multipion processes (discussed at the end of Sec. VI) provides empirical evidence that the scattering lengths *are* small.

provided that for the ρ mass term we use

$$-\tfrac{1}{2}m_\rho^2[\varrho_\mu+2g_0^{-1}\lambda^2(1+\lambda^2\boldsymbol{\pi}^2)^{-1}(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})]^2. \quad (1.7)$$

As a consequence, the contribution of ρ exchange to π-π scattering is effectively cancelled near threshold by a direct $(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})^2$ interaction. Perhaps, by taking this cancellation into account, it will be possible to construct a model of π-π scattering which applies from threshold up to the ρ mass.

It would be interesting to see what kind of nonlinear realizations are possible for a general symmetry group.

II. TRANSFORMATION OF THE PION FIELD

We will first show that *the nonlinear transformation induced by chiral $SU(2)\times SU(2)$ on the pion field is unique*, up to possible redefinition of the fields.

The operators of chiral $SU(2)\times SU(2)$ will be denoted T_a, X_a, with $a=1, 2, 3$; they satisfy the familiar commutation relations

$$[T_a,T_b]=i\epsilon_{abc}T_c, \quad (2.1)$$

$$[T_a,X_b]=i\epsilon_{abc}X_c, \quad (2.2)$$

$$[X_a,X_b]=i\epsilon_{abc}T_c. \quad (2.3)$$

We will allow the transformation induced by X_a on the pion field π_b to be completely general in form, i.e.,

$$[X_a,\pi_b]=-if_{ab}(\boldsymbol{\pi}), \quad (2.4)$$

where $f_{ab}(\boldsymbol{\pi})$ is as yet an arbitrary function. On the other hand, isotopic spin is an ordinary symmetry, and its action on $\boldsymbol{\pi}$ will be required to take the usual form:

$$[T_a,\pi_b]=i\epsilon_{abc}\pi_c. \quad (2.5)$$

The restrictions imposed by the commutation relations (2.1)–(2.3) on the transformation function $f_{ab}(\boldsymbol{\pi})$ can be most easily determined by using the various Jacobi identities of $\boldsymbol{\pi}$ with pairs of generators. First we use

$$[T_a,[X_b,\pi_c]]\equiv[X_b,[T_a,\pi_c]]+[[T_a,X_b],\pi_c]. \quad (2.6)$$

With Eqs. (2.2)–(2.4), and (2.5), this gives

$$[T_a,f_{bc}(\boldsymbol{\pi})]=i\epsilon_{acd}f_{bd}(\boldsymbol{\pi})+i\epsilon_{abd}f_{dc}(\boldsymbol{\pi}),$$

or in other words,

$$\frac{\partial f_{bc}(\boldsymbol{\pi})}{\partial\pi_d}\epsilon_{ade}\pi_e=f_{bd}(\boldsymbol{\pi})\epsilon_{acd}+f_{dc}(\boldsymbol{\pi})\epsilon_{abd}. \quad (2.7)$$

This just says that $f_{bc}(\pi)$ is an isotopic tensor—not a surprising result.

The other useful Jacobi identity is for $\boldsymbol{\pi}$ with a pair of X's:

$$[X_a,[X_b,\pi_c]]-[X_b,[X_a,\pi_c]]=[[X_a,X_b],\pi_c]. \quad (2.8)$$

With Eqs. (2.3)–(2.5) this gives

$$[X_a,f_{bc}(\boldsymbol{\pi})]-[X_b,f_{ac}(\boldsymbol{\pi})]=-i\epsilon_{abd}\epsilon_{dce}\pi_e,$$

or, more explicitly,

$$\frac{\partial f_{bc}(\boldsymbol{\pi})}{\partial\pi_d}f_{ad}(\boldsymbol{\pi})-\frac{\partial f_{ac}(\boldsymbol{\pi})}{\partial\pi_d}f_{bd}(\boldsymbol{\pi})=\delta_{ac}\pi_b-\delta_{bc}\pi_a. \quad (2.9)$$

Now we solve Eqs. (2.7) and (2.9). As already remarked, Eq. (2.7) is merely the statement that $f_{bc}(\boldsymbol{\pi})$ is an isotopic tensor. Further, $f_{bc}(\boldsymbol{\pi})$ has even parity, so it must be even in $\boldsymbol{\pi}$, and hence takes the form

$$f_{bc}(\boldsymbol{\pi})=\delta_{bc}f(\boldsymbol{\pi}^2)+\pi_b\pi_c g(\boldsymbol{\pi}^2), \quad (2.10)$$

with f and g arbitrary functions of $\boldsymbol{\pi}^2$. Direct calculation gives

$$\frac{\partial f_{bc}(\boldsymbol{\pi})}{\partial\pi_d}f_{ad}(\boldsymbol{\pi})-\frac{\partial f_{ac}(\boldsymbol{\pi})}{\partial\pi_d}f_{bd}(\boldsymbol{\pi})=(fg-2ff'-2\pi^2gf')(\delta_{ac}\pi_b-\delta_{bc}\pi_a),$$

so our other differential equation, Eq. (2.9), imposes *one* further relation on f and g:

$$g(\boldsymbol{\pi}^2)=\frac{1+2f(\boldsymbol{\pi}^2)f'(\boldsymbol{\pi}^2)}{f(\boldsymbol{\pi}^2)-2\boldsymbol{\pi}^2f'(\boldsymbol{\pi}^2)}. \quad (2.11)$$

[A prime denotes differentiation with respect to the argument $\boldsymbol{\pi}^2$.] Our conclusion is that the most general pion transformation law is given by Eqs. (2.4) and (2.5), where $f_{bc}(\boldsymbol{\pi})$ has the form (2.10), with $g(\boldsymbol{\pi}^2)$ specified in terms of $f(\boldsymbol{\pi}^2)$ by Eq. (2.11).

We promised to show that the pion transformation law is essentially unique. Let us consider the effect on the transformation function $f_{ab}(\boldsymbol{\pi})$ of a redefinition $\boldsymbol{\pi}\to\boldsymbol{\pi}^*$ of the pion field.[13] Since the new pion field $\boldsymbol{\pi}^*$ must be an isovector satisfying Eq. (2.5), the most general redefinition is of the form

$$\pi_a^*=\pi_a\Phi(\boldsymbol{\pi}^2). \quad (2.12)$$

We can easily calculate that $\boldsymbol{\pi}^*$ has a chiral transformation law of the same form as $\boldsymbol{\pi}$; i.e.,

$$[X_a,\pi_b^*]=-i\{\delta_{ab}f^*(\boldsymbol{\pi}^{*2})+\pi_a^*\pi_b^*g^*(\boldsymbol{\pi}^{*2})\}, \quad (2.13)$$

where

$$f^*(\boldsymbol{\pi}^{*2})=f(\boldsymbol{\pi}^2)\Phi(\boldsymbol{\pi}^2) \quad (2.14)$$

and

$$g^*(\boldsymbol{\pi}^{*2})=[g(\boldsymbol{\pi}^2)\Phi(\boldsymbol{\pi}^2)+2f(\boldsymbol{\pi}^2)\Phi'(\boldsymbol{\pi}^2)+2\boldsymbol{\pi}^2g(\boldsymbol{\pi}^2)\Phi'(\boldsymbol{\pi}^2)]\Phi^{-2}(\boldsymbol{\pi}^2). \quad (2.15)$$

We see that *by redefining $\boldsymbol{\pi}$ we can make the function f^* anything we like*; Eq. (2.15) then determines the corresponding function g^* in such a way that Eq. (2.11) will still hold.

In particular, we may define the pion field $\boldsymbol{\pi}$ (now dropping superscripts) so that

$$f(\boldsymbol{\pi}^2)=(1/2\lambda)(1-\lambda^2\boldsymbol{\pi}^2), \quad (2.16)$$

with λ an arbitrary constant. Then Eq. (2.11) gives

$$g(\boldsymbol{\pi}^2)=\lambda\,, \tag{2.17}$$

and the pion transformation law is

$$[X_a,\pi_b]=-(i/\lambda)\{\tfrac{1}{2}(1-\lambda^2\boldsymbol{\pi}^2)\delta_{ab}+\lambda^2\pi_a\pi_b\}\,. \tag{2.18}$$

It is easy to show that this is precisely the transformation law (with $\lambda=G/2m_N$) satisfied by the "new pion field" $\boldsymbol{\pi}'\equiv\lambda\boldsymbol{\varepsilon}$ defined in our previous work,[1] and is also the rule adopted by Schwinger.[5] Alternatively, we could also define $\boldsymbol{\pi}$ so that

$$f(\boldsymbol{\pi}^2)=\lambda^{-1}\,, \tag{2.19}$$

in which case Eq. (2.11) would again give

$$g(\boldsymbol{\pi}^2)=\lambda\,, \tag{2.20}$$

and the pion transformation law would be

$$[X_a,\pi_b]=-i\lambda^{-1}\{\delta_{ab}+\lambda^2\pi_a\pi_b\}\,. \tag{2.21}$$

It is to be stressed that (2.18) and (2.21) do not represent inequivalent realizations of $SU(2)\times SU(2)$; these transformation laws, and all other possible pion transformation laws, can be derived from one another by suitable redefinitions of the pion field.

III. TRANSFORMATION OF OTHER FIELDS

Now that we have at hand a $\boldsymbol{\pi}$ field which transforms nonlinearly under chiral $SU(2)\times SU(2)$, what can we do with it? The lesson taught by current algebra is that a symmetry like chirality does not manifest itself in linear γ_5 invariance relations (which for instance would require the vanishing of the nucleon mass), but rather it determines relations between an arbitrary process $\alpha\to\beta$ and the related processes $\alpha\to\beta+n\pi$. The attractive feature of a nonlinear-field transformation law is that it allows us to build this interpretation of chiral symmetry into the Lagrangian from the beginning. Suppose an arbitrary field ψ has a chiral transformation law of the form

$$[X_a,\psi]=v_{ab}(\boldsymbol{\pi})t_b\psi\,, \tag{3.1}$$

where t_b is the Hermitian isospin matrix appropriate to ψ, i.e.,

$$[T_b,\psi]=-t_b\psi\,. \tag{3.2}$$

Then any isospin-invariant function of ψ (not its derivatives) will also be chiral-invariant; for instance $\psi^\dagger\psi$ commutes with T_a and hence with X_a, and chirality will *not* tell us that the mass of ψ vanishes. What it *does* tell us is explored in Sec. IV. In this section we will answer some necessary preliminary questions: Does there exist a function $v_{ab}(\boldsymbol{\pi})$ for which (3.1) and (3.2) are a self-consistent realization of $SU(2)\times SU(2)$? And if so, what is it?

The consistency requirements that must be satisfied by the transformation rules (3.1), (3.2) are embodied in the Jacobi identities analogous to Eqs. (2.6) and (2.8):

$$[T_a,[X_b,\psi]]\equiv[X_b,[T_a,\psi]]+[[T_a,X_b],\psi]\,,$$
$$[X_a,[X_b,\psi]]-[X_b,[X_a,\psi]]\equiv[[X_a,X_b],\psi]\,.$$

The first identity just tells us that $v_{ab}(\boldsymbol{\pi})$ is an isotopic tensor, i.e.,

$$\frac{\partial v_{bc}(\boldsymbol{\pi})}{\partial\pi_d}\epsilon_{ade}\pi_e=v_{bd}(\boldsymbol{\pi})\epsilon_{acd}+v_{dc}(\boldsymbol{\pi})\epsilon_{abd}\,. \tag{3.3}$$

The second identity gives

$$[X_a,v_{be}(\boldsymbol{\pi})]t_e-[X_b,v_{ae}(\boldsymbol{\pi})]t_e+v_{bd}(\boldsymbol{\pi})v_{ac}(\boldsymbol{\pi})[t_d,t_c]=-i\epsilon_{abe}t_e\,.$$

We use the pion transformation law (2.4) and the isospin commutation rule $[t_d,t_c]=i\epsilon_{dce}t_e$ to put this in the form of a differential equation for $v_{ab}(\boldsymbol{\pi})$:

$$\frac{\partial v_{be}(\boldsymbol{\pi})}{\partial\pi_d}f_{ad}(\boldsymbol{\pi})-\frac{\partial v_{ae}(\boldsymbol{\pi})}{\partial\pi_d}f_{bd}(\boldsymbol{\pi})=-v_{ac}(\boldsymbol{\pi})v_{bd}(\boldsymbol{\pi})\epsilon_{cde}+\epsilon_{abe}\,. \tag{3.4}$$

The function v_{bc} carries negative parity, so it must be odd in $\boldsymbol{\pi}$; with (3.3), this restricts its form to

$$v_{ab}(\boldsymbol{\pi})=\epsilon_{abc}\pi_c v(\boldsymbol{\pi}^2)\,. \tag{3.5}$$

Inserting (3.5) and (2.10) in (3.4), we obtain a nonlinear differential equation for $v(\boldsymbol{\pi}^2)$:

$$\pi_d[\epsilon_{bed}\pi_a-\epsilon_{aed}\pi_b]\times\{v(\boldsymbol{\pi}^2)g(\boldsymbol{\pi}^2)+2v'(\boldsymbol{\pi}^2)[f(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2g(\boldsymbol{\pi}^2)]\}+2\epsilon_{abe}v(\boldsymbol{\pi}^2)f(\boldsymbol{\pi}^2)=\pi_e\pi_d\epsilon_{bad}v^2(\boldsymbol{\pi}^2)+\epsilon_{abe}\,.$$

It is convenient to rewrite this using the identity

$$\boldsymbol{\pi}^2\epsilon_{abe}\equiv\epsilon_{abc}\pi_c\pi_e+\epsilon_{bec}\pi_c\pi_a+\epsilon_{eac}\pi_c\pi_b\,.$$

We then find

$$\pi_d(\epsilon_{bed}\pi_a-\epsilon_{aed}\pi_b)\times\{v(\boldsymbol{\pi}^2)g(\boldsymbol{\pi}^2)+2v'(\boldsymbol{\pi}^2)[f(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2g(\boldsymbol{\pi}^2)]-v^2(\boldsymbol{\pi}^2)\}+\epsilon_{abe}[-2v(\boldsymbol{\pi}^2)f(\boldsymbol{\pi}^2)-\boldsymbol{\pi}^2v^2(\boldsymbol{\pi}^2)+1]=0\,. \tag{3.6}$$

For this to be possible, both functions in brackets must vanish; i.e.,

$$vg+2v'(f+\boldsymbol{\pi}^2g)-v^2=0\,, \tag{3.7}$$

$$-2vf-\boldsymbol{\pi}^2v^2+1=0\,. \tag{3.8}$$

The solution of the second equation is

$$v(\boldsymbol{\pi}^2)=-\{f(\boldsymbol{\pi}^2)+[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{1/2}\}^{-1}\,. \tag{3.9}$$

We leave it to the reader's pertinacity to show that (3.7) is also satisfied when $v(\boldsymbol{\pi}^2)$ and $g(\boldsymbol{\pi}^2)$ are given by (3.9) and (2.11), respectively. Thus (3.1) is indeed a possible chiral transformation law, with $v_{ab}(\boldsymbol{\pi})$ given *uniquely* by

$$v_{ab}(\boldsymbol{\pi})=\epsilon_{abc}\pi_c\{f(\boldsymbol{\pi}^2)+[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{1/2}\}^{-1}\,, \tag{3.10}$$

where $f(\boldsymbol{\pi}^2)$ is the function appearing in the pion transformation equations.

It may be noted that a pion field defined to transform according to Eq. (2.18) will have $v(\boldsymbol{\pi}^2)$ equal to a constant λ, so here (3.1) takes the particularly simple form

$$[\mathbf{X},\psi]=\lambda(\mathbf{t}\times\boldsymbol{\pi})\psi. \tag{3.11}$$

For this reason, this particular choice of the pion transformation rule is somewhat more convenient than other, equivalent choices.

IV. COVARIANT DERIVATIVES

We are now going to see how it is possible to construct a chiral-invariant Lagrangian out of pion fields $\boldsymbol{\pi}$ and other fields ψ which transform according to the rules (2.4) and (3.1). As already remarked, there is no problem in coupling ψ's with each other as long as field derivatives or pion fields do not enter; any isoscalar function of the ψ's alone will be chiral-invariant. [This corresponds to the fact that chiral symmetry tells us nothing about baryon masses, baryon-baryon scattering, etc.] Our task is to learn what to do with the pion fields' derivatives.

First, the pions. It would be possible to treat $\boldsymbol{\pi}$ like any other field ψ if its transformation law had the same form, i.e., if

$$-if_{ab}(\boldsymbol{\pi})=v_{ac}(\boldsymbol{\pi})(t_c)_{bd}\pi_d.$$

But this is not true, so it is not possible to make chiral-invariant interactions by merely coupling $\boldsymbol{\pi}$'s with each other and with ψ's to find form isoscalars. However, we do not run into this difficulty with $\partial_\mu\boldsymbol{\pi}$. That is, we can define a covariant derivative

$$D_\mu\pi_a\equiv d_{ab}(\boldsymbol{\pi})\partial_\mu\pi_b, \tag{4.1}$$

which transforms like an ordinary ψ field; i.e.,

$$[X_b,D_\mu\pi_c]=-iv_{ab}(\boldsymbol{\pi})\epsilon_{bcd}D_\mu\pi_d, \tag{4.2}$$

$$[T_a,D_\mu\pi_c]=-i\epsilon_{acd}D_\mu\pi_d. \tag{4.3}$$

[We are using the appropriate isospin matrix for pions, $(t_b)_{ce}=-i\epsilon_{bce}$.] It is perfectly straightforward, though rather tedious, to show that the function $d_{ab}(\boldsymbol{\pi})$ for which (4.2) and (4.3) are satisfied is uniquely given (up to a multiplicative constant) by

$$d_{ab}(\boldsymbol{\pi})\propto[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{-1/2}\delta_{ab} +[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{-1}[2f'(\boldsymbol{\pi}^2)-v(\boldsymbol{\pi}^2)]\pi_a\pi_b, \tag{4.4}$$

where $f(\boldsymbol{\pi}^2)$ is the function appearing in the pion transformation equations and $v(\boldsymbol{\pi}^2)$ is the function appearing in the transformation law of other fields;

$$v(\boldsymbol{\pi}^2)=\{f(\boldsymbol{\pi}^2)+[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{1/2}\}^{-1}.$$

The covariant derivative is therefore

$$D_\mu\boldsymbol{\pi}\propto[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{-1/2}\partial_\mu\boldsymbol{\pi} +[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{-1}[f'(\boldsymbol{\pi}^2)-\tfrac{1}{2}v(\boldsymbol{\pi}^2)]\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi}^2. \tag{4.5}$$

By virtue of Eqs. (4.2) and (4.3), any isoscalar function of $D_\mu\boldsymbol{\pi}$ and ψ's will automatically be chiral-invariant. In particular, chirality allows a gradient-coupling pion-nucleon interaction proportional to

$$\bar{N}i\gamma_5\gamma_\mu\boldsymbol{\tau}ND_\mu\boldsymbol{\pi}. \tag{4.6}$$

Also, the free-pion part of the Lagrangian is contained in the self-coupling

$$-\tfrac{1}{2}D_\mu\boldsymbol{\pi}\cdot D^\mu\boldsymbol{\pi}. \tag{4.7}$$

For a pion field defined to transform according to Eq. (2.18) the covariant pion derivative (4.1) is given by

$$D_\mu\boldsymbol{\pi}=(1+\lambda^2\boldsymbol{\pi}^2)^{-1}\partial_\mu\boldsymbol{\pi}, \tag{4.8}$$

so the pion-nucleon interaction is

$$(G_0/2m_N)\bar{N}i\gamma_5\gamma_\mu\boldsymbol{\tau}N(1+\lambda^2\boldsymbol{\pi}^2)^{-1}\partial_\mu\boldsymbol{\pi}, \tag{4.9}$$

and the pion kinematic Lagrangian is contained in

$$-\tfrac{1}{2}(1+\lambda^2\boldsymbol{\pi}^2)^{-2}\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi}. \tag{4.10}$$

These agree precisely with our previous results,[1] obtained by regarding chirality as a broken linear symmetry. We note once again that chirality forces the interaction of nucleons with single pions to be accompanied by interactions with three pions, five pions, etc., and correspondingly that the kinematic pion term must be accompanied with self-interaction terms representing pion-pion scattering, $2\pi\rightarrow4\pi$, etc.

We will now pause to consider what value should be given to the constant λ. (Had we reserved our full freedom to redefine the pion field, λ would be arbitrary, but in choosing the pion kinematic term to be given by Eq. (4.10) we have committed ourselves to a particular conventional normalization of π, and the value of λ thus has a meaning.) The axial-vector current defined by Noether's theorem is

$$A_a{}^\mu\equiv-2\frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi_b)}f_{ab}(\boldsymbol{\pi})-2i\sum_\psi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}v_{ab}(\boldsymbol{\pi})t_b\psi. \tag{4.11}$$

Referring back to (4.7) and (2.16), we see that this contains a term $\partial^\mu\pi_a$ with coefficient λ^{-1}. Hence, if $A_a{}^\mu$ is to be identified with the axial-vector current of weak interactions, we must take

$$\lambda=F_{\pi0}{}^{-1}, \tag{4.12}$$

where $F_{\pi0}$ is the unrenormalized value of the usual pion decay amplitude. Note also that if the Lagrangian contains a pion-nucleon interaction (4.9), then $A_a{}^\mu$ will contain a term

$$-(G_0/2m_N\lambda)\bar{N}i\gamma_5\gamma^\mu\tau_aN,$$

so the unrenormalized weak coupling constants satisfy the Goldberger-Treiman relation:

$$(g_A/g_V)_0=G_0/2m_N\lambda=G_0F_{\pi0}/2m_N. \tag{4.13}$$

It is noteworthy that nothing in this formalism forces the unrenormalized ratio $(g_A/g_V)_0$ to be unity.

There remains the problem of incorporating derivatives of the ψ's into a chiral-invariant Lagrangian. We note that $\partial_\mu\psi$ does not transform like ψ; i.e.,

$$[X_a,\partial_\mu\psi]=v_{ab}(\boldsymbol{\pi})t_b\partial_\mu\psi+\frac{\partial v_{ab}(\boldsymbol{\pi})}{\partial\pi_c}t_b\partial_\mu\pi_c\psi. \quad (4.14)$$

Therefore, we will try to construct a covariant derivative

$$D_\mu\psi\equiv\partial_\mu\psi+iM_c(\boldsymbol{\pi})(\partial_\mu\pi_c)\psi, \quad (4.15)$$

such that

$$[X_a,D_\mu\psi]=v_{ab}(\boldsymbol{\pi})t_bD_\mu\psi. \quad (4.16)$$

Here $M_c(\boldsymbol{\pi})$ is a matrix like t_c which acts on the suppressed isospin indices of ψ. Our problem is to find an $M_c(\boldsymbol{\pi})$ for which (4.16) is satisfied; once this is accomplished, we can build a chiral-invariant Lagrangian by coupling $D_\mu\psi$ with ψ and $D_\mu\boldsymbol{\pi}$ in any isospin-invariant way.

Comparing (4.16) with (4.14), we see that the condition to be satisfied by $M_c(\pi)$ is

$$iv_{ab}(\boldsymbol{\pi})[t_b,M_c(\boldsymbol{\pi})] =\frac{\partial v_{ab}(\boldsymbol{\pi})}{\partial\pi_c}t_b+\frac{\partial M_c(\boldsymbol{\pi})}{\partial\pi_d}f_{ad}(\boldsymbol{\pi})+M_d(\boldsymbol{\pi})\frac{\partial f_{ad}(\boldsymbol{\pi})}{\partial\pi_c}. \quad (4.17)$$

One particular solution of Eq. (4.17) is

$$M_c(\boldsymbol{\pi})=\epsilon_{cde}t_d\pi_e[f^2(\boldsymbol{\pi}^2)-\boldsymbol{\pi}^2]^{-1/2}v(\boldsymbol{\pi}^2), \quad (4.18)$$

where $f(\boldsymbol{\pi}^2)$ is the function appearing in the pion transformation law; and as before,

$$v(\boldsymbol{\pi}^2)\equiv[f(\boldsymbol{\pi}^2)+(f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2)^{1/2}]^{-1}.$$

To (4.18) we can add any solution $M_c^{(0)}$ of the homogeneous equation

$$iv_{ab}[t_b,M_c^{(0)}]=\frac{\partial M_c^{(0)}}{\partial\pi_d}f_{ad}+M_d^{(0)}\frac{\partial f_{ad}}{\partial\pi_c};$$

but this just corresponds to introducing the covariant derivative $D_\mu\boldsymbol{\pi}$ of the pion field into interactions, and need not be considered as a separate possibility. Using (4.18) in (4.15), the covariant derivative of a general field ψ is

$$D_\mu\psi=\partial_\mu\psi+iv(\boldsymbol{\pi}^2)[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{-1/2}\mathbf{t}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\psi. \quad (4.19)$$

For a pion field defined to transform according to Eq. (2.18) this is

$$D_\mu\psi=\partial_\mu\psi+2i\lambda^2(1+\lambda^2\boldsymbol{\pi}^2)^{-1}\mathbf{t}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\psi. \quad (4.20)$$

The kinematic Lagrangian for ψ must be constructed out of $D_\mu\psi$. Using (4.20) with ψ for a Dirac field of arbitrary isospin, this gives the terms

$$-\bar\psi\gamma^\mu\partial_\mu\psi-2i\lambda^2(1+\lambda^2\boldsymbol{\pi}^2)^{-1}\ \gamma^\mu\mathbf{t}\psi\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}). \quad (4.21)$$

We recognize here the multipion interactions derived earlier from the broken-symmetry approach,[1] and in particular we observe that the term $-2i\lambda^2\bar\psi\gamma^\mu\mathbf{t}\psi\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})$, with λ given by (4.13), yields the universal pion scattering lengths[16] derived originally from current algebra.

To summarize what we have learned: a Lagrangian will be chiral-invariant if it is isospin-invariant and constructed out of the pion covariant derivative $D_\mu\boldsymbol{\pi}$, any general fields ψ [transforming according to Eq. (3.1)], and their covariant derivatives $D_\mu\psi$.

V. CONVENTIONAL FIELDS

We were originally led to introduce fields with nonlinear transformation rules as a substitute for larger chiral multiplets which transform linearly. We will now show how this process may be reversed. That is, we will use the nonlinearly transforming fields $\boldsymbol{\pi}$, ψ discussed in the last two sections to construct fields, for pions and for other particles of arbitrary isospin, with conventional linear-transformation properties.

Our construction is based on the following:

Lemma: Let x_a, t_a be an arbitrary $N\times N$ matrix representation of the $SU(2)\times SU(2)$ algebra; i.e.,

$$[t_a,t_b]=i\epsilon_{abc}t_c, \quad (5.1)$$

$$[t_a,x_b]=i\epsilon_{abc}x_c, \quad (5.2)$$

$$[x_a,x_b]=i\epsilon_{abc}t_c. \quad (5.3)$$

Then there exists an $N\times N$ matrix function $\Lambda(\pi)$ such that

$$[X_a,\Lambda(\boldsymbol{\pi})]=-x_a\Lambda(\boldsymbol{\pi})-\Lambda(\boldsymbol{\pi})v_{ab}(\boldsymbol{\pi})t_b, \quad (5.4)$$

where $v_{ab}(\boldsymbol{\pi})$ is the function (3.10). (We remind the reader that X_a is not a matrix, but a Hilbert-space operator which does not commute with $\boldsymbol{\pi}$.)

Proof: Using the pion transformation rule (2.4) let us write (5.4) as a Lie differential equation

$$-if_{ab}(\boldsymbol{\pi})\frac{\partial\Lambda(\boldsymbol{\pi})}{\partial\pi_b}=-x_a\Lambda(\boldsymbol{\pi})-\Lambda(\boldsymbol{\pi})v_{ab}(\boldsymbol{\pi})t_b. \quad (5.5)$$

This is soluble if and only if it satisfies an integrability condition:

$$f_{cd}\frac{\partial}{\partial\pi_d}\left(f_{ab}\frac{\partial\Lambda}{\partial\pi_b}\right)-f_{ab}\frac{\partial}{\partial\pi_b}\left(f_{cd}\frac{\partial\Lambda}{\partial\pi_d}\right) =\left(f_{cd}\frac{\partial f_{ae}}{\partial\pi_d}-f_{ab}\frac{\partial f_{ce}}{\partial\pi_b}\right)\frac{\partial\Lambda}{\partial\pi_e}. \quad (5.6)$$

It is easy to show that (5.5) does satisfy (5.6) because f_{ab} and v_{ab} satisfy (2.9) and (3.4).

[16] Y. Tomozawa, Nuovo Cimento **46A**, 707 (1966); S. Weinberg, Ref. 14; B. Hamprecht (to be published); K. Raman and E. C. G. Sudarshan, Phys. Rev. **154**, 1499 (1967); A. P. Balachadran, M. Gundzik, and F. Nicodemi, Nuovo Cimento **44**, 1257 (1966).

Now, let us use $\Lambda(\boldsymbol{\pi})$ to construct a conventional chiral quadruplet Π_α of fields representing the pion and a 0^+ meson. Here α is an index running over the values 1, 2, 3, 0, and for t_a and x_a we use the $(\frac{1}{2},\frac{1}{2})$ representation:

$$(t_a)_{bc}=-i\epsilon_{abc}\,, \tag{5.7}$$

$$(t_a)_{b0}=(t_a)_{0b}=(t_a)_{00}=0\,, \tag{5.8}$$

$$(x_a)_{b0}=-(x_a)_{0b}=i\delta_{ab}\,, \tag{5.9}$$

$$(x_a)_{bc}=(x_a)_{00}=0\,, \tag{5.10}$$

the indices a, b, c running over 1, 2, 3. We can define a unit four-vector n_α which points in the 0 direction, and construct the four-vector Π_α as

$$\Pi_\alpha=\Lambda_{\alpha\beta}(\boldsymbol{\pi})n_\beta\sigma\,, \tag{5.11}$$

where σ is either a constant or a 0^+ chiral-invariant field. [There is no distinction if the mass of the σ field is sufficiently large.[1] If σ is a constant, then (5.11) really defines only three independent fields Π_a, the fourth being given[10] by $\Pi_0=(\sigma^2-\boldsymbol{\Pi}^2)^{1/2}$. If σ is a field,[9] then (5.11) defines four independent fields Π_α in terms of the four fields $\boldsymbol{\pi}$, σ.] The chiral transformation law for Π_α can be determined from (5.4). Since n_β is annihilated by $(t_a)_{\alpha\beta}$, the last term in (5.4) does not contribute here, and we find that

$$[X_a,\Pi_\alpha]=-(x_a)_{\alpha\beta}\Pi_\beta\,, \tag{5.12}$$

so Π_α is indeed a chiral four-vector. This is hardly surprising, for we already knew that a linearly transforming four-vector Π_α can be used to define a nonlinearly transforming triplet $\boldsymbol{\pi}$, and we have shown in Sec. II that such a pion triplet is essentially unique.

Next, consider a field ψ which transforms according to the nonlinear rule (3.1):

$$[X_a,\psi]=v_{ab}(\boldsymbol{\pi})t_b\psi\,.$$

In close analogy with (5.11), define a conventional chiral multiplet Ψ as

$$\Psi=\Lambda(\boldsymbol{\pi})\psi\,. \tag{5.13}$$

From (3.1) and (5.4) we find immediately that

$$[X_a,\Psi]=-x_a\Psi\,, \tag{5.14}$$

so Ψ indeed transforms linearly under chiral $SU(2)\times SU(2)$.

The reader can easily verify that the conventional chiral-invariant Lagrangians constructed from Π_α, $\partial_\mu\Pi_\alpha$, Ψ, and $\partial_\mu\Psi$ may always be re-expressed as functions of $D_\mu\pi$, ψ, and $D_\mu\psi$.

VI. SYMMETRY BREAKING

We will now return to the nonlinear formalism sketched in Secs. II–IV, and use it to discuss the problem of chiral symmetry breaking by the pion mass. There is no doubt that the pion mass does break chirality, for we have seen in Sec. IV that pion fields can only enter the Lagrangian accompanied with at least one derivative. The question is, when we add a term $-\frac{1}{2}m_\pi{}^2\boldsymbol{\pi}^2$ to the Lagrangian, should we stop there or should we also add terms proportional to $m_\pi{}^2(\boldsymbol{\pi}^2)^2$, $m_\pi{}^2(\boldsymbol{\pi}^2)^3$, etc.? We must beware of rejecting such terms on grounds of simplicity alone, for any such hypothesis has meaning only for a particular definition of the pion field. That is, if the only term in the Lagrangian which breaks $SU(2)\times SU(2)$ is the pion-mass term $-\frac{1}{2}m_\pi{}^2\boldsymbol{\pi}^2$, and we define a new pion field $\boldsymbol{\pi}^*$ so that $\boldsymbol{\pi}=\boldsymbol{\pi}^*(1+\alpha\boldsymbol{\pi}^{*2}+\cdots)$, then the symmetry-breaking term will be expressed in terms of $\boldsymbol{\pi}^*$ as

$$-\tfrac{1}{2}m_\pi{}^2\boldsymbol{\pi}^{*2}(1+2\alpha\boldsymbol{\pi}^{*2}+\cdots)\,,$$

and the π-π scattering lengths will be different from what they would be if the symmetry-breaking term were $-\frac{1}{2}m_\pi{}^2\boldsymbol{\pi}^{*2}$. For this reason, chiral symmetry alone can only predict one linear combination $2a_0-5a_2$ of the scattering lengths; the ratio a_0/a_2 cannot be determined without a more definite statement of how chirality is broken.[15]

It is natural to characterize the chiral-symmetry-breaking term in the Lagrangian in terms of its chiral transformation properties; in this way we at least avoid making hypotheses which depend on how we define the pion field. Suppose that the symmetry-breaking term is a function $\mathcal{L}_N(\boldsymbol{\pi}^2)$ which transforms according to the $(N/2,\,N/2)$ representation of $SU(2)\times SU(2)$, or equivalently, using the isomorphism of $SU(2)\times SU(2)$ with $O(4)$, suppose that

$$\mathcal{L}_N=t_{00\cdots0}{}^{(N)}\,, \tag{6.1}$$

where $t_{\alpha\beta\cdots\gamma}{}^{(N)}$ is a traceless symmetric tensor of rank N. The ordinary rules of tensor analysis then give

$$[X_a,\mathcal{L}_N]=-iNt_{a0\cdots0}{}^{(N)}$$

and

$$[X_b,[X_a,\mathcal{L}_N]]=-iN\{-i(N-1)t_{ab0\cdots0}{}^{(N)}+i\delta_{ab}t_{00\cdots0}{}^{(N)}\}\,.$$

But since $t^{(N)}$ is traceless, we have

$$t_{aa0\cdots0}{}^{(N)}=-t_{00\cdots0}{}^{(N)}\,,$$

and so

$$[X_a,[X_a,\mathcal{L}_N]]=N(N+2)\mathcal{L}_N\,. \tag{6.2}$$

We will use (6.2) to distinguish the different possible $\mathcal{L}_N$.

Let us now construct an $\mathcal{L}_N$ which transforms according to Eq. (6.1), i.e., which satisfies (6.2). For convenience, we will adopt a pion field which is defined to transform according to Eq. (2.18); i.e.,

$$[X_a,\pi_b]=-i\lambda^{-1}\{\tfrac{1}{2}(1-\lambda^2\boldsymbol{\pi}^2)\delta_{ab}+\lambda^2\pi_a\pi_b\}\,.$$

We then have

$$[X_a,\boldsymbol{\pi}^2]=-i\lambda^{-1}(1+\lambda^2\boldsymbol{\pi}^2)\pi_a\,,$$

so for an arbitrary function of $\boldsymbol{\pi}^2$

$$[X_a,\mathfrak{F}(\boldsymbol{\pi}^2)]=-i\lambda^{-1}(1+\lambda^2\boldsymbol{\pi}^2)\pi_a\mathfrak{F}'(\boldsymbol{\pi}^2),$$

$$[X_a,[X_a,\mathfrak{F}(\boldsymbol{\pi}^2)]]=-\tfrac{1}{2}\lambda^{-2}(1+\lambda^2\boldsymbol{\pi}^2)(3+\lambda^2\boldsymbol{\pi}^2)\mathfrak{F}'(\boldsymbol{\pi}^2)-\lambda^{-2}(1+\lambda^2\boldsymbol{\pi}^2)^2\boldsymbol{\pi}^2\mathfrak{F}''(\boldsymbol{\pi}^2).$$

Thus Eq. (6.2) just amounts to a second-order differential equation for $\mathfrak{L}(\boldsymbol{\pi}^2)$:

$$(1+\lambda^2\boldsymbol{\pi}^2)^2\boldsymbol{\pi}^2\mathfrak{L}_N''(\boldsymbol{\pi}^2)+\tfrac{1}{2}(1+\lambda^2\boldsymbol{\pi}^2)(3+\lambda^2\boldsymbol{\pi}^2)\mathfrak{L}_N'(\boldsymbol{\pi}^2)+N(N+2)\lambda^2\mathfrak{L}_N(\boldsymbol{\pi}^2)=0. \quad (6.3)$$

Since this differential equation is singular at $\boldsymbol{\pi}^2=0$, its regular solution is unique up to an over-all constant, which we can fix by requiring that the term linear in $\boldsymbol{\pi}^2$ have coefficient $-\frac{1}{2}m_\pi^2$. The solution may then be expressed as a power series in $\lambda^2\boldsymbol{\pi}^2$:

$$\mathfrak{L}_N(\boldsymbol{\pi}^2)=-\tfrac{1}{2}m_\pi^2\{-3[2N(N+2)\lambda^2]^{-1}+\boldsymbol{\pi}^2-\tfrac{1}{5}[N(N+2)+2]\lambda^2(\boldsymbol{\pi}^2)^2+(1/105)[2N^2(N+2)^2+20N(N+2)+27]\times\lambda^4(\boldsymbol{\pi}^2)^3+\cdots\}. \quad (6.4)$$

The constant term is of course without physical significance. The quartic term contributes to the π-π scattering lengths; in conjunction with the kinematic term (4.10), it gives

$$2a_0+a_2=\tfrac{3}{5}L[N(N+2)+2], \quad (6.5)$$

$$2a_0-5a_2=6L, \quad (6.6)$$

where

$$L\equiv\frac{m_\pi\lambda^2}{2\pi}\simeq\frac{G^2m_\pi}{8\pi m_N^2}\left(\frac{g_V}{g_A}\right)^2\simeq 0.115\; m_\pi^{-1}.$$

Higher terms in Eq. (6.4) contribute to more complicated processes, like $2\pi\to 4\pi$, etc. In the previously considered[14] case, $N=1$, where $\mathfrak{L}_N$ and $\partial_\mu A_a{}^\mu$ form a chiral four-vector, it is possible to sum up all these terms by finding a solution of Eq. (6.3) in closed form:

$$\mathfrak{L}_1(\boldsymbol{\pi}^2)=-\tfrac{1}{2}m_\pi^2(1+\lambda^2\boldsymbol{\pi}^2)^{-1}\boldsymbol{\pi}^2+\text{const.}, \quad (6.7)$$

in agreement (what else?) with our previous results.[1]

It is easy to derive the same scattering lengths by the methods of current algebra. Apart from the commutators of current components, what is needed in general in current-algebra calculations[17] is a knowledge of the "σ terms"; e.g.,

$$\sigma_{ab}\equiv[X_a,\partial_\mu A_b{}^\mu],$$

$$\sigma_{abc}\equiv[X_a,\sigma_{bc}],\cdots. \quad (6.8)$$

Since the divergence of the axial current is given by

$$\partial_\mu A_c{}^\mu=-i[X_c,\mathfrak{L}_N],$$

we learn from Eq. (6.2) that

$$\sigma_{abb}=N(N+2)\partial_\mu A_a{}^\mu. \quad (6.9)$$

Also, applying the Jacobi identity and the chiral commutation relations to Eq. (6.8), we find

$$\sigma_{abc}-\sigma_{bac}=\delta_{bc}\partial_\mu A_a{}^\mu-\delta_{ac}\partial_\mu A_b{}^\mu. \quad (6.10)$$

Equations (6.9) and (6.10) provide enough information to compute the π-π scattering matrix with *three* pions off the mass shell.

The numerical values of a_0 and a_2 obtained from Eqs. (6.5) and (6.6) are presented in Table I. The values for $N=1$ are those derived previously by myself,[14] while those for $N=2$ were derived by Meire and Sugawara.[18] The last two rows give the values obtained by assuming that the symmetry-breaking term in $\mathfrak{L}$ is just $-\frac{1}{2}m_\pi^2\boldsymbol{\pi}^2$, with $\boldsymbol{\pi}$ defined to transform according to Eqs. (2.18) or (2.21), respectively. The last row corresponds to Schwinger's model.[5] (Schwinger uses a pion field that transforms according to Eq. (2.18), but gets $a_0/a_2=-\frac{3}{2}$ because he assumes that the pion field *so defined* is proportional to the divergence of the axial-vector current.

Evidently chiral symmetry alone does not suffice to fix a_0 and a_2 separately, without a specific hypothesis as to how the symmetry is broken. Since some such hypothesis is needed, it seems most reasonable to assume that chirality is broken by a term in the Lagrangian whose chiral transformation properties are as simple as possible[19] (rather than by a term which looks like a simple function of some arbitrary defined field). This leads to the choice $N=1$; the symmetry-breaking term $\mathfrak{L}_1$ then forms a chiral four-vector with $\partial_\mu A_a{}^\mu$, and the scattering lengths have the previously derived values $a_0=0.20$, $a_2=-0.06$.

Only experiment can decide whether this is right. Unfortunately, the π-π scattering lengths are not so easy to measure. The high-energy–pion-production experiments which purport to measure π-π scattering can really only do so if the peripheral graph dominates over the other "absorption" graphs. This is true at small momentum transfer if the two pions are produced at a relative velocity at which they can interact strongly (as in ρ production), but if the π-π scattering lengths are as small as we think they are then peripheral production does *not* dominate when the pions are produced at low relative velocities. For similar reasons,

[17] See, e.g., H. Abarbanel and S. Nussinov, Ann. Phys. (N. Y.) **42**, 467 (1967); also Refs. 2 and 14. It has been suggested by L. S. Kissingler [Phys. Rev. Letters **18**, 861 (1967)] that σ terms might be responsible for the $I=2$ amplitude in $K_{2\pi}$ decay. However, the usual current-algebra calculation neglects "gradient coupling" terms which are probably just as large as the σ terms, and can easily account for the observed rate of $K^+\to\pi^++\pi^0$. In fact, the real puzzle is why $\Delta I=\frac{1}{2}$ works so *well* in the nonleptonic K decays.

[18] F. T. Meiere and M. Sugawara, Phys. Rev. **153**, 1702 (1967). They did not make any explicit assumption about σ terms or the transformation of symmetry-breaking terms in the Lagrangian, but instead arbitrarily set $a_2=0$.

[19] This assumption is analogous to the assumption that $SU(3)$ is broken by an octet term in the Lagrangian, made by M. Gell-Mann [California Institute of Technology Synchrotron Laboratory Report CTSL-20, 1961 (unpublished)], and by S. Okubo [Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962)].

experiments on η decay and τ decay can verify that the π-π scattering lengths are small; but they cannot provide numerical values. The only hope appears to lie in a comparison of data on K_{e4} decay with the Watson-theorem calculations of Cabibbo and Maksymovich,[20] or in a comparison of data on $\pi+N\rightarrow 2\pi+N$ at *low* energy with the current-algebra calculation of Chang.[2] In both cases a tremendous improvement in statistics will be needed before accurate values of the π-π scattering lengths can be obtained. However, the success of current algebra[2,21] in accounting for the existing data on these processes, as well as τ decay[22] and π-N scattering,[16] provides ample evidence that the scattering lengths are small.

VII. ϱ MESONS

Our introduction of covariant differentiation in Sec. IV was reminiscent of the Yang-Mills theory of gauge fields.[23] We can make the analogy even closer by introducing ρ mesons to take the place of the direct vector interaction of pion pairs. Consider a general field ψ whose chiral transformation properties are given by Eq. (3.1):

$$[X_a,\psi]=v_{ab}(\boldsymbol{\pi})t_b\psi.$$

We may define a covariant derivative

$$\mathfrak{D}_\mu\psi\equiv\partial_\mu\psi-ig_0\mathbf{t}\cdot\boldsymbol{\varrho}_\mu\psi\,, \tag{7.1}$$

and require that $\mathfrak{D}_\mu\psi$ transform like ψ; i.e.,

$$[X_a,\mathfrak{D}_\mu\psi]=v_{ab}(\boldsymbol{\pi})t_b\mathfrak{D}_\mu\psi. \tag{7.2}$$

This imposes on $\boldsymbol{\varrho}$ the transformation law

$$[X_a,\rho_{b\mu}]=-iv_{ad}(\boldsymbol{\pi})\epsilon_{dbc}\rho_{c\mu}-ig_0^{-1}\partial_\mu v_{ab}(\boldsymbol{\pi}). \tag{7.3}$$

The last term looks like what we would expect from a gauge transformation of the second kind, and it has similar consequences. In particular, the covariant derivative $\mathfrak{D}_\nu\rho_{b\mu}$ does *not* transform like ψ or $\mathfrak{D}_\mu\psi$, but we can nevertheless define a covariant curl:

$$\mathfrak{F}_{b\mu\nu}\equiv\partial_\mu\rho_{b\nu}-\partial_\nu\rho_{b\mu}+g_0\epsilon_{bcd}\rho_{c\mu}\rho_{d\nu}\,, \tag{7.4}$$

such that

$$[X_a,\mathfrak{F}_{b\mu\nu}]=-iv_{ad}(\pi)\epsilon_{dbc}\mathfrak{F}_{c\mu\nu}. \tag{7.5}$$

The kinematic ρ-meson part of the Lagrangian is thus contained in

$$-\tfrac{1}{4}\boldsymbol{\mathfrak{F}}_{\mu\nu}\cdot\boldsymbol{\mathfrak{F}}^{\mu\nu}, \tag{7.6}$$

just as in the Yang-Mills theory.

[20] N. Cabibbo and A. Maksymovich, Phys. Rev. **137**, B438 (1965).

[21] S. Weinberg, Phys. Rev. Letters **17**, 336 (1966).

[22] M. Suzuki, Phys. Rev. **144**, 1154 (1966); Y. Hara and Y. Nambu, Phys. Rev. Letters **16**, 865 (1966); D. K. Elias and J. C. Taylor, Nuovo Cimento **44**, 518 (1966); S. K. Bose and S. N. Biswas, Phys. Rev. Letters **16**, 340 (1966); B. M. K. Nefkens, Phys. Letters **22**, 94 (1966); H. D. I. Abarbanel, Phys. Rev. **153**, 1547 (1967); C. Bouchiat and P. Meyer, Phys. Letters **22**, 198 (1966); L. J. Clavelli, Phys. Rev. **160**, 1384 (1967).

[23] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

Of course, we do not want to follow the gauge theory so far that we leave out the ρ-meson mass. The mass term can be introduced without violating chiral invariance if we note that the transformation (7.3) allows the construction of a field

$$\phi_{b\mu}\equiv\rho_{b\mu}+g_0^{-1}v(\boldsymbol{\pi}^2)[f^2(\boldsymbol{\pi}^2)+\boldsymbol{\pi}^2]^{-1/2}\epsilon_{abc}\pi_c\partial_\mu\pi_a \tag{7.7}$$

which transforms like the ψ's; i.e.,

$$[X_a,\phi_{b\mu}]=-iv_{ad}(\boldsymbol{\pi})\epsilon_{dbc}\phi_{c\mu}. \tag{7.8}$$

[Note that $\phi_{b\mu}$ has to transform this way because the difference between $\mathfrak{D}_\mu\psi$ and $D_\mu\psi$ is just $-ig\mathbf{t}\cdot\boldsymbol{\phi}_\mu\psi$.] The mass term then is

$$-\tfrac{1}{2}m_\rho^2\boldsymbol{\phi}_\mu^2. \tag{7.9}$$

If we define the pion field so that it transforms according to Eq. (2.18), then this is[24]

$$-\tfrac{1}{2}m_\rho^2[\boldsymbol{\varrho}_\mu+2g_0^{-1}\lambda^2(1+\lambda^2\boldsymbol{\pi}^2)^{-1}(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})]^2. \tag{7.10}$$

The complete Lagrangian then may be supposed to consist of (7.6) plus (7.9), or (7.10) plus the pion terms (4.7), or (4.10) and (6.4), or (6.7) plus an unknown function of ψ, $\mathfrak{D}_\mu\psi$, and $D_\mu\boldsymbol{\pi}$.

The $\rho\pi\pi$ coupling (for *soft* pions, not necessarily in ρ decay) is given by (7.10) as

$$-2m_\rho^2g_\rho^{-1}\lambda^2\boldsymbol{\varrho}_\mu\cdot(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi}). \tag{7.11}$$

Thus ρ exchange automatically accounts for the values of the πN scattering lengths given current algebra.[16] It may be surprising that the $\rho\pi\pi$ coupling constant does not automatically come out equal to the $\rho\bar{N}N$ coupling constant g_0; but it should be kept in mind that $\boldsymbol{\varrho}_\mu$ is not coupled to $\boldsymbol{\pi}$ like an ordinary gauge field. If we require as a separate condition that the ρ couples universally to pions and nucleons,[25] then we obtain for g_0 the familiar value[26]

$$g_0^2=2m_\rho^2\lambda^2=2m_\rho^2F_{\pi0}^{-2}.$$

We can also now answer a question that has raised some doubts[27] about the validity of the current-algebra calculation[14] of the π-π scattering length: How is it that we are able to get an answer without any explicit reference to the contribution of ρ exchange? We see from (7.11) that at low energy, where the ρ propagator

[24] Observe that if we let $m_0\rightarrow 0$ the pion decouples from the ρ meson, and hence from all other hadrons. This corresponds to the remark of Higgs, that Goldstone bosons are not required by a "broken" symmetry like chirality when the theory contains a gauge particle of zero mass; see P. W. Higgs, Phys. Letters **12**, 132 (1964); Phys. Rev. Letters **13**, 508 (1964); Phys. Rev. **145**, 1156 (1966). Also see F. Englert and R. Brout, Phys. Rev. Letters **13**, 321 (1964); G. S. Guralnik, C. R. Hagen, and J. W. B. Hibble, *ibid.* **13**, 585 (1964); J. W. B. Hibble, Phys. Rev. **155**, 1554 (1967). It should be noted that our $\boldsymbol{\varrho}_\mu$ is the $\boldsymbol{\varrho}'_\mu$ of Wess and Zumino (Ref. 5) while their $\boldsymbol{\varrho}_\mu$ is our $\boldsymbol{\phi}_\mu$.

[25] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).

[26] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966); F. J. Gilman and H. J. Schnitzer, *ibid.* **150**, 1362 (1966); J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966); M. Ademollo, Nuovo Cimento **46**, 156 (1966).

[27] D. F. Greenberg (private communication).

can be taken as m_ρ^{-2}, the three ρ-exchange graphs contribute an effective 4π interaction

$$4m_\rho^2 g_\rho^{-2}\lambda^4(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})^2.$$

But this is exactly cancelled by the $(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})^2$ term arising directly from (7.10). Thus, if we wish to add ρ-exchange terms to the results of current algebra for low-energy π-π scattering, we must also add compensating terms whose effect is to convert the propagator $(q^2+m_\rho^2)^{-1}$ into $-(q^2/m_\rho^2)(q^2+m_\rho^2)^{-1}$. The total contribution made by ρ exchange plus compensating terms to low-energy π-π scattering is of fourth order in m_π, and hence negligible.

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Test of Time-Reversal Invariance in the Decay Process $K^+ \to \mu^+ + \nu + e^+ + e^-$†

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CP violation occurring in the $K_L \to 2\pi$ decay is assumed to take place when the electromagnetic and weak interactions occur simultaneously. An analysis of the $K^+ \to \mu^+ + \nu + e^+ + e^-$ decay is presented which shows that the measurement of the muon polarization in this decay is a practical way of determining this effect. The reduction of the principal background from $K_{\mu 3}$ is discussed. Such an experiment would also provide valuable information on structure radiation.

I. INTRODUCTION

AMONG the many theories and models that have been proposed since the discovery of the decay mode $K_L^0 \to \pi^+\pi^-$,[1] those that invoke the electromagnetic interaction[2] are of particular interest. These hypotheses that the electromagnetic interaction is related to CP violation, can explain "naturally" the factor

$$|\eta_{+-}| = [P(K_L^0 \to \pi^+\pi^-)/\Gamma(K_S^0 \to \pi^+\pi^-)]^{1/2} \approx \alpha/\pi$$

and the large admixture of the $\Delta I = \frac{3}{2}$ amplitude in the process $K_L^0 \to \pi\pi$.[3] The results of the recent experiments on weak decay processes $K_L^0 \to \pi\mu\nu$,[4] $\Lambda \to p\pi^-$,[5] and Ne^{19} β decay[6] showing no evidence of CP violation for these decay modes may serve as indirect support for these hypotheses.

On the other hand, the experiments with relatively high statistics, such as the upper limit[7] for $\eta \to \pi^0 e^+ e^-$ and the charge asymmetry[8] between the two charged pions in $\eta \to \pi^+\pi^-\pi^0$, show that the present experimental status is at least consistent with the conventional C- and P-conserving electromagnetic theory. Thus, even though a definite conclusion should not be drawn until further experimental information on Compton scattering[9] and other processes becomes available, we are tempted to think that the ordinary C- and P-conserving electromagnetic interaction can describe nature fairly well to the order of $g_{st}e$, where g_{st} is the strong-interaction coupling constant.

However, we are aware that there is no experiment which checks definitely the possibility of CP-violating electromagnetic interaction accompanied by weak inter-

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[1] $K_L^0 \to \pi^+ + \pi^-$ was first observed by J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters **13**, 138 (1964). For a summary of the experimental situation, see V. L. Fitch, Rapporteur's Talk, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, Calif., 1966* (University of California Press, Berkeley, 1967), p. 63.

[2] Possibility of the connection between CP violation and electromagnetic interaction is discussed by many authors; see, for example, J. Bernstein, G. Feinberg, and T. D. Lee, Phys. Rev. **139**, B1650 (1965).

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[4] For the most recent experimental result, see K. K. Young, M. J. Longo, and J. A. Helland, Phys. Rev. Letters **18**, 806 (1967).

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[7] D. Berley, E. L. Hart, D. C. Rahm, D. L. Stonehill, B. Thevenet, W. J. Willis, and S. S. Yamamoto, Phys. Rev. **142**, 893 (1966); see also C. Baglin, A. Bezaguet, B. Degrange, F. Jacquet, P. Musset, U. Nguyen-Khac, and G. Nihoul-Boutang, Phys. Letters **22**, 219 (1966).

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[9] Generally it is difficult to detect C- and T-violating but P-conserving electromagnetic effects among hadrons. See, for example, K. Hiida and T. Ebata, Phys. Rev. **154**, 1337 (1967).