Low-Energy Nucleon Compton Scattering*†

R. KÖBERLE
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois
(Received 22 September 1967)

In the case of nucleon Compton scattering, theory and experiment disagree at the photoproduction threshold and at the peak of the first pion-nucleon resonance (1236 MeV) at 90° c.m. scattering angle. Dispersion relations are used to calculate the scattering amplitudes of this process in the low-energy region in terms of the nucleon Born pole, photoproduction, π^0, and 2π exchange. Several currently accepted models for $I=J=0$ $\pi\pi$ scattering are considered. Using a nonresonating or a resonating $\pi\pi$ phase shift with resonance width $\gtrsim 100$ MeV and position $\gtrsim 600$ MeV, we are able to account for the discrepancy between theory and experiment around the photoproduction threshold. At the 90° c.m. resonance peak, the situation remains essentially unchanged.

I. INTRODUCTION

DISPERSION relations combined with the unitarity of the S matrix have had considerable success in their application to nucleon Compton scattering.[1] It has been shown that, in the region including the first pion-nucleon resonance, the main contribution to the absorptive part of the process comes from single-pion photoproduction, the pion-nucleon system being in relative s and p states. Yet some systematic discrepancy seemed to exist around the photoproduction threshold and at the resonance peak at 90° c.m. angle, where the experimental cross section was consistently lower than the theoretical one.[2] This showed up in dispersion-theoretic and isobar-model calculations, although the latter could not be expected to explain the details of the energy dependence. It has been suggested that the two-pion exchange contribution may account for the disagreement,[3] although no reliable estimate has been given.

The aim of the present paper is to include several details and to try to account for discrepancies between theory and experiment below 400 MeV photon laboratory energies. To this end, a fixed-angle dispersion relation obtained from the Mandelstam representation[4] is used for the photon-nucleon scattering amplitudes. By means of unitarity the discontinuities across the cuts are expressed in terms of the possible intermediate states in the process $\gamma N\to\gamma N$ and $N\bar{N}\to 2\gamma$. On the right-hand cut, we include the one-nucleon and the pion-nucleon intermediate state, the former leading to the Born poles and the latter requiring a knowledge of photoproduction. Since we are neglecting processes of second order in e^2, our formulas do not lead to integral equations. On the left-hand cut, we retain π_0 and 2π exchange, so that we need to know the amplitudes for $2\pi\to 2\gamma$ and $N\bar{N}\to 2\pi$.

These discontinuities are fed in the dispersion relations, whose subtractions are estimated by appealing to the low-energy theorem.[5]

In Sec. II the kinematics is written and some comments are made on the selection of suitable amplitudes for a numerical calculation. Section III contains the Mandelstam representation and the fixed-angle dispersion relation. The unitarity condition for the direct and crossed channels is exploited in Secs. IV and V and the contributions of the different intermediate states is discussed, including several models for $I=J=0$ $\pi\pi$ scattering. In Sec. VI we compare our results with experiment and state the conclusions.

II. KINEMATICS

Consider the process

$$\gamma_1+N_1\to\gamma_2+N_2 \quad (s\text{ channel}),$$

where the incoming momenta of γ_1 and N_1 are k_1 and p_1 and the outgoing momenta of γ_2 and N_2 are k_2 and p_2, respectively.

Define the following scalars:

$$s=-(p_1+k_1)^2=-(p_2+k_2)^2, \qquad (2.1)$$

$$u=-(p_1-k_2)^2=-(p_2-k_1)^2, \qquad (2.2)$$

$$t=-(p_1-p_2)^2=-(k_1-k_2)^2, \qquad (2.3)$$

which satisfy $s+u+t=2m^2$, where m is the nucleon mass. s is the square of the s-channel c.m. energy, while

* Work supported in part by the U. S. Atomic Energy Commission.

† Submitted to the Department of Physics, The University of Chicago, in partial fulfillment of the requirements for the Ph.D. degree.

[1] M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. **95**, 1612 (1954); R. H. Capps, *ibid.* **106**, 1031 (1957); **108**, 1032 (1957); T. Akiba and I. Sato, Progr. Theoret. Phys. (Kyoto) **19**, (1958); L. I. Lapidus and Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **37**, 1714 (1959); **38**, 201 (1960) [English transls.: Soviet Phys.—JETP **10**, 1213 (1960); **11**, 147 (1960)]; M. Jacob and J. Mathews, Phys. Rev. **117**, 854 (1960); V. F. Müller, Z. Physik **170**, 114 (1962); A. P. Contogouris, Phys. Rev. **124**, 912 (1961); Nuovo Cimento **25**, 104 (1962); D. Holliday, Ann. Phys. (N.Y.) **24**, 289 (1963); **24**, 319 (1963).

[2] E. R. Gray and A. O. Hanson, Phys. Rev. **160**, 1212 (1967). Further reference to experimental work can be found in this paper.

[3] A. C. Hearn and E. Leader, in Proceedings of the International Conference on Nucleon Structure, Stanford, California, 1963 (unpublished); A. P. Contogouris and A. Vernagelakis, Phys. Letters **6**, 103 (1963).

[4] A. C. Hearn and E. Leader, Phys. Rev. **126**, 789 (1962).

[5] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954).

u and t are the energies squared of the following channels:

$$\gamma_2+N_1\to\gamma_1+N_2 \quad (u \text{ channel}),$$
$$N_1+\bar{N}_2\to\gamma_1+\gamma_2 \quad (t \text{ channel}).$$

In terms of s-channel c.m. momentum p and angle θ, we have

$$s=[(p^2+m^2)^{1/2}+p]^2, \tag{2.4}$$
$$u=-2p^2(1+\cos\theta)+[(p^2+m^2)^{1/2}-p]^2, \tag{2.5}$$
$$t=-2p^2(1-\cos\theta). \tag{2.6}$$

It is useful to define the following[6] orthogonal vectors:

$$\begin{aligned} K_\mu&=\tfrac{1}{2}(k_1+k_2)_\mu,\\ Q_\mu&=(k_1-k_2)_\mu=(p_2-p_1)_\mu,\\ P_\mu&=\tfrac{1}{2}(p_1+p_2)_\mu,\\ P_\mu'&=P_\mu-P\cdot KK_\mu/K^2,\\ N_\mu&=i\epsilon_{\mu\nu\rho\sigma}P_\nu'K_\rho Q_\sigma. \end{aligned} \tag{2.7}$$

Normalize the T matrix as

$$\langle f|S|i\rangle=\delta_{f,i}+(2\pi)^4 i\delta^{(4)}(p_1+k_1-p_2-k_2)\times(16p_{10}p_{20}k_{10}k_{20})^{-1/2}\langle f|T|i\rangle. \tag{2.8}$$

In the s channel, we have

$$\langle f|T|i\rangle=\epsilon_\nu^{(2)*}\bar{u}_2(p_2)T_{\mu\nu}u_1(p_1)\epsilon_\mu^{(1)}, \tag{2.9}$$

where $\epsilon^{(i)}$ are the photon polarization vectors and the u_i satisfy

$$(i\gamma\cdot p_i+m)u_i(p_i)=0,\quad i=1,2. \tag{2.10}$$

Requiring Lorentz, gauge, parity, and charge-conjugation invariance, the six independent amplitudes of $T_{\mu\nu}$ may be written[4]

$$\begin{aligned} T_{\mu\nu}=&A_1(s,t,u)(P_\mu'P_\nu'/P'^2)+A_2(s,t,u)(N_\mu N_\nu/N^2)\\ &+A_3(s,t,u)[(P_\mu'N_\nu-P_\nu'N_\mu)/(P'^2N^2)^{1/2}]i\gamma_5\\ &+A_4(s,t,u)(P_\mu'P_\nu'/P'^2)i\gamma\cdot K+A_5(s,t,u)N_\mu N_\nu/N^2\\ &\times i\gamma\cdot K+A_6(s,t,u)\frac{P_\mu'N_\nu+P_\nu'N_\mu}{(P'^2N^2)^{1/2}}i\gamma_5 i\gamma\cdot K\\ =&\sum_i T_{\mu\nu}{}^{(i)}A_i(s,t,u), \end{aligned} \tag{2.11}$$

with $(P'^2N^2)^{1/2}\equiv\frac{1}{2}(m^4-su)$.

The isospin decomposition of each amplitude is

$$A_i=A_i{}^sI+A_i{}^V\tau_3, \tag{2.12}$$

I and τ_3 referring to the nucleon isospin space. Crossing symmetry states that

$$A_i(s,t,u)=\eta_iA_i(u,t,s),$$

with

$$\eta_i=+1 \quad\text{for}\quad i=1,2,3,6,$$
$$\eta_i=-1 \quad\text{for}\quad i=4,5. \tag{2.13}$$

[6] R. E. Prange, Phys. Rev. **110**, 240 (1958).

The decomposition (2.11) is free of kinematical singularities and rather convenient for theoretical calculations. It can easily be seen though, for example, in the Born approximation, that the differential cross section $d\sigma/d\Omega$, given by

$$\begin{aligned} d\sigma/d\Omega=&(1/64\pi^2s)\{[2m^2+2p^2(1-\cos\theta)]\\ &\times(|A_1|^2+|A_2|^2)+2p^2(1-\cos\theta)|A_3|^2-2p^2W\\ &\times[p(1-\cos\theta)-W](|A_4|^2+|A_5|^2)+2p^2W^2\\ &\times(1+\cos\theta)|A_6|^2+m[2m^2-2W^2+2p^2(1-\cos\theta)]\\ &\times\mathrm{Re}(A_1A_4{}^*+A_2A_5{}^*)\}, \end{aligned} \tag{2.14}$$

where $W=s^{1/2}$, is a very sensitive function of A_i and as such not suited for numerical calculations. For this purpose, we introduce helicity amplitudes[7] $\langle r_2,\lambda_2|T|r_1,\lambda_1\rangle$, where $|r_i,\lambda_i\rangle$ represents a state with nucleon helicity r_i and photon helicity λ_i. Six independent amplitudes and their partial-wave expansion may be defined as follows:

$$\begin{aligned} \Phi_1&\equiv\langle\tfrac{1}{2},1|T|\tfrac{1}{2},1\rangle=\frac{1}{p}\sum_J(J+\tfrac{1}{2})\Phi_{-\frac{1}{2},-\frac{1}{2}}{}^Jd_{-\frac{1}{2},-\frac{1}{2}}{}^J(\theta),\\ \Phi_2&\equiv\langle-\tfrac{1}{2},-1|T|\tfrac{1}{2},1\rangle=\frac{1}{p}\sum_J(J+\tfrac{1}{2})\Phi_{-\frac{1}{2},\frac{1}{2}}{}^Jd_{-\frac{1}{2},\frac{1}{2}}{}^J(\theta),\\ \Phi_3&\equiv\langle\tfrac{1}{2},-1|T|\tfrac{1}{2},1\rangle=\frac{1}{p}\sum_J(J+\tfrac{1}{2})\Phi_{-\frac{1}{2},\frac{3}{2}}{}^Jd_{-\frac{1}{2},\frac{3}{2}}{}^J(\theta),\\ \Phi_4&\equiv\langle-\tfrac{1}{2},1|T|\tfrac{1}{2},1\rangle\\ &=\frac{1}{p}\sum_J(J+\tfrac{1}{2})\Phi_{-\frac{1}{2},-\frac{3}{2}}{}^Jd_{-\frac{1}{2},\frac{3}{2}}{}^J(\theta),\\ \Phi_5&\equiv\langle-\tfrac{1}{2},1|T|-\tfrac{1}{2},1\rangle\\ &=\frac{1}{p}\sum_J(J+\tfrac{1}{2})\Phi_{-\frac{3}{2},-\frac{3}{2}}{}^Jd_{-\frac{3}{2},-\frac{3}{2}}{}^J(\theta),\\ \Phi_6&\equiv\langle\tfrac{1}{2},-1|T|-\tfrac{1}{2},1\rangle\\ &=\frac{1}{p}\sum_J(J+\tfrac{1}{2})\Phi_{-\frac{3}{2},\frac{3}{2}}{}^Jd_{-\frac{3}{2},\frac{3}{2}}{}^J(\theta). \end{aligned} \tag{2.15}$$

We may relate the Φ_i to the A_i;

$$\begin{aligned} \Phi_1&=\cos\tfrac{1}{2}\theta[m(A_2-A_1)-WP(A_5-A_4)+2pWA_6],\\ \Phi_2&=-\sin\tfrac{1}{2}\theta[E(A_2+A_1)-mp(A_5-A_4)-2pA_3],\\ \Phi_3&=\cos\tfrac{1}{2}\theta[m(A_2+A_1)-Wp(A_5+A_4)],\\ \Phi_4&=-\sin\tfrac{1}{2}\theta[E(A_2-A_1)-mp(A_5-A_4)],\\ \Phi_5&=\cos\tfrac{1}{2}\theta[m(A_2-A_1)-Wp(A_5-A_4)-2pWA_6],\\ \Phi_6&=\sin\tfrac{1}{2}\theta[E(A_2+A_1)-mp(A_5+A_4)+2pA_3]. \end{aligned} \tag{2.16}$$

The differential cross section is given by

$$d\sigma/d\Omega=\tfrac{1}{2}(1/64\pi^2s)[|\Phi|^2+|\Phi_2|^2+2|\Phi_3|^2+2|\Phi_4|^2+|\Phi_5|^2+|\Phi_6|^2]. \tag{2.17}$$

[7] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

The following expansion of the scattering amplitude between Pauli spinors is often useful:

$$\langle\gamma_2,N_2|T|\gamma_1,N_1\rangle=\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2^*f_1+(\hat{p}_1\times\boldsymbol{\varepsilon}_1)\cdot(\hat{p}_2\times\boldsymbol{\varepsilon}_2^*)f_2 +i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_1\times\boldsymbol{\varepsilon}_2^*f_3+i\boldsymbol{\sigma}\cdot(\hat{p}_2\times\boldsymbol{\varepsilon}_2^*)\times(\hat{p}_1\times\boldsymbol{\varepsilon}_1)f_4 +i[\boldsymbol{\sigma}\cdot\hat{p}_1(\hat{p}_2\times\boldsymbol{\varepsilon}_2^*)\cdot\boldsymbol{\varepsilon}_1-\boldsymbol{\sigma}\cdot\hat{p}_2(\hat{p}_1\times\boldsymbol{\varepsilon}_1)\cdot\epsilon_2^*]f_5 +i[\boldsymbol{\sigma}\cdot\hat{p}_2(\hat{p}_2\times\boldsymbol{\varepsilon}_2^*)\cdot\boldsymbol{\varepsilon}_1-\boldsymbol{\sigma}\cdot\hat{p}_1(\hat{p}_1\times\boldsymbol{\varepsilon}_1)\cdot\boldsymbol{\varepsilon}_2^*]f_6, \quad (2.18)$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the initial and final c.m. nucleon momenta.

The f_i are related to the A_i as

$$\begin{aligned}
(\sin^2\theta)f_1&=C_1(A_1\cos\theta+A_2)+C_2(A_4\cos\theta+A_5),\\
(\sin^2\theta)f_2&=-[C_1(A_1+A_2\cos\theta)+C_2(A_4+A_5\cos\theta)],\\
f_3&=-[(m-E)A_1+p(m-W)A_4],\\
f_4&=(m-E)A_2+p(m-W)A_5,\\
(\sin^2\theta)f_5&=(m-E)(A_1\cos\theta+A_2)+p(m-W)\\
&\quad\times(A_4\cos\theta+A_5)-pA_3(1+\cos\theta)\\
&\quad-pWA_6(1-\cos\theta),\\
(\sin^2\theta)f_6&=-(m-E)(A_1+A_2\cos\theta)-p(m-W)\\
&\quad\times(A_4+A_5\cos\theta)+pA_3(1+\cos\theta)\\
&\quad-pWA_6(1-\cos\theta),
\end{aligned} \quad (2.19)$$

where

$$C_1=(E+m)-\cos\theta(E-m),$$
$$C_2=-p[(W+m)+\cos\theta(W-m)],$$

and $E=(p^2+m^2)^{1/2}$ is the c.m. nucleon energy.

A helicity expansion can also be made in the t channel. Six amplitudes $\langle\lambda_1,\lambda_2|T|r_1,\bar{r}_2\rangle$ and their partial-wave expansions can be defined as

$$\begin{aligned}
G_1&\equiv\langle 1,1|T|\tfrac{1}{2},\tfrac{1}{2}\rangle=\frac{1}{(p_3k_3)^{1/2}}\sum_J(J+\tfrac{1}{2})G_{00}{}^Jd_{00}{}^J(\psi),\\
G_2&\equiv\langle 1,-1|T|\tfrac{1}{2},\tfrac{1}{2}\rangle=\frac{1}{(p_3k_3)^{1/2}}\sum_J(J+\tfrac{1}{2})G_{2,0}{}^Jd_{2,0}{}^J(\psi),\\
G_3&\equiv\langle 1,-1|T|\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{1}{(p_3k_3)^{1/2}}\sum_J(J+\tfrac{1}{2})\bar{G}_{0,0}{}^Jd_{0,0}{}^J(\psi),\\
G_4&\equiv\langle 1,-1|T|\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{1}{(p_3k_3)^{1/2}}\sum_J(J+\tfrac{1}{2})G_{2,1}{}^Jd_{2,1}{}^J(\psi),\\
G_5&\equiv\langle -1,1|T|\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{1}{(p_3k_3)^{1/2}}\sum_J(J+\tfrac{1}{2})G_{-2,1}{}^Jd_{-2,1}{}^J(\psi),\\
G_6&\equiv\langle 1,1|T|\tfrac{1}{2},-\tfrac{1}{2}\rangle=\frac{1}{(p_3k_3)^{1/2}}\sum_J(J+\tfrac{1}{2})G_{0,1}{}^Jd_{0,1}{}^J(\psi),
\end{aligned} \quad (2.20)$$

where p_3 and k_3 are t-channel nucleon and photon c.m. 3-momenta and ψ is the t-channel c.m. scattering angle. Their relations with s, t, and u are

$$\begin{aligned}
s&=-2k_3^2-m^2-2k_3\cos\psi(k_3^2+m^2)^{1/2},\\
u&=-2k_3^2-m^2+2k_3\cos\psi(k_3^2+m^2)^{1/2},\\
t&=4k_3^2=4(p_3^2+m^2).
\end{aligned} \quad (2.21)$$

The G_i's are related to the A_i's as follows:

$$\begin{aligned}
G_1&=p_3(A_1+A_2)-k_3m\cos\psi(A_4+A_5)+2k_3A_3,\\
G_2&=p_3(-A_1+A_2)-k_3m\cos\psi(-A_4+A_5),\\
G_3&=p_3(A_1+A_2)-k_3m\cos\psi(A_4+A_5)-2k_3A_3,\\
G_4&=-k_3\sin\psi(A_4-A_5)-2p_3k_3\sin\psi A_6,\\
G_5&=-k_3^2\sin\psi(A_4-A_5)+2p_3k_3\sin\psi A_6,\\
G_6&=k_3^2\sin\psi(A_4+A_5).
\end{aligned} \quad (2.22)$$

III. ANALYTICITY PROPERTIES

We assume that the six functions $A_i(s,t,u)$ satisfy the Mandelstam representation

$$A_i(s,t,u)=P_i+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}ds'\int_{(m+\mu)^2}^{\infty}du'\frac{\chi_i(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt' \times\frac{\rho_i(s',t')}{t'-t}\left(\frac{1}{s'-s}+\frac{\eta_i}{s'-u}\right), \quad (3.1)$$

where μ is the pion mass and P_i represents the nucleon and pion pole terms:

$$P_i=R_i\left(\frac{1}{s-m^2}+\frac{\eta_i}{u-m^2}\right)+\frac{r_i}{t-u^2}. \quad (3.2)$$

The absorptive parts in the various channels are given by

$$A_i{}^s(s,t,u)\equiv\operatorname{Im}A_i(s,t,u)\ \text{ for } s\geqslant(m+\mu)^2 =\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}du'\frac{\chi_i(s,u')}{u'-u}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\rho_i(s,t')}{t'-t} \quad (3.3)$$

and

$$A_i{}^t(s,t,u)\equiv\operatorname{Im}A_i(s,t,u)\quad\text{for}\quad t\geqslant 4\mu^2 =\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\rho_i(s',t)\left(\frac{1}{s'-s}+\frac{\eta_i}{s'-u}\right). \quad (3.4)$$

Following Hearn and Leader,[4] we define the functions

$$A_i{}^s(x,y)=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}du'\frac{\chi_i(x,u')}{u'-u(x)}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\rho(x,t')}{t'-y}\quad\text{for } x\geqslant(m+\mu)^2, \quad (3.5)$$

$$A_i^t(x,y)=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\rho_i(s',y)\left(\frac{1}{s'-x}+\frac{\eta_i}{s'-u(x)}\right)$$

$$\text{for } y\geq 4\mu^2, \quad (3.6)$$

where

$$u(x)\equiv x^{-1}[m^4-\tfrac{1}{2}(x-m^2)^2(1+\cos\theta)],$$
$$t(x)\equiv -[(x-m^2)^2/2x](1-\cos\theta). \quad (3.7)$$

The definitions (3.5) and (3.6) are convenient in order to obtain a fixed $\cos\theta$ dispersion relation in which right- and left-hand cuts are separated. In order to do this, apply the identity

$$\int\frac{f(x',y(x))}{x'-x-i\epsilon}dx\equiv\int\frac{f(x',y(x'))}{x'-x-i\epsilon}+P\int\frac{f(x',y(x))}{x'-x}dx-P\int\frac{f(x',y(x'))}{x'-x}dx'$$

to both denominators in Eq. (3.1). After identifying the various contributions, we can write

$$A_i(s,\cos\theta)=P_i+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{A_i^s(s',t(s'))}{s'-s}+\frac{2\eta_i}{\pi}\int_{(m+\mu)^2}^{\infty}du'\frac{1}{u_+(u')-u_-(u')}$$
$$\times\left\{\frac{u_+(u')}{s(1+\cos\theta)-u_+(u')}A_i^s\left[u',t\left(\frac{u_+(u')}{1+\cos\theta}\right)\right]-\frac{u_-(u')}{s(1+\cos\theta)-u_-(u')}A_i^s\left[u',t\left(\frac{u_-(u')}{1+\cos\theta}\right)\right]\right\}$$
$$+\frac{2}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{1}{t_+(t')-t_-(t')}\left\{\frac{t_+(t')}{s(1-\cos\theta)-t_+(t')}A_i^t\left[\frac{t_+(t')}{1-\cos\theta},t'\right]-\frac{t_-(t')}{s(1-\cos\theta)-t_-(t')}A_i^t\left[\frac{t_-(t')}{1-\cos\theta},t'\right]\right\}, \quad (3.8)$$

where

$$u_\pm(x)=m^2(1+\cos\theta)-x\pm\{[m^2(1+\cos\theta)-x]^2+m^4\sin^2\theta\}^{1/2},$$
$$t_\pm(x)=m^2(1-\cos\theta)-x\pm\{[m^2(1-\cos\theta)-x]^2-m^4(1-\cos\theta)^2\}^{1/2}. \quad (3.9)$$

We can easily see the singularity structure of (3.8) in the s plane. The right-hand cut, $s\geq(m+\mu)^2$ comes from the first integral in (3.8). The denominators $s(1+\cos\theta)-u_\pm(u')=0$ give cuts for

$$-\infty\leq s\leq \mathcal{U}_-(\cos\theta) \quad (3.10)$$

and

$$0\leq s\leq \mathcal{U}_+(\cos\theta),$$

where

$$\mathcal{U}_\pm(\cos\theta)=u_\pm[(m+\mu)^2]/(1+\cos\theta). \quad (3.11)$$

The cut $0\leq s\leq \mathcal{U}_+$ is called "crossed physical cut." In the third integral, if

$$4\mu^2>2m^2(1-\cos\theta),$$

the cuts are real, since

$$[m^2(1-\cos\theta)-t']^2-m^4(1-\cos\theta)^2>0$$

and run along

$$-\infty\leq s\leq \mathcal{T}_-(\cos\theta)$$

and

$$\mathcal{T}_+(\cos\theta)\leq s\leq 0, \quad (3.12)$$

where

$$\mathcal{T}_\pm(\cos\theta)=t_\pm(4\mu^2)/(1-\cos\theta).$$

If

$$4\mu^2<t'<2m^2(1-\cos\theta),$$

the cuts are complex at $|s|=m^2$, namely,

$$(\mathrm{Re}s+i\,\mathrm{Im}s)(1-\cos\theta)=t_\pm(t')$$

gives

$$\mathrm{Re}s=m^2-4\mu^2/(1-\cos\theta),$$
$$\mathrm{Im}s=\pm(m^4-\mathrm{Re}s)^{1/2}, \quad (3.13)$$

On the other hand, if

$$4\mu^2<2m^2(1-\cos\theta),$$

the cut is real along

$$-\infty\leq s\leq 0.$$

If on the circle cut we put $s=m^2e^{i\alpha}$, then t is given by

$$t=4m^2\sin^2(\tfrac{1}{2}\theta)\sin^2(\tfrac{1}{2}\alpha). \quad (3.14)$$

In the second integral, only contributions from the crossed physical cut $0\leq s\leq\mathcal{U}_+(\cos\theta)$ will be taken into account, since in the interval $-\infty\leq s\leq\mathcal{U}_-(\cos\theta)$, we would need the function

$$A_i^s\left[u',t\left(\frac{u_-\cos\theta}{1+\cos\theta}\right)\right]$$

for unphysical values of the scattering angle, so that the Legendre expansion does not converge.

In the third integral, only part of the circle cut can be calculated, using a partial-wave expansion in the t channel. This expansion converges for $-40°\lesssim\alpha\lesssim 40°$, and from Eq. (3.14) we see that at $\alpha\sim 40°$, $\theta\sim 90°$ energies as high as $t\sim m^2$ become important. Since we will in any case only take 2π effects into account, i.e., $t<9\mu^2$, we see that the convergence requirements do not pose serious limitations.

It is known that Eq. (3.8), which has been written without subtraction, has to satisfy a low-energy theorem,[5] which states that to first order in photon energy the scattering of photons on spin-$\frac{1}{2}$ particles depends only on the static charge and magnetic form factors. The result is

$$T=\chi_2^*\{-2e^2F_1^2\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2^*-(e^2/m)F_1^2pi\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2^*$$
$$-(e^2/m)(F_1+F_2)^2pi\boldsymbol{\sigma}\cdot(\hat{p}_2\times\boldsymbol{\epsilon}_2^*)\times(\hat{p}_1\times\boldsymbol{\varepsilon}_1)$$
$$-(e^2/m)pF_1(F_1+F_2)\times i[\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}_2(\hat{p}_2\times\boldsymbol{\varepsilon}_2^*)\cdot\boldsymbol{\varepsilon}_1$$
$$-\boldsymbol{\sigma}\cdot\hat{p}_1\boldsymbol{\varepsilon}_2^*\cdot(\hat{\mathbf{p}}_1\times\boldsymbol{\varepsilon}_1)]\}\chi_1, \quad (3.15)$$

where F_1 and F_2 are defined in Eq. (4.4) and $e^2=4\pi/137$. Using Eqs. (2.18) and (2.19), we get for fixed $\cos\theta$ and $p\to 0$

$$A_1\to(e^2/m)F_1^2(\cos\theta-1),$$
$$A_2\to(e^2/m)F_2(2F_1+F_2),$$
$$A_3\to(e^2/2m)F_2(2F_1+F_2)+(e^2/2m)(1-\cos\theta)(F_1+F_2),$$
$$A_4\to-(e^2/m)F_1^2/p, \quad (3.16)$$
$$A_5\to(e^2/m)(F_1+F_2)^2p^{-1},$$
$$A_6\to(e^2/2m)F_2^2+(e^2/2m)F_1(F_1+F_2)(1-\cos\theta).$$

Comparing with Eq. (4.7), we see that A_1, A_4, and A_5 have the correct low-energy behavior, but that A_2, A_3, and A_6 need a subtraction.[8]

IV. *s*-CHANNEL UNITARITY

From the unitarity condition of the S matrix

$$SS^\dagger=S^\dagger S=I, \quad (4.1)$$

and the definition (2.8), we obtain

$$\langle f|(T-T^\dagger)/2i|i\rangle=\tfrac{1}{2}(2\pi)^4\sum_a N_a^2 \times\delta^{(4)}(P_f-Pi)\langle f|T|a\rangle\langle a|T^\dagger|i\rangle, \quad (4.2)$$

where the sum extends over all intermediate states $|a\rangle$ with permissible quantum numbers and

$$N_a=(2E_a)^{-1/2}.$$

Introducing Eq. (2.11) into Eq. (4.2) we get

$$\sum_{i=1}^{6}\mathrm{Im}A_i(s,t,u)\langle N_2,\gamma_2|T_{\mu\nu}{}^{(i)}|N_1,\gamma_1\rangle =\tfrac{1}{2}(2\pi)^4\sum_a\frac{1}{2E_a}\delta^{(4)}(P_f-P_i) \times\langle N_2,\gamma_2|T|a\rangle\langle a|T^\dagger|N_1,\gamma_1\rangle. \quad (4.3)$$

In the s channel, the sum over a will be limited to the nucleon and pion-nucleon terms.

The matrix element $\langle\gamma_2,N_2|T|N_1\rangle$ can be written as

$$\langle k_2,p_2|T|p_1\rangle=ie\bar{u}(p_2)[\gamma_\mu F_1((p_1-p_2)^2)-\sigma_{\mu\nu}(p_2-p_1)_\nu \times F_2((p_1-p_2)^2)/2m]u(p_1)\epsilon_\mu, \quad (4.4)$$

where

$$F_1\equiv F_1(0)=\tfrac{1}{2}(I+\tau_3),$$
$$F_2\equiv F_2(0)=\tfrac{1}{2}(\mu_p+\mu_n)I+\tfrac{1}{2}(\mu_p-\mu_n)\tau_3,$$

and μ_p, μ_n are the proton and neutron anomalous magnetic moments,

$$\mu_p=1.79\,,$$
$$\mu_n=-1.91\,.$$

Substituting Eq. (4.4) into Eq. (4.3) and using the identities

$$\bar{u}(p_2)\gamma\cdot N\gamma\cdot Ku(p_1)=-(N^2P'^2)^{1/2}\bar{u}(p_2)\gamma_5u(p_1),$$
$$\bar{u}(p_2)\gamma\cdot Nu(p_1)=2iK^2\bar{u}(p_2)\gamma_5\gamma\cdot Ku(p_1), \quad (4.5)$$

we find

$$A_i{}^{IN}=\pi\delta(m^2-s)e^2B_i,$$

where

$$B_1=-2mF_1^2,$$
$$B_2=0,$$
$$B_3=-mF_1(F_1+F_2),$$
$$B_4=F_1^2,$$
$$B_5=-(F_1+F_2)^2,$$
$$B_6=-F_1(F_1+F_2). \quad (4.6)$$

It follows that the contribution of the s and u channels to the P_i of Eq. (3.2) is

$$P_i=\frac{(1-\cos\theta)e^2B_i}{2m^2+(s-m^2)(1+\cos\theta)} \quad \text{for } i=1,2,3,6,$$

$$P_i=e^2B_i\left[\frac{2}{s-m^2}+\frac{\cos\theta-1}{2m^2+(s-m^2)(1+\cos\theta)}\right] \quad \text{for } i=4,5. \quad (4.7)$$

The pion-nucleon intermediate state in Eq. (4.3) requires the knowledge of pion photoproduction. The unitarity equation (4.3) may be integrated trivially if we expand both sides in terms of helicity amplitudes. For the process $\gamma_1+N_1\to\pi+N_2$ define the following[4] independent helicity amplitudes $\langle s_2|T|s_1,\mu\rangle$, where s_i, μ are the nucleon and photon helicities, respectively:

$$\psi_1\equiv\langle\tfrac{1}{2}|T|\tfrac{1}{2},1\rangle=(pq)^{-1/2}\sum_J(J+\tfrac{1}{2})\psi_{-\frac{1}{2},\frac{1}{2}}{}^Jd_{-\frac{1}{2},\frac{1}{2}}{}^J(\theta)\,,$$
$$\psi_2\equiv\langle-\tfrac{1}{2}|T|\tfrac{1}{2},1\rangle=(pq)^{-1/2}\sum_J(J+\tfrac{1}{2})\psi_{-\frac{1}{2},-\frac{1}{2}}{}^Jd_{-\frac{1}{2},-\frac{1}{2}}{}^J(\theta)\,,$$
$$\psi_3\equiv\langle\tfrac{1}{2}|T|-\tfrac{1}{2},1\rangle=(pq)^{-1/2}\sum_J(J+\tfrac{1}{2})\psi_{-\frac{3}{2},\frac{1}{2}}{}^Jd_{-\frac{3}{2},\frac{1}{2}}{}^J(\theta)\,, \quad (4.8)$$
$$\psi_4\equiv\langle-\tfrac{1}{2}|T|-\tfrac{1}{2},1\rangle=(pq)^{1/2}\sum_J(J+\tfrac{1}{2})\psi_{-\frac{3}{2},-\frac{1}{2}}{}^Jd_{-\frac{3}{2},-\frac{1}{2}}{}^J(\theta)\,,$$

[8] One could, of course, also leave Eqs. (3.8) as they are and impose the limits (3.7) on A_2, A_3, and A_6.

where q is the πN c.m. momentum. The unitarity condition now becomes

$$(8\pi W)\mathrm{Im}\phi_{s,s'}{}^J=\tfrac{1}{2}\sum_{s_N=\pm 1/2}\psi_{ss_N}{}^J\psi_{s's_N}{}^{J*}. \tag{4.9}$$

For each photoproduction amplitude, the following isospin decomposition can be made:

$$\psi_i=\psi_i{}^{(+)}\mathcal{G}_\beta{}^{(+)}+\psi_i{}^{(-)}\mathcal{G}_\beta{}^{(-)}+\psi_i{}^{(0)}\mathcal{G}_\beta{}^{(0)}, \tag{4.10}$$

where

$$\mathcal{G}_\beta{}^{(+)}=\delta_{\beta 3},\quad \mathcal{G}_\beta{}^{(-)}=i\epsilon_{\beta 3\gamma}\tau_\gamma,\quad \mathcal{G}_\beta{}^{(0)}=\tau_\beta.$$

Introducing Eq. (4.10) into Eq. (4.9), we get

$$\begin{aligned}\psi_{ss_N}\psi_{s's_N}{}^*=[\psi_{ss_N}{}^{(+)}\psi_{s's_N}{}^{(+)*}+2\psi_{ss_N}{}^{(-)}\psi_{s's_N}{}^{(-)*}+3\psi_{ss_N}{}^{(0)}\psi_{s's_N}{}^{(0)*}]I\\ +[\psi_{ss_N}{}^{(+)}\psi_{s's_N}{}^{(0)*}+\psi_{ss_N}{}^{(0)}\psi_{s's_N}{}^{(+)*}-2\psi_{ss_N}{}^{(-)}\psi_{s's_N}{}^{(0)*}-2\psi_{ss_N}{}^{(0)}\psi_{s's_N}{}^{(-)*}]\tau_3.\end{aligned} \tag{4.11}$$

The relation of the partial helicity amplitudes to the multipoles $E_{l\pm}$, $M_{l\pm}$ as defined by CGLN[9] is

$$\begin{aligned}\psi_{-\frac{1}{2},\frac{1}{2}}{}^J=-\psi_{\frac{1}{2},-\frac{1}{2}}{}^J&=(\tfrac{1}{2}pq)^{1/2}[l(M_{l+}-E_{(l+1)-})+(l+2)(E_{l+}+M_{(l+1)-})],\\ \psi_{-\frac{1}{2},-\frac{1}{2}}{}^J=-\psi_{\frac{1}{2},\frac{1}{2}}{}^J&=(\tfrac{1}{2}pq)^{1/2}[l(M_{l+}+E_{(l+1)-})+(l+2)(E_{l+}-M_{(l+1)-})],\\ \psi_{-\frac{3}{2},\frac{1}{2}}{}^J=-\psi_{\frac{3}{2},-\frac{1}{2}}{}^J&=(\tfrac{1}{2}pq)^{1/2}[l(l+1)]^{1/2}[-E_{l+}+M_{l+}-E_{(l+1)-}-M_{(l+1)-}],\\ \psi_{-\frac{3}{2},-\frac{1}{2}}{}^J=-\psi_{\frac{3}{2},\frac{1}{2}}{}^J&=(\tfrac{1}{2}pq)^{1/2}[l(l+1)]^{1/2}[-E_{l+}+M_{l+}+E_{(l+1)-}+M_{(l+1)-}],\end{aligned} \tag{4.12}$$

with[10]

$$J=l+\tfrac{1}{2}.$$

For $E_{l\pm}$, $M_{l\pm}$ the results of Donnachie and Shaw[11] were used. They started from fixed-t dispersion relations for the invariant amplitudes $A_i(s,t,u)$, from which integral equations for the multipoles were projected out.[9] Only multipoles leading to final s, p, and d waves were retained. Knowledge of the pion-nucleon phase shifts allows the coupled integral equations for the dominant M_{1+} and E_{0+} transitions to be solved by iteration. The other multipoles were evaluated in the (Born$+M_{1+}{}^{(3)}$) approximation. A more recent solution of these equations by conformal mapping techniques,[12] which improves the evaluation of the $E_{1+}{}^{(3)}$ and $M_{1-}{}^{(0,1)}$ multipoles, gives the same results as far as Compton scattering with the present experimental accuracy is concerned. Equations (4.11) and (4.12) are then introduced in Eq. (4.9), which allows one to evaluate the contribution from the cut $0\leq s\leq$(cutoff) in Eq. (3.8). The integrals were cut off smoothly, going to zero at about $s=3m^2$ and it was checked that the result was essentially independent of the cutoff.

V. t-CHANNEL UNITARITY

For this channel we retain the π^0 and 2π exchange.[13] The relevant matrix elements for π^0 exchange are

$$\langle\gamma_1(k),\gamma_2(k')|T|\pi^0\rangle=i(G/\mu^2)\epsilon_{\mu\nu\rho\sigma}\epsilon_\mu\epsilon_\nu'k_\rho k_\sigma, \tag{5.1}$$

$$\langle\pi^0|T|N(p_1),\bar{N}(p_2)\rangle=ig\bar{v}(p_2)\gamma_5 u(p_1), \tag{5.2}$$

with $G=\pm 8(\pi/\tau\mu)^{1/2}$, where $\tau=0.89\times 10^{-16}$ sec is the $\pi^0\to 2\gamma$ lifetime.[14]

Introducing Eqs. (5.1) and (5.2) into Eq. (4.2), we get

$$\begin{aligned}A_i{}^t(s,t,u)&=0\quad\text{for}\quad i\neq 3,\\ A_3{}^t(s,t,u)&=\pi(\tfrac{1}{2}gG)\mu\delta(\mu^2-t).\end{aligned} \tag{5.3}$$

The sign in Eq. (5.3) is chosen in agreement with Lapidus and Kuang-Chao,[15] i.e., $gG<0$.

In order to calculate the 2π contribution to Eq. (4.2), the isospin $I=0$ s-wave $N\bar{N}\to 2\pi$ and $2\pi\to 2\gamma$ partial-wave amplitudes will be needed. To this end, a dispersion relation following from the Mandelstam representation[16] will be written. On the right-hand cut we will restrict ourselves again to the 2π s wave, i.e., we consider $N\bar{N}\to 2\pi\to 2\pi$ and $2\pi\to 2\pi\to 2\gamma$. Since no reliable information on $I=0$ $\pi\pi$ scattering exists, this amplitude will be parametrized by an N/D decomposition This determines the discontinuity across the right-hand cut. We then solve in the usual fashion[16,17] for the $N\bar{N}\to 2\pi$ and $2\pi\to 2\gamma$ amplitudes in terms of the left-hand cut discontinuities of these amplitudes. The left-hand cuts will be approximated by the respective Born poles.

Since in this channel we are dealing with the annihilation $N\bar{N}\to 2\gamma$, the unitarity condition becomes

$$\sum_{i=1}^{6}\mathrm{Im}A_i(s,t,u)\langle\gamma_1\gamma_2|T_{\mu\nu}{}^{(i)}|N_1\bar{N}_2\rangle=\tfrac{1}{2}(2\pi)^4\sum_a\frac{1}{2E_a}\delta^{(4)}(P_f-P_i)\times\langle\gamma_1\gamma_2|T|a\rangle\langle a|T^\dagger|N_1\bar{N}_2\rangle. \tag{5.4}$$

[9] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).
[10] A sign error should be noted in Ref. 4, Eq. (4.18).
[11] A. Donnachie and G. Shaw, Ann. Phys. (N.Y.) **37**, 333 (1966).
[12] F. A. Berends, A. Donnachie, and D. L. Weaver, Nuclear Phys. (to be published).
[13] The η pole may be included trivially, but its contribution is negligible.
[14] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).
[15] L. I. Lapidus and Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **41**, 294 (1961) [English transl.: Soviet Phys.—JETP **14**, 210 (1962)].
[16] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).
[17] R. Omnès, Nuovo Cimento **8**, 316 (1958).

Retaining only the 2π state in Eq. (5.4), we obtain

$$\sum_{i=1}^{6} \mathrm{Im}A_i(s,t,u)\langle\gamma_1\gamma_2|T_{\mu\nu}{}^{(i)}|N_1\bar{N}_2\rangle = \frac{1}{64\pi^2}\left(\frac{t-4\mu^2}{t}\right)^{1/2}\int d\Omega_{\mathbf{q}} \times\langle\gamma_1\gamma_2|T|\pi_1\pi_2\rangle\langle\pi_1\pi_2|T^\dagger|N_1\bar{N}_2\rangle, \quad (5.5)$$

where the integration is over the solid angle of $\mathbf{q}$, the 2π c.m. momentum. Equation (5.5) will be integrated with the aid of the helicity expansion (2.20), so we write down the relevant formulas.

For $2\pi \to N\bar{N}$, we have the usual decomposition into invariant amplitudes;

$$\langle N(p_1),\bar{N}(p_2)|T|\pi(q_1),\pi(q_2)\rangle = \bar{v}(p_2)[-A+\tfrac{1}{2}i(q_1-q_2)\cdot\gamma B]u(p_1), \quad (5.6)$$

where each amplitude has the isospin decomposition

$$A_{\alpha\beta}=A^{(+)}\delta_{\alpha\beta}+A^{(-)}\tfrac{1}{2}[\tau_\alpha,\tau_\beta]. \quad (5.7)$$

The relation of A, B to the helicity amplitudes $\langle s,\bar{s}|T|2\pi\rangle$ is

$$\begin{aligned}\mathfrak{F}_1&\equiv\langle\tfrac{1}{2},\tfrac{1}{2}|T|2\pi\rangle=2(pA+mq\cos\psi B),\\ \mathfrak{F}_2&\equiv\langle\tfrac{1}{2},-\tfrac{1}{2}|T|2\pi\rangle=-2qE\sin\psi B,\end{aligned} \quad (5.8)$$

where q, p, E, and ψ are the pion momentum, the nucleon momentum and energy, and the scattering angle in the c.m. system, respectively.

The partial-wave expansion of (5.7) is

$$\mathfrak{F}_1=(pq)^{-1/2}\sum_J(J+\tfrac{1}{2})\mathfrak{F}_+{}^J(t)P_J(\cos\psi),$$
$$\mathfrak{F}_2=(pq)^{-1/2}\sum_J(J+\tfrac{1}{2})\mathfrak{F}_-{}^J(t)\sin\psi P_J{}'(\cos\psi)[J(J+1)]^{-1/2}. \quad (5.9)$$

$\mathfrak{F}_+{}^J(t)$ is related to $f_+{}^J(t)$ as defined in Ref. 16 by

$$\mathfrak{F}_+{}^J(t)=-16\pi(q/p)^{1/2}(pq)^Jf_+{}^J(t). \quad (5.10)$$

Similarly, for $2\gamma\to2\pi$ we have

$$\langle\pi(q_1),\pi(q_2)|T|\gamma(k_1),\gamma(k_2)\rangle = \epsilon_\mu{}^{(1)}\epsilon_\nu{}^{(2)}[B_1P_\mu{}'P_\nu{}'/P'^2+B_2N_\mu N_\nu/N^2], \quad (5.11)$$

where

$$K=\tfrac{1}{2}(k_2-k_1),\quad P=\tfrac{1}{2}(q_1-q_2),\quad P'=P-P\cdot KK/K^2,$$
$$Q=-(q_1+q_2),\quad N_\mu=i\epsilon_{\mu\nu\rho\sigma}P_\nu{}'K_\rho Q_\sigma.$$

The isospin decomposition is

$$B_i=B_i{}^{(0)}2\delta_{\alpha\beta}+B_i{}^{(2)}(\delta_{\alpha\beta}-3\delta_{\alpha3}\delta_{\beta3}). \quad (5.12)$$

The helicity amplitudes $\langle2\pi|T|\lambda_1,\lambda_2\rangle$ are

$$\begin{aligned}\mathcal{G}_1&\equiv\langle2\pi|T|1,1\rangle=-\tfrac{1}{2}(B_1+B_2),\\ \mathcal{G}_2&\equiv\langle2\pi|T|1,-1\rangle=\tfrac{1}{2}(B_1-B_2).\end{aligned} \quad (5.13)$$

The partial-wave expansion of (5.12) is

$$\mathcal{G}_1=(kq)^{-1/2}\sum_J(J+\tfrac{1}{2})\mathcal{G}_+{}^J(t)P_J(\cos\psi'),$$
$$\mathcal{G}_2=(kq)^{-1/2}\sum_J(J+\tfrac{1}{2})\times\mathcal{G}_-{}^J(t)\frac{P_J{}''(\cos\psi')\sin^2\psi'}{[(J-1)J(J+1)(J+2)]^{1/2}}, \quad (5.14)$$

where k is the photon c.m. momentum and ψ' is the scattering angle. As in Eq. (5.9) we define

$$g_+{}^J(t)=(qk)^{-1/2}(kq)^{-J}\mathcal{G}_+{}^J(t). \quad (5.15)$$

Substituting now Eqs. (5.7), (5.8), (5.12), and (5.13) into Eq. (5.5), we see that only the $I=0$ part contributes. Hence, if we neglect d and higher waves, we obtain

$$\mathrm{Im}G_1=\mathrm{Im}G_2=-\tfrac{3}{8}(q^2/kp)^{1/2}f_+{}^{0*}(t)g_+{}^0(t), \quad (5.16)$$

all other G_i's being zero. From Eq. (2.22) finally follows

$$A_1{}^t(s,t,u)=A_2{}^t(s,t,u) = -\tfrac{3}{4}\left(\frac{t-4\mu^2}{t}\right)^{1/2}\frac{f_+{}^{0*}(t)g_+{}^0(t)}{t-4m^2}. \quad (5.17)$$

At this stage, we assume the usual analyticity properties for $f_+{}^0(t)$ and $g_+{}^0(t)$.[16] $f_+{}^0(t)$ has cuts along $-\infty\le t\le a$ and $4\mu^2\le t\le\infty$, where $a=4\mu^2(1-\mu^2/4m^2)$. Similarly, $g_+{}^0(t)$ has cuts along $-\infty\le t\le0$ and $4\mu^2\le t\le\infty$. From unitarity it follows that both have the phase $\exp[i\delta_0(t)]$ when $4\mu^2\le t\le16\mu^2$, where $\delta_0(t)$ is the $I=J=0$ $\pi\pi$ phase shift. We decompose this $\pi\pi$ partial-wave amplitude according to

$$A(t)=[t/(t-4\mu^2)]^{1/2}e^{i\delta_0(t)}\sin\delta_0(t)=N(t)/D(t), \quad (5.18)$$

such that

$$\begin{aligned}\mathrm{Im}N(t)&=D(t)\,\mathrm{Im}A(t)\quad &&\text{for } t\le0\\ &=0 &&\text{for } t>0,\\ \mathrm{Im}D(t)&=-\mathrm{N}(t)[(t-4\mu^2)/t]^{1/2} &&\text{for } t\ge4\mu^2\\ &=0 &&\text{for } t<4\mu^2.\end{aligned} \quad (5.19)$$

Since $D(t)$ has the phase $\exp[-i\delta_0(t)]$ for $4\mu^2\le t\le16\mu^2$, $D(t)f_+{}^0(t)$ and $D(t)g_+{}^0(t)$ are real in this interval and we have the approximate solution

$$f_+{}^0(t)=\frac{1}{\pi D(t)}\int_{-\infty}^{a}dt'\frac{D(t')\mathrm{Im}f_+{}^0(t')}{t'-t}, \quad (5.20)$$

$$g_+{}^0(t)=\frac{1}{\pi D(t)}\int_{-\infty}^{0}dt'\frac{D(t')\mathrm{Im}g_+{}^0(t')}{t'-t}. \quad (5.21)$$

Two subtractions will be made in Eq. (5.20) at $t=0$, corresponding to forward $\pi - N$ scattering, giving

$$f_+{}^0(t) = D^{-1}(t)\Big\{ D(t=0)\mathrm{Re}f_+{}^0(0) + t\frac{\partial}{\partial t''}[D(t'')\mathrm{Re}f_+{}^0(t'')]_{t''=0} + \frac{t^2}{\pi}\frac{\partial}{\partial t''}P \times \int_{-\infty}^{a} dt' \frac{D(t')\mathrm{Im}f_+{}^0(t')}{(t'-t'')(t'-t)}\Big|_{t''=0}\Big\}. \quad (5.22)$$

The subtraction constants have been determined by Menotti,[18] Morgan,[19] and Vick[19] from forward πN scattering by the method of Ball and Wong.[20] They obtain

$$\mathrm{Re}f_+{}^0(0) = -2.7 \pm 0.5,$$
$$\mathrm{Re}f_+{}^{0\prime}(0) = 2.9 \pm 0.9. \quad (5.23)$$

In the remaining integral, the Born approximation[16] will be used for $\mathrm{Im}f_+{}^0(t)$

$$\mathrm{Im}f_+{}^0(t) = -\tfrac{1}{8}g^2 m \frac{t-2\mu^2}{[(4m^2-t)(4\mu^2-t)]^{1/2}}, \quad (5.24)$$

with $g^2/4\pi = 14.6$. In (5.21), $\mathrm{Im}g_+{}^0(t)$ will be approximated by the pion poles [see Fig. 1(a)]. In order to satisfy gauge invariance, the seagull graph [see Fig. 1(b)] must also be included. Accordingly, we make one subtraction in Eq. (5.21) at $t=4\mu^2$, the subtraction constant containing the seagull graph,

$$g_+{}^0(t) = -\tfrac{2}{3}e^2\Big[D(t=4\mu^2) + 4\mu^2(t-4\mu^2) \times \int_{-\infty}^{0} dt' \frac{D(t')}{(t'-t)(t'-4\mu^2)[t'(t'-4\mu^2)]^{1/2}}. \quad (5.25)$$

The t-channel contribution to $\gamma N \rightarrow \gamma N$ can now be calculated, once a model is used to evaluate $\delta_0(t)$ in Eq. (5.18).

There have been several attempts ot obtain information on $I=0$ $\pi\pi$ scattering from πN scattering,[19,21] K_{e4} decays,[22] K_0 mass differences,[23,24] etc.

We consider three models in order to provide the necessary information in Eq. (5.18).

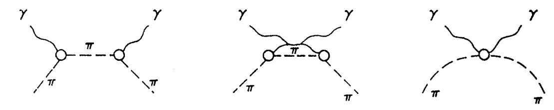

FIG. 1. (a) Pion pole terms contributing to pion Compton scattering; (b) Seagull graph in pion Compton scattering.

A. Nonresonating Phase Shift $\delta_0(t)$ from πN Data

In this approach, dispersion relations for partial-wave amplitudes are used, which are of the form

$$\mathrm{Re}f_l(s) = P_l(s) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{\mathrm{Im}f_l(s')}{s'-s} + \frac{1}{\pi}\int_{0}^{(m-\mu)^2} ds' \frac{\mathrm{Im}f_l(s')}{s'-s} + \Delta_l(s). \quad (5.26)$$

As in Eq. (3.8) $P_l(s)$ stands for the nucleon Born pole, the first integral is due to rescattering and comes from the right-hand cut, and the second integral comes from the crossed physical cut. These three terms can be evaluated in terms of the πN coupling constant and known πN phase shifts. $\Delta_l(s)$ contains the contributions from the circle cut $|s| = m^2 - \mu^2$ and $-\infty \leq s \leq 0$. The distant singularities, i.e., $-\infty \leq s \leq 0$ are approximated by a constant plus pole term, somewhere on the cut $-\infty \leq s \leq 0$, and the discontinuity across the circle cut is supposed to be dominated by the $\pi\pi$ contribution.

In the spirit of the effective-range formula, the discontinuity in $\mathrm{Im}A(t)$ [Eq. (5.19)] is replaced by a δ function at $t=t_1$, $t_1<0$. Thus

$$\mathrm{Im}A(t) = -\pi\Gamma\delta(t-t_1), \quad (5.27)$$

where Γ is a constant. Equation (5.19) gives

$$D(t) = 1 - \frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty} dt' \left(\frac{t'-4\mu^2}{t}\right)^{1/2} \frac{N(t')}{(t'-t_0)(t'-t)}, \quad (5.28)$$

where one subtraction has been made at $t=t_0$ and we have chosen $D(t_0)=1$. Substituting Eq. (5.27) into Eq. (5.19) and the result in (5.28), we obtain

$$D(t) = 1 - \frac{\Gamma}{\pi}(t-t_1)\int_{4\mu^2}^{\infty} dt' \times \left(\frac{t'-4\mu^2}{t'}\right)^{1/2} \frac{1}{(t'-t_1)^2(t'-t)}. \quad (5.29)$$

Integrating (5.29) gives

$$\mathrm{Re}D(t) = 1 + \frac{\Gamma}{\pi}\left(\frac{F(t)-F(t_1)}{t-t_1} - F'(t_1)\right), \quad (5.30)$$

where

$$F(t) = x\ln\left|\frac{1+x}{1-x}\right| \quad \text{for } t>4\mu^2 \text{ and } t<0$$
$$= (2/y)\arctan y \quad \text{for } 0<t<4\mu^2, \quad (5.31)$$

---

[18] P. Menotti, Nuovo Cimento **23**, 931 (1962).

[19] See J. Hamilton, in *Strong Interactions and High Energy Physics, Scottish Universities Summer School, 1963*, edited by R. G. Moorhouse (Oliver and Boyd, Edinburgh, 1964), p. 334; J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

[20] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **6**, 29 (1961).

[21] G. Lovelace, R. M. Heinz, and A. Donnachie, Phys. Letters **22**, 332 (1966).

[22] See, e.g., C. Kacser, P. Singer, and T. N. Truong, Phys. Rev. **137**, B1605 (1965).

[23] T. N. Truong, Phys. Rev. Letters **17**, 1102 (1966); K. Kang and D. J. Land, *ibid.* **18**, 503 (1967).

[24] R. Rockmore and T. Yao, Phys. Rev. Letters **18**, 501 (1967).

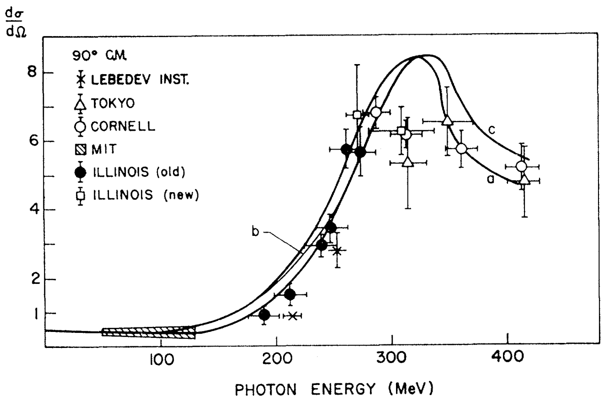

FIG. 2. Differential cross sections at 90° in the c.m. system. All cross sections are given in units of $r_p^2=(e^2/4\pi m)^2$, the square of the classical proton electromagnetic radius. Curve labeled (a) correspond to older calculations, without including the $\pi\pi$ interaction. Our results correspond to (b), without $\pi\pi$ interaction, and to (c), where this contribution is included, using the $\pi\pi$ phase shift of Hamilton *et al.* (Ref. 19).

with

$$x=[(t-4\mu^2)/t]^{1/2} \text{ and } y=[-t/(t-4\mu^2)]^{1/2}. \quad (5.32)$$

Since the right side of Eq. (5.26) can also be evaluated in terms of πN phase shifts, a fit to the discrepancies $\Delta_l(s)$ can now be made, and the parameters Γ and t_1 determined. The phase shifts obtained are positive, rise to a maximum of about 30°, and fall off to about 15° by $t=50\mu^2$. The scattering length obtained is

$$a_0 \lesssim 1.3. \quad (5.33)$$

B. Resonating $\pi\pi$ Phase Shift

A resonance in the $I=J=0$ channel has been suggested for several reasons.[21,23] We will make two subtractions in the D function of Eq. (5.28) and require the condition $\mathrm{Re}D(t_R)=0$, where $t=t_R$ is the resonance position. With a constant N function, $N=N_0$, Eq. (5.19) gives for $D(t)$

$$D(t)=D(t_0)+[(t-t_0)/(t_0-t_1)][D(t_0)-D(t_1)] -\frac{N_0}{\pi}(t-t_0)(t-t_1)\int_{4\mu^2}^{\infty}dt'\left(\frac{t'-4\mu^2}{t'}\right)^{1/2} \times[(t'-t)(t'-t_0)(t'-t_1)]^{-1}. \quad (5.34)$$

This integral can be done trivially as in Eq. (5.29), and the arbitrary parameters can be expressed in terms of t_R and the scattering length a_0. Using

$$[(t-4\mu^2)/t]^{1/2}\cot\delta_0(t)=N(t)/\mathrm{Re}D(t), \quad (5.35)$$

we obtain

$$\left(\frac{t-4\mu^2}{t}\right)^{1/2}\cot\delta_0(t)=\frac{1}{a_0}\left(1-\frac{t-4\mu^2}{t_R-4\mu^2}\right) +\frac{1}{\pi}\left[F(t)-\frac{t-4\mu^2}{t_R-4\mu^2}F(t_R)\right], \quad (5.36)$$

where $F(t)$ has been defined in Eq. (5.31).

C. $\pi\pi$ Phase Shift from ρ Exchange

In this model, the s-wave $\pi\pi$ interaction is supposed to be driven by ρ exchange in the crossed s and u channels. The crossing-symmetric Born amplitude for ρ exchange is[24]

$$-32\pi T_{\mathrm{Born}}(s,t,u)=\delta_{ab}\delta_{cd}f_\rho^2\left[\frac{t-s}{u-m_\rho^2}+\frac{t-u}{s-m_\rho^2}\right] +\delta_{ad}\delta_{bc}f_\rho^2\left[\frac{u-s}{t-m_\rho^2}+\frac{u-t}{s-m_\rho^2}\right] +\delta_{ac}\delta_{bd}\left[\frac{s-u}{t-m_\rho^2}+\frac{s-t}{u-m_\rho^2}\right], \quad (5.37)$$

where the $\rho\pi\pi$ coupling constant is $f_\rho^2/4\pi=2.16$. Projecting out the s wave, we get

$$N(\nu)_{\mathrm{Born}}=-\frac{f_\rho^2}{4\pi}\left\{\frac{(2\nu+\mu^2+m_\rho^2/4)}{\nu}Q_0\left(1+\frac{m_\rho^2}{2\nu}\right)-\tfrac{1}{2}\right\}, \quad (5.38)$$

where

$$\nu=\tfrac{1}{4}t-\mu^2. \quad (5.39)$$

The left-hand cut in (5.19) is now replaced by a pole,[25] so that the N function becomes

$$N(\nu)=-5\lambda+(\nu-\nu_0)B_0(\omega_0+\nu_0)/(\omega_0+\nu), \quad (5.40)$$

where λ is the value of $A(\nu)$ at the symmetry point $\nu=\nu_0=-\frac{2}{3}$, ω_0 gives the position of the pole, and B_0 is proportional to the residue. For $\lambda=0$, Eq. (5.39) is the

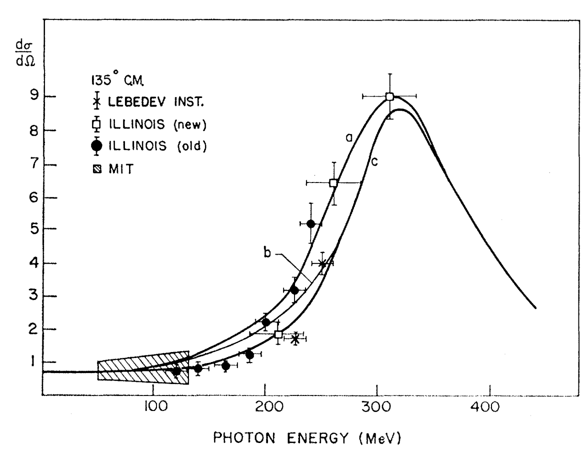

FIG. 3. Differential cross sections at 135° in the c.m. system, in units of $r_p^2=(e^2/4\pi m)^2$.

[25] B. R. Desai, Phys. Rev. Letters 6, 497 (1961).

one-pole approximation to Eq. (5.38). Substituting Eq. (5.39), into Eq. (5.28) we obtain

$$D(\nu)=1+[B_0(\omega_0+\nu_0)-5\lambda][F(\nu)-F(\nu_0)]/\pi, \quad (5.41)$$

where $F(\nu)$ is obtained from $F(t)$ by inserting Eq. (5.39) into Eq. (5.31). Rockmore and Yao obtain as their best fit a scattering length of $a_0 \gtrsim 0.8$, $\lambda \simeq -0.1$ and no resonance is required.

VI. SUMMARY AND CONCLUSIONS

A dispersion-theoretic analysis of low-energy Compton scattering has been given assuming that singularities far away from the physical region have a negligible effect. In this spirit the absorptive parts of the scattering amplitude have been calculated in terms of the nucleon and π^0 pole, photoproduction, and the $I=J=0$ $\pi\pi$ interaction. The latter has been estimated using some currently accepted models.

Our theoretical results are compared with experiment at 90° and 135° c.m. angle (see Figs. 2 and 3). Curves labeled (a) correspond to older calculations, without including the $\pi\pi$ interaction, e.g., Contogouris.[2] Our results correspond to (b), without $\pi\pi$ interaction and to (c), where this contribution is included, using the $\pi\pi$ phase shift of Hamilton *et al.*[19] We conclude that the discrepancy around the photoproduction threshold, where the experimental data were lower than the theoretical ones, can be accounted for once the $\pi\pi$ contribution is included. With the nonresonating $\pi\pi$ phase shift of Hamilton *et al.*[19] and the resonating one discussed under model 3, Sec. V., the agreement is essentially the same, if a resonance of width $\Gamma \lesssim 100$ MeV at an energy $E_R \lesssim 600$ MeV is excluded. The $\pi\pi$ scattering amplitude from ρ exchange gives too small a contribution to $\gamma p \to \gamma p$ due to the increasing value of the N function for $t \geq 4\mu^2$.

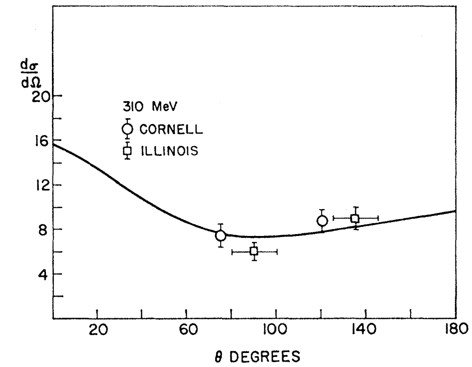

FIG. 4. Angular distribution at 310 MeV as a function of the angle in the c.m. system in units of r_p^2.

At 90° c.m. and resonance energy our values, as those of all other theoretical calculations, are higher than the results of Ref. 2. This can also be seen in Fig. 4, where the angular distribution is plotted at 310-MeV photon laboratory energy.

In conclusion it can be said that theory and experiment are in agreement at low energy except at 90° c.m. and resonance energy, but the experimental uncertainties should be reduced before a definite conclusion can be drawn.

ACKNOWLEDGMENTS

It is the author's pleasure to thank Professor Y. Nambu for his sponsorship and for having suggested an investigation of Compton scattering, and Professor G. Shaw for having made available to him the details of the work in Ref. 11. The numerical computations were carried out on an IBM 7094 computer at the Institute for Computer Research at the University of Chicago.