Rev. Letters **19**, 1064 (1967)]. The tree diagrams we refer to in Sec. IV are precisely what Schwinger called skeletal interactions [J. Schwinger, Phys. Rev. **158**, 1391 (1967)]. We understand that S. Coleman and B. Zumino have considered the problem we dealt with in Sec. IV [S. Coleman and B. Zumino (to be published)].

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Daughter Regge Trajectories in the Van Hove Model*

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The Van Hove model of Regge poles is generalized to include propagator self-energy insertions and used to study unequal-mass daughter trajectories. The first daughter trajectory is found to have negative slope at $t=0$.

I. INTRODUCTION

RECENTLY the study of Feynman diagrams has shed new light on the origin and behavior of Regge poles in relativistic quantum mechanics. Van Hove[1] has suggested a simple model in which the amplitude for Regge exchange is given by the sum of the one-particle exchange diagrams for the set of particles lying on an infinitely rising Regge trajectory. Durand[2] has emphasized the close correspondence between the daughter trajectories found by Freedman and Wang[3] in unequal-mass scattering and the lower spin components that are carried by off-mass-shell Feynman propagators for particles with spin.

We wish in this paper to show that the Van Hove model when studied for unequal external masses and generalized to include self-energy insertions on the propagators of the exchanged particles leads to and gives information about moving daughter trajectories. Our results, while model-dependent, suggest that only in accidental cases are the daughter trajectories expected to move parallel to the parent trajectory. In particular we find the first daughter has negative slope at $t=0$ for $\alpha_D(0)>-\frac{5}{2}$.

Lest the reader get lost below in the technical details of higher spin, let us first state the plan and simple physical ideas of our work. We first consider the unequal-mass scattering $m_1+m_1\rightarrow m_2+m_2$ computed with bare Feynman propagators for the exchanged particles. We find that the singularities at $t=0$ of the leading Regge-pole contribution are cancelled by fixed daughter poles. As is well known, fixed poles in the angular momentum plane are incompatible with (t-channel) unitarity. It is natural to hope, therefore, that when the Van Hove model is unitarized, the fixed daughter poles will turn into moving daughter trajectories. Our calculations show that this is precisely what happens, and we find an expression which determines the first daughter trajectory.

II. FIXED DAUGHTER POLES

We begin by studying the unequal-mass scattering $m_1+m_1\rightarrow m_2+m_2$ as $s\rightarrow\infty$ with momenta as defined in Fig. 1. In order to avoid undue complications we have throughout confined our attention to the leading and first daughter trajectories. The amplitude for the ex-

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[1] L. Van Hove, Phys. Letters **24B**, 183 (1967). Durand [Loyal Durand III, Phys. Rev. **161**, 1610 (1967)] has studied the smoothness conditions which are required in order to obtain Regge-type behavior from an infinite set of t-channel diagrams. In particular he has pointed out that the "particles" which are exchanged need not actually occur as physical resonances. They can be poles on the second sheet with negative mass squared as would occur for trajectories which turn over at some finite value of t. Hence, the requirement of infinitely rising trajectories is not necessary for obtaining Regge behavior. The authors wish to thank Professor Durand for helpful discussions on this point.

[2] Loyal Durand III, Phys. Rev. **154**, 1537 (1967).

[3] D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1956 (1967).

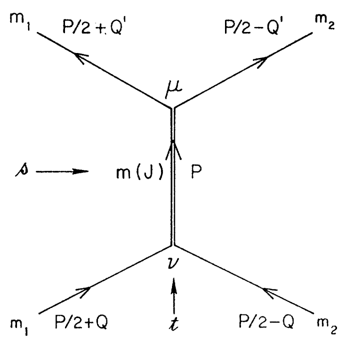

FIG. 1. Spin-J exchange contribution to $m_1+m_1 \to m_2+m_2$ scattering.

change of a spin-J particle is[4]

$$\mathfrak{M}(J)=\frac{g^2(J)b(J)Q'^{\mu_1}\cdots Q'^{\mu_J}}{m^2(J)-P^2}\times(-1)^J\Gamma_{\mu_1\cdots\mu_J;\,\nu_1\cdots\nu_J}(m^2(J))Q^{\nu_1}\cdots Q^{\nu_J}\,,\quad(1)$$

where

$$b(J)=(2J+1)!!/J!=(2J+1)!/2^J(J!)^2$$

and $(-1)^J\Gamma^J(m^2(J))$ is the numerator of the spin-J Feynman propagator.[5] The argument m^2 in $\Gamma^J(m^2)$ means that the momentum factors appear as $P_{\mu_i}P_{\nu_j}/m^2$ rather than $P_{\mu_i}P_{\nu_j}/P^2$. Thus for $P^2\neq m^2(J)$, $\mathfrak{M}(J)$ does not describe pure spin-J exchange but has in addition spin $J-1$, $J-2$, $\cdots$ components. These are present in precisely the right amounts to guarantee that $\mathfrak{M}(J)$ is well behaved at $P^2=0$.

Equation (1) may be rewritten[2] in terms of a Legendre polynomial

$$\mathfrak{M}(J)=\frac{(2J+1)g^2(J)\bar{q}^{2J}P_J(\bar{z})}{m^2(J)-t}\,,\quad(2)$$

where

$$t=P^2\,,$$

$$4\bar{q}^2=-4\left(Q^2-\frac{(P\cdot Q)^2}{m^2(J)}\right)=\left[t-2(m_1^2+m_2^2)+\frac{(m_1^2-m_2^2)^2}{m^2(J)}\right]\quad(3)$$

and

$$\bar{z}=-\left[Q\cdot Q'-\frac{P\cdot Q\;P\cdot Q'}{m^2(J)}\right]\Big/\bar{q}^2=\left[s-u+\frac{(m_1-m_2^2)^2}{m^2(J)}\right]\Big/4\bar{q}^2.\quad(4)$$

In order to reveal the angular momentum content of Eq. (2) it is useful to expand it in terms of Legendre functions of argument z, where z is the t-channel center-of-mass scattering angle:

$$\bar{q}^{2J}P_J(\bar{z})=q^{2J}P_J(z)-(2J-1)\frac{(m_1^2-m_2^2)^2}{4t}\times\frac{m^2(J)-t}{m^2(J)}q^{2(J-1)}P_{J-1}(z)+\cdots.\quad(5)$$

The quantities q^2 and z are given by Eqs. (3) and (4) with $m^2(J)$ replaced by t.

The amplitude for Regge-pole exchange is given then, according to Van Hove, by[1,6]

$$R=\mathfrak{M}(\text{Regge})=\sum_{J=0}^{\infty}\mathfrak{M}(J)=\sum_{J=0}^{\infty}\left[\frac{(2J+1)g^2(J)}{m^2(J)-t}q^{2J}P_J(z)\right]-\sum_{J=1}^{\infty}\left[\frac{(2J+1)(2J-1)g^2(J)(m_1^2-m_2^2)^2q^{2(J-1)}}{(4t)m^2(J)}P_{J-1}(z)\right]+\cdots\quad(6a)$$

$$=\frac{i}{2}\int_C\frac{dJ(2J+1)}{\sin\pi J}q^{2J}P_J(-z)\left[\frac{g^2(J)}{m^2(J)-t}-\frac{g^2(J+1)(2J+3)(m_1^2-m_2^2)^2}{(4t)m^2(J+1)}+\cdots\right].\quad(6b)$$

We assume that the coupling $g^2(J)$ has no singularities which prevent us from opening the contour C from its original position about the $\text{Re}J\geq 0$ axis to some vertical line in the left-hand J plane. The amplitude then takes

[4] Our rules for vertices follow from an effective interaction Hamiltonian

$$H_I=g(J)[b(J)]^{1/2}\Psi_{\mu_1\cdots\mu_J}(x)\phi_1(x)\times\left(\frac{\overrightarrow{\partial}_{\mu_1}-\overleftarrow{\partial}_{\mu_1}}{2}\right)\left(\frac{\overrightarrow{\partial}_{\mu_2}-\overleftarrow{\partial}_{\mu_2}}{2}\right)\cdots\left(\frac{\overrightarrow{\partial}_{\mu_J}-\overleftarrow{\partial}_{\mu_J}}{2}\right)\phi_2(x),$$

where $\Psi_{\mu_1\cdots\mu_J}$, ϕ_1 ,and ϕ_2 are the Hermitian fields of the particles $m(J)$, m_1, and m_2, respectively. The factor of $[b(J)]^{1/2}$ has been introduced to simplify subsequent formulas. It seems natural in the Van Hove model to assume that $g^2(J)$ rather than $g^2(J)b(J)$ has good analyticity in J.

[5] If we define $G_{\mu\nu}=g_{\mu\nu}-P_\mu P_\nu/m^2$, then

$$\Gamma^0=1,\ \Gamma^1=G_{\mu\nu},\ \Gamma^2=\tfrac{1}{2}(G_{\mu_1\nu_1}G_{\mu_2\nu_2}+G_{\mu_1\nu_2}G_{\mu_2\nu_1})-\tfrac{1}{3}G_{\mu_1\mu_2}G_{\nu_1\nu_2},$$

etc. For an expression for general J, see footnote 8 of Ref. 2.

the form

$$R=\frac{-g^2(\alpha(t))[2\alpha(t)+1]}{\sin\pi\alpha(t)}\pi\frac{d\alpha(t)}{dt}q^{2\alpha(t)}P_{\alpha(t)}(-z)+\frac{g^2(\alpha(0))[2\alpha(0)+1]}{\sin(\pi[\alpha(0)-1])}\pi\frac{d\alpha(0)}{dt}[2\alpha(0)-1]\times\frac{(m_1{}^2-m_2{}^2)^2}{4t}q^{2[\alpha(0)-1]}P_{\alpha(0)-1}(-z)+\cdots. \quad (7)$$

The first term in Eq. (7) is the contribution of the leading Regge trajectory at $m^2(J)-t=0$, i.e., at $J=\alpha(t)$. The second term of Eq. (7) arises from the pole in the integrand of Eq. (6b) at $m^2(J+1)=0$. Its form is precisely that of the first daughter trajectory. Rather than a true moving trajectory, however, we have a fixed daughter pole at $J=\alpha(0)-1$. By carrying the expansion further in Eq. (6b), it is easy to show that the second, third, etc., daughter trajectories are also fixed poles in the simple Van Hove model.[7] The presence of the daughter poles means that the usual high-energy behavior is obtained even at $t=0$.

III. MOVING DAUGHTER TRAJECTORIES

Let us now extend the Van Hove model so that it satisfies two-particle unitarity in the t channel. The technique for doing this is well known. We must replace the bare Feynman propagators in Eq. (1) by the full propagators.

The full propagator for a particle of integer spin J is given by (see Fig. 2):

$$\Delta_{\mu;\nu}{}^J(t)=\frac{\Gamma_{\mu;\,\nu}{}^J(m^2)}{t-m^2}+\frac{\Gamma_{\mu;\,\lambda}{}^J(m^2)}{t-m^2}\sum{}_J{}^{\lambda;\sigma}(t)\Delta_{\sigma;\,\nu}{}^J(t)\,, \quad (8)$$

where μ stands for the set of indices $\mu_1, \mu_2, \mu_3\cdots\mu_J$, etc.

The self-energy function $\sum_J{}^{\lambda;\,\sigma}(t)$ is symmetric under interchange of any of its indices. It can be written in terms of $J+1$ invariant amplitudes in the form

$$\sum{}_J{}^{\lambda;\sigma}(t)=-\frac{g^2(J)}{J!}\tilde{A}_J(t)\sum_{r=0}^{[J/2]}\{g^{\lambda_1\sigma_1}g^{\lambda_2\sigma_2}\cdots g^{\lambda_J\sigma_J}\}_r -\frac{g^2(J)}{J!}(2J+1)\tilde{B}_J(t)\sum_{i=1}^{J}\sum_{r=0}^{[J/2]}\{P^{\lambda_i}P^{\sigma_i}g^{\lambda_1\sigma_1}\cdots[g^{\lambda_i\sigma_i}]\cdots g^{\lambda_J\sigma_J}\}_r-\frac{g^2(J)}{J!}\frac{(2J+1)(2J-1)}{3}\tilde{C}_J(t)\times\sum_{i\neq j}^{J}\sum_{r=0}^{[J/2]}\{P^{\lambda_i}P^{\sigma_i}P^{\lambda_j}P^{\sigma_j}g^{\lambda_1\sigma_1}\cdots[g^{\lambda_i\sigma_i}]\cdots[g^{\lambda_j\sigma_j}]\cdots g^{\lambda_J\sigma_J}\}_r\cdots, \quad (9)$$

where $[g^{\lambda_i\sigma_i}]$ means that the symbol $g^{\lambda_i\sigma_i}$ does not appear. The notation $\{\cdots\}_r$ is that of Durand.[2,8] The invariant amplitudes $\tilde{A}_J(t)$, $\tilde{B}_J(t)$, $\cdots$ have no kinematic singularities. Each of them is an analytic function of t with a cut running from $(m_1+m_2)^2$ to infinity. In particular, there are no singularities at $t=0$ nor are there any relations among the amplitudes at that point.

In order to simplify the algebra, it is convenient to introduce a different set of invariant amplitudes by writing $\sum_J{}^{\lambda;\,\sigma}(t)$ in terms of the orthogonal projection operators

$$\theta^{\lambda\sigma}=g^{\lambda\sigma}-P^\lambda P^\sigma/P^2\,,\qquad \tau^{\lambda\sigma}=P^\lambda P^\sigma/P^2. \quad (10)$$

We have

$$\sum{}_J{}^{\lambda;\sigma}=-\frac{g^2(J)}{J!}A_J(t)\sum_{r=0}^{[J/2]}\{\theta^{\lambda_1\sigma_1}\theta^{\lambda_2\sigma_2}\cdots\theta^{\lambda_J\sigma_J}\}_r -\frac{g^2(J)}{J!}(2J+1)B_J(t)\sum_{i=1}^{J}\sum_{r=0}^{[J/2]}\{\tau^{\lambda_i\sigma_i}\theta^{\lambda_1\sigma_1}\cdots[\theta^{\lambda_i\sigma_i}]\cdots\theta^{\lambda_J\sigma_J}\}_r-\frac{g^2(J)}{J!}\frac{(2J+1)(2J-1)}{3}C_J(t)\times\sum_{i\neq j}^{J}\sum_{r=0}^{[J/2]}\{\tau^{\lambda_i\sigma_i}\tau^{\lambda_j\sigma_j}\theta^{\lambda_1\sigma_1}\cdots[\theta^{\lambda_i\sigma_i}]\cdots[\theta^{\lambda_j\sigma_j}]\cdots\theta^{\lambda_J\sigma_J}\}_r\cdots, \quad (11)$$

where

$$A_J(t)=\tilde{A}_J(t),\quad B_J(t)=\frac{\tilde{A}_J(t)}{2J+1}+t\tilde{B}_J(t),\quad C_J(t)=\frac{3}{(2J+1)(2J-1)}\tilde{A}_J(t)+\frac{6}{(2J-1)}t\tilde{B}_J(t)+t^2\tilde{C}_J(t)\,,\ \cdots. \quad (12)$$

[6] For convenience we will ignore the trivial complications of signature; it can easily be added at the end. Since the daughter trajectory serves to cancel a term from the leading trajectory it clearly must have the same phase, and hence the opposite signature.

[7] J. C. Taylor (to be published) has already examined the Van Hove model in the unequal-mass case without self-energy insertions. The case studied by Taylor has also been studied independently by both J. D. Bjorken and M. B. Halpern (private communications). If one does not expand $P_J(\bar{z})$ in terms of $P_J(z)$, as we do in Eq. (5), one finds that the Regge high-energy behavior is achieved at $t=0$ by virtue of a contribution from the J-plane cut of $P_J(\bar{z})$. This cut is not, however, a cut in the complex angular momentum (Regge) plane of the partial-wave amplitude. Instead the structure is that of a set of fixed daughter poles as we have described.

[8] The bracket $\{g^{\lambda_1\sigma_1}g^{\lambda_2\sigma_2}\cdots g^{\lambda_J\sigma_J}\}_r$ is a product of J, g symbols symmetrized with respect to either the λ_i or σ_i. In r distinct pairs, the left index of one g is interchanged with the right index of the other $g^{\lambda_i\sigma_i}g^{\lambda_j\sigma_j}\rightarrow g^{\lambda_i\lambda_j}g^{\sigma_i\sigma_j}$. This interchange is to be done in all ways which give distinct terms. In this latter regard our notation differs from that of Durand.

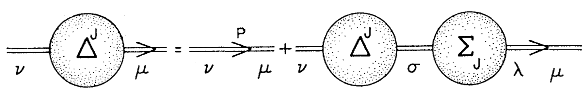

FIG. 2. The full propagator for a spin-J particle.

The amplitudes $A_J(t), B_J(t), \cdots$ have the same analytic properties as the twiddle amplitudes. However, there are J relations among them at $t=0$:

$$A_J(0)=(2J+1)B_J(0)$$
$$=\tfrac{1}{3}(2J+1)(2J-1)C_J(0)=\cdots. \quad (13)$$

These relations arise simply because we have expressed $\sum_J{}^{\lambda;\,\sigma}(t)$ in terms of the projection operators $\theta^{\mu\nu}$ and $\tau^{\mu\nu}$ which have poles at $t=0$. Equation (13) merely ensures that $\sum_J{}^{\lambda;\,\sigma}(t)$ itself has no singularity at $t=0$.

To study the leading Regge trajectory and the first daughter it suffices to extract the spin-J and spin-$(J-1)$ parts of $\Delta_{\mu;\,\nu}{}^J$. To this end we expand $\Gamma_{\mu;\,\nu}{}^J(m^2)$ in the form

$$\Gamma_{\mu;\,\nu}{}^J(m^2)=\Gamma_{\mu;\,\nu}{}^J(P^2)-\frac{1}{J}\left(\frac{t-m^2}{m^2}\right)\sum_{i,j}\tau_{\mu_i\nu_i}$$
$$\times\Gamma_{\mu_1\cdots[\mu_i]\cdots\mu_J;\,\nu_1\cdots[\nu_j]\cdots\nu_J}{}^{J-1}(P^2)$$
$$+(\text{operators which project onto states with angular momentum } J-2,\ J-3,\cdots,0), \quad (14)$$

where the subscripts in square brackets do not occur. The first two terms on the right-hand side of Eq. (14) are projection operators onto states of angular momentum J and $J-1$, respectively. They are orthogonal to each other and to all other terms in the expansion of $\Gamma_{\mu;\,\nu}{}^J(m^2)$.

If we write

$$\Delta_{\mu;\,\nu}{}^J(t)=D_J(t)\Gamma_{\mu;\,\nu}{}^J(P^2)-D_{J-1}(t)\frac{1}{J}$$
$$\times\sum_{i,j}\tau_{\mu_i\nu_j}\Gamma_{\mu[\mu_i];\nu[\nu_j]}{}^{J-1}(P^2)+\cdots \quad (15)$$

and substitute Eqs. (11), (14), and (15) into Eq. (8) we find

$$D_J(t)=\frac{1}{t-m^2(J)+g^2(J)A_J(t)}, \quad (16a)$$

$$D_{J-1}(t)=\frac{1}{m^2(J)-(2J+1)g^2(J)B_J(t)}. \quad (16b)$$

It is clear from Eqs. (16a) and (16b) that the relation $(2J+1)B_J(0)=A_J(0)$ is precisely the one required to prevent Δ^J from having a $1/t$ singularity. Higher-order $1/t$ singularities are also cancelled by virtue of the other relations of Eq. (13). The full propagator $\Delta_{\mu;\,\nu}{}^J(t)$ has a simple pole at $t=M^2(J)$ the physical (renormalized) mass with a residue that fixes the coupling constant renormalization. These effects only come from D_J and are as follows:

$$M^2(J)=m^2(J)-g^2(J)A_J(M^2(J)), \quad (17)$$

$$G^2(J)=g^2(J)[1+g^2(J)(dA_J/dt)(M^2(J))]^{-1}. \quad (18)$$

In Eq. (18), $G(J)$ denotes the renormalized coupling constant.

These renormalizations are most easily handled by writing a dispersion relation for $A_J(t)$ twice subtracted at $t=M^2(J)$. We have then

$$g^2(J)D_J(t)=\frac{G^2(J)}{[t-M^2(J)]\{1+[t-M^2(J)]G^2(J)\bar{A}_J(t)\}}, \quad (19)$$

where

$$\bar{A}_J(t)=\frac{1}{\pi}\int_{(m_1+m_2)^2}^{\infty}\frac{dt'\,\mathrm{Im}A_J(t')}{[t'-M^2(J)]^2(t'-t)}. \quad (20)$$

It is convenient also to write a dispersion relation for $B_J(t)$ once subtracted at $t=0$

$$B_J(t)=A_J(0)/(2J+1)+t\bar{B}_J(t),$$
$$\bar{B}_J(t)=\frac{1}{\pi}\int_{(m_1+m_2)^2}^{\infty}dt'\frac{\mathrm{Im}B_J(t')}{t'(t'-t)}, \quad (21)$$

and Eq. (13) has been used to fix the subtraction constant. Finally we write

$$g^2(J)D_{J-1}(t)=\frac{G^2(J)}{M^2(J)[1-M^2(J)G^2(J)\bar{A}_J(0)]-(2J+1)tG^2(J)\bar{B}_J(t)}. \quad (22)$$

For Regge exchange we have in place of Eq. (6b)

$$R=+\frac{i}{2}\int_C\frac{dJ(2J+1)q^{2J}P_J(-z)}{\sin\pi J}\left[\frac{G^2(J)}{[M^2(J)-t]\{1+[t-M^2(J)]G^2(J)\bar{A}_J(t)\}}\right.$$
$$\left.-\frac{(m_1{}^2-m_2{}^2)^2(2J+3)G^2(J+1)}{4t\{M^2(J+1)[1-M^2(J+1)G^2(J+1)\bar{A}_{J+1}(0)]-(2J+3)G^2(J+1)tB_{J+1}(t)\}}+\cdots\right]. \quad (23)$$

When we open the contour C we pick up the leading Regge pole at $J=\alpha(t)$, where $M^2(\alpha(t))=t$, from the first term on the right-hand side of Eq. (23). In principle we could compute $M^2(J)$ from Eq. (17), and hence $\alpha(t)$, once $m^2(J)$ and $g^2(J)$ were given. Since $A_J(M^2)$ has a cut for $M^2\geq(m_1+m_2)^2$ we note that the resulting trajectory would properly become complex above threshold, $t=(m_1+m_2)^2$. Here we will simply take $\alpha(t)$ as given and, moreover, assume sufficient analyticity in $G^2(J)$ to permit deformation of the contour.

From the second term on the right-hand side of Eq. (23) we pick up a pole at

$$M^2(J+1)[1-M^2(J+1)\bar{A}_{J+1}(0)] \\ -(2J+3)G^2(J+1)t\bar{B}_{J+1}(t)=0. \quad (24)$$

Solving Eq. (24) for J gives the trajectory of the first daughter[9] $J=\alpha_D(t)$.

While it is essentially impossible to solve for $\alpha_D(t)$ exactly some properties are clear. At $t=0$, Eq. (24) is satisfied by $M^2(J+1)=0$ which gives the expected result $\alpha_D(0)=\alpha(0)-1$. This follows directly from Eq. (13), i.e., from the fact that $\Delta_{\mu;\,\nu}{}^J(0)$ is finite.

The factor $t\bar{B}_{J+1}(t)$ in Eq. (24) guarantees that the daughter trajectory will move as a function of t. From the over-all sign of the second term in Eq. (23) it is clear that if $\alpha_D(t)$ were to reach zero, it would give rise to a ghost state. Thus it is of interest to study the slope of α_D. For small t we can write

$$\alpha_D(t)=\alpha(0)-1+\alpha_D'(0)t+\cdots, \quad (25)$$

and we find

$$\alpha_D'(0)=\alpha'(0)G^2(\alpha(0))[2\alpha(0)+1]\bar{B}_{\alpha(0)}(0). \quad (26)$$

In order to proceed it is necessary to adopt a model which will enable us to say something about $\bar{B}_J(t)$. Since we are interested in small values of t we shall make the physical assumption that the dispersion integral for $\bar{B}_J(t)$ [Eq. (21)] is dominated by the two-particle intermediate states. In other words, we shall require that the scattering amplitude satisfies two-particle unitarity exactly in the t channel, but neglect multiparticle intermediate states. This requirement uniquely determines $\mathrm{Im}\sum_J{}^{\lambda;\,\sigma}(t)$. We have[10]

$$\mathrm{Im}\sum_J{}^{\lambda;\sigma}(t)=\frac{(-)^{J+1}}{8\pi^2}g^2(J)b(J)\int d^4k\;\delta_+((\tfrac{1}{2}P+k)^2-m_1^2) \\ \times\delta_+((\tfrac{1}{2}P-k)^2-m_2^2)k^{\lambda_1}k^{\lambda_2}\cdots k^{\lambda_J}k^{\sigma_1}\cdots k^{\sigma_J}, \quad (27)$$

so that

$$\mathrm{Im}A_J(t)=q(t)^{2J+1}/8\pi t^{1/2}, \\ \mathrm{Im}B_J(t)=-(m_1^2-m_2^2)^2q(t)^{2J-1}/32\pi t^{3/2}, \quad (28)$$

where

$$4q^2(t)=t-2(m_1^2+m_2^2)+(m_1^2-m_2^2)^2/t.$$

For $J\geq 2$ the dispersion integrals for $\bar{A}_J(t)$ and $\bar{B}_J(t)$ will diverge. In a more sophisticated model this divergence is presumably removed by the form factor associated with the $m(J)\leftrightarrow m_1+m_2$ vertex. Such a form factor is generated by the many-particle intermediate-state contributions to the vertex. Here we shall merely crudely simulate this by cutting off all divergent integrals.

Since $\alpha(0)\leq 1$ we can let the cutoff go to infinity in Eq. (26) and then have

$$\alpha_D'(0)=-\alpha'(0)G^2(\alpha(0))[2\alpha(0)+1](m_1^2-m_2^2)^2 \\ \times\frac{1}{32\pi}\int_{(m_1+m_2)^2}^{\infty}\frac{dt'q(t')^{2\alpha(0)-1}}{t'^{5/2}}. \quad (29)$$

So, for $\alpha(0)>-\frac{1}{2}$, the slope of the first daughter is negative at $t=0$, and is therefore unlikely to give rise to a ghost.[11] It should be noted that the sign of $\mathrm{Im}B_J(t)$ could not be changed by including multiparticle intermediate states. The sign of $\alpha_D'(0)$ can only be changed if it is necessary to make a second subtraction in $B_J(t)$.[12]

From Eqs. (27) and (28) we see that $\mathrm{Im}B_J(t)$ will always be proportional to $(m_1^2-m_2^2)^2$ as long as we take into account only two-particle intermediate states. As a result, in the equal-mass case the daughter trajectory will only move if we take into account multiparticle effects.

[9] In obtaining our results we have assumed that the factor $\{1+[t-m^2(J)]G^2(J)\bar{A}_J(t)\}$ in the first denominator of Eq. (23) never vanishes. This is satisfied provided $g^2(J)$ considered as a function of the renormalized coupling constant $G^2(J)$, through Eq. (18), satisfies $g^2(J)>0$. Such a requirement sets an upper bound on $G^2(J)$ and is the usual requirement that the theory has no ghosts. We remark also that this same condition guarantees that the coefficient of $M^2(J+1)$ in Eq. (24) is always positive and hence that the daughter term of Eq. (23) is also ghost free.

[10] The requirement that the amplitude satisfies two-particle unitarity is equivalent, in the language of Feynman diagrams, to including only the contribution of the bubble diagrams in $\Sigma_J{}^{\lambda;\,\sigma}(t)$. In this approximation $\Sigma_J{}^{\lambda;\,\sigma}(t)$ is given by the divergent integral

$$\Sigma_J{}^{\lambda;\,\sigma}(t)=\frac{i(-)^J}{(2\pi)^4}g^2(J)b(J)\int\frac{d^4k\;k^{\lambda_1}k^{\lambda_2}\cdots k^{\lambda_J}k^{\sigma_1}\cdots k^{\sigma_J}}{[(\frac{1}{2}P+k)^2-m_1^2][(\frac{1}{2}P-k)^2-m_2^2]},$$

Proceeding formally we could obtain Eq. (13) from the $O(4)$ invariance of the Feynman integral at $t=0$, and Eqs. (27) and (28) from Cutkosky's rules. It should be noted that the requirement that the amplitude satisfy two-particle unitarity does not determine $\mathrm{Im}\Sigma_J{}^{\lambda;\,\sigma}(t)$ uniquely since one could always add to the interaction Hamiltonian given in footnote 4 terms of the form

$$H_I'=g'(J)[b(J)]^{1/2}\partial^{\mu_1}\Psi_{\mu_1\mu_2\cdots\mu_J}(x) \\ \times\phi_1(x)\left(\frac{\overrightarrow{\partial}_{\mu_2}-\overleftarrow{\partial}_{\mu_2}}{2}\right)\cdots\left(\frac{\overrightarrow{\partial}_{\mu_J}-\overleftarrow{\partial}_{\mu_J}}{2}\right)\phi_2(x).$$

Such terms would contribute to all of the self-energy functions except $A_J(t)$. Although these terms would change the magnitude of $\alpha_D'(0)$, it is easy to see that they would not change its sign. They would also leave our other qualitative results unchanged.

[11] This result actually holds for intercepts of the leading trajectory down to $\alpha(0)>-\frac{3}{2}$. To see this, note that the integral in Eq. (29) has a simple pole at $\alpha(0)=-\frac{1}{2}$ coming from the lower limit of integration. Thus the combination

$$(2\alpha(0)+1)\int_{(m_1+m_2)^2}^{\infty}\frac{dt'q(t')^{2\alpha(0)-1}}{t'^{5/2}}$$

is positive for $1\geq\alpha(0)>-\frac{3}{2}$. For $-\frac{3}{2}>\alpha(0)>-\frac{5}{2}$ the first daughter has positive slope at $t=0$, but the pole at $J=-\frac{3}{2}$ of $\bar{B}_J(0)$ prevents $\alpha_D(0)$ from ever getting above $-\frac{5}{2}$. So again the attempt of the first daughter trajectory to introduce a ghost into the theory is thwarted.

[12] It should be emphasized that the fact we write the dispersion relation for $B_J(t)$ with one subtraction is not just a luxury, but is also a necessity. Equation (28) shows that $\mathrm{Im}B_J$ is negative and proportional to $(m_1^2-m_2^2)^2$. Since $\mathrm{Im}A_J$ is positive and does not vanish as $m_1\to m_2$ it is impossible to satisfy the relation $A_J(0)=(2J+1)B_J(0)$ with unsubtracted dispersion relations for both A_J and B_J.

In general it is difficult to say much about $\alpha_D(t)$ away from $t=0$. In the weak-coupling limit we can solve Eq. (24) to first order in G^2:

$$\alpha_D(t)=\alpha(0)-1-\alpha'(0)G^2(\alpha(0))[2\alpha(0)+1](m_1{}^2-m_2{}^2)^2 \times\frac{t}{32\pi}\int_{(m_1+m_2)^2}^{\infty}dt'\frac{q(t')^{2\alpha(0)-1}}{t'^{3/2}(t'-t)}. \quad (30)$$

In this case we note that as $t\rightarrow\pm\infty$ the daughter trajectory goes to a constant even though the leading trajectory may be infinitely rising

$$\alpha_D(+\infty)=\alpha_D(-\infty)=\alpha(0)-1+\alpha'(0)G^2(\alpha(0)) \times\frac{(m_1{}^2-m_2{}^2)^2}{32\pi}\int_{(m_1+m_2)^2}^{\infty}\frac{dt'q(t')^{2\alpha(0)-1}}{t'^{3/2}}, \quad (31)$$

provided $\alpha(0)<1$. For the case $\alpha(0)-1$ the dispersion integral in Eq. (31) is logarithimically divergent, indicating a sensitivity to the detailed behavior at large t about which we can say nothing with confidence.

IV. CONCLUSIONS

We have generalized the Van Hove model of Regge poles and used it to study the first daughter trajectory away from $t=0$. It should be emphasized that in our model the existence of daughter trajectories at $t=0$ depends only on the facts that the full propagator for a spin-J particle has a simple pole at $t=M^2(J)$ with residue $\Gamma_{\mu;\,\nu}{}^J(M^2(J))$ and that the self-energy functions are finite at $t=0$. In order to study the daughter trajectories away from $t=0$ it is necessary to adopt a model which will enable us to calculate the self-energy functions. We have made the approximation that the self-energy functions are dominated at $t=0$ by the two-particle intermediate-state contributions. Under this assumption the first daughter trajectory has a negative slope at $t=0$. This eliminates the worry that this daughter trajectory would introduce a ghost state should it cross $\alpha_D(t)=0$.

The mass dependence of our results is perhaps worthy of a few comments. If the external masses are set equal, the daughter trajectories are uncoupled from the scattering amplitude. On the other hand, if the internal particle masses are set equal, the daughters become fixed poles in the angular momentum plane. This is not surprising since a model with unequal-mass external particles and equal-mass internal particles violates t-channel unitarity. Similar behavior was also obtained by Swift[13] who studied the Bethe-Salpeter equation for unequal-mass scattering. Also, like Swift, we find at $\alpha_D'(0)\sim G^2\alpha'(0)$ with the great difference that $\alpha_D'(0)<0$ in our case.[11]

It is trivial to extend our results to the general case of four unequal masses $m_1+m_3\rightarrow m_2+m_4$. In this case the self-energy functions A_J, B_J, $\cdots$ receive additive contributions from the thresholds at $(m_1+m_2)^2$ and $(m_3+m_4)^2$. The general properties of the resulting first daughter trajectory are completely unchanged.

The second and further daughter trajectories can of course also be studied by our method. For the second daughter the relevant parts of the self-energy equation [Eq. (8)] reduce to a 2×2 matrix equation. For further daughters the complexity escalates rapidly. Since the second daughter trajectory will involve, among other things, the function $C_J(t)$ [Eq. (11)] whose imaginary part is positive, one may expect it to have positive slope at $t=0$. It will also be interesting to study the generalized Van Hove model in cases in which the external particles have spin. In such cases the fixed conspirator poles found by Taylor[7] will turn into moving conspirator trajectories.

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[13] Arthur R. Swift, Phys. Rev. Letters **18**, (1967); Phys. Rev. **166**, 1621 (1968).