Phenomenological Lagrangian for Field Algebra, Hard Pions, and Radiative Corrections

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The phenomenological Lagrangian approach to the current-field algebra, and its advantages and defects, are discussed. The computational rules are stated, and justification is offered. Based on the phenomenological field-algebra Lagrangian, we consider the electromagnetic effects on the pion mass (for finite pion mass), the ρ-meson mass, and the pion β decay.

I. INTRODUCTION

RECENTLY there have been several papers[1-7] dealing with the phenomenological Lagrangian approach to current[8] (or field[9]) algebra. In view of this, we must offer a raison d'être for still another paper on the subject.[10] The purpose of this paper is to spell out, in a manner as precise as possible, what phenomenological Lagrangians are intended to be used for, and what the computational rules are. It is our feeling that these points are treated inadequately in the extant literature. We shall offer a justification for the computation rules, which are designed to fulfill a limited set of requirements consistent with the viewpoints of the field algebra, and of the hypothesis of partially conserved axial-vector current (PCAC).[11,12] For the sake of lucidity we shall consider a particularly simple model, in which the world is made of pions, ρ mesons and $A1$ mesons; but many results of this paper (particularly of Sec. IV) are of greater generality than the presentation in the context of this model might indicate. We shall not discuss functional transformations on the fields and their ramifications—a subject discussed exhaustively elsewhere.[4,5]

Broadly speaking, a phenomenological Lagrangian, which incorporates the current algebra and PCAC, is a device for reproducing the current-algebra results within the framework of the conventional perturbative Lagrangian field theory. The Lagrangian one writes down in this context is in no sense to be understood as the basis of a complete field theory of the subparticle world; rather the machinery of the Lagrangian field theory is utilized only to the extent of constructing "smooth" off-shell amplitudes that satisfy constraints imposed on them by the current-field algebra and PCAC.

At this point, a question arises inevitably as to whether the Lagrangian approach is anything more than an algorithm for reproducing current-field algebra results. As developed so far, it is nothing but that, although it is a useful device not only as a computational short cut but also, as has been emphasized elsewhere,[5] as a means of eliciting ambiguities and shortcomings of the usual analyses based on current algebra. It is however hoped, perhaps too optimistically, that the phenomenological Lagrangian method might provide us with a formalism for a more satisfactory description of particle phenomena; by this we mean, particularly, the possibility of implementing unitarity together with low-energy theorems derived from current-field algebra. Although we do not expect that the present approach will lead to a fundamental understanding of the laws that govern the subparticle world, we may, and do, hope to achieve a limited goal of unraveling and exploiting regularities that are manifest in low-energy particle phenomena.

The plan of this article is as follows. We shall concentrate on the dynamics of pions and the chiral $SU(2)\otimes SU(2)$ symmetry. In Sec. II, discussion is made of the construction of a phenomenological Lagrangian that consists of the pion fields only and satisfies the $SU(2)\otimes SU(2)$ algebra and the PCAC condition. This is preliminary to ensuing discussions. Applications of this Lagrangian have already been reported elsewhere.[5] In Sec. III, a Lagrangian is constructed, which includes pions, ρ mesons, and isotriplet axial-vector mesons (A_1 mesons), and satisfies the field algebra and PCAC. Computation rules for matrix elements are presented in Sec. IV, which ensure that the resulting amplitudes satisfy the generalized Ward identities of the kind discussed by Schnitzer and Weinberg,[13] and other constraints of the field algebra

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[1] S. Weinberg, Phys. Letters **18**, 188 (1967).
[2] J. Schwinger, Phys. Letters **24B**, 473 (1967); and (to be published).
[3] J. A. Cronin, University of Chicago Report, 1967 (unpublished).
[4] J. Wess and B. Zumino, Phys. Rev. **163**, 1727 (1967).
[5] W. A. Bardeen and B. W. Lee, Canadian Summer Institute Lectures (to be published by W. A. Benjamin, Inc., New York, 1967).
[6] L. S. Brown, Phys. Rev. **163**, 1802 (1967).
[7] P. Chang and F. Gürsey, Phys. Rev. **164**, 1752 (1967).
[8] M. Gell-Mann, Physics **1**, 63 (1964).
[9] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).
[10] We would like to record that, only near or after the completion of this work, we came to know of the works of J. Schwinger (to be published), Ref. 2; J. Wess and B. Zumino, Ref. 4; P. Chang and F. Gürsey, Ref. 7; G. C. Wick and B. Zumino, Ref. 28; M. Halpern and G. Segrè, Ref. 29; V. Weisskopf, Ref. 34.
[11] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **39**, 703 (1960) [English transl.: Soviet Phys.—JETP **12**, 492 (1961)].
[12] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).
[13] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).

and PCAC. The justification for our rules follows from lemmas 1 and 2 in this section. Section V is devoted to the discussion of electromagnetic effects. Electromagnetism is superimposed upon the strong dynamics in such a way that both gauge invariance and the field-current identity are satisfied. Electromagnetic effects on the pion mass, ρ-meson mass and the pion β decay are considered and discussed. Section VI contains further discussions on the phenomenological Lagrangian approach.

II. CHIRAL DYNAMICS

We shall briefly review the ingredients that go into the usual current algebra analyses.

(1) One assumes the existence of vector and axial-vector currents, $V_\mu{}^\alpha(x)$ and $A_\mu{}^\alpha(x)$, whose time components satisfy the local chiral algebra [e.g., $SU(2)\otimes SU(2)$] at equal times. For $SU(2)\otimes SU(2)$, one has (g, β being isospin indices)

$$\begin{aligned}[][V_0{}^\alpha(\mathbf{x},0),\, V_0{}^\beta(\mathbf{x}',0)]&=i\epsilon^{\alpha\beta\gamma}V_0{}^\gamma(\mathbf{x},0)\delta^3(\mathbf{x}-\mathbf{x}'),\\ [V_0{}^\alpha(\mathbf{x},0),\, A_0{}^\beta(\mathbf{x}',0)]&=i\epsilon^{\alpha\beta\gamma}A_0{}^\gamma(\mathbf{x},0)\delta^3(\mathbf{x}-\mathbf{x}'),\\ [A_0{}^\alpha(\mathbf{x},0),\, A_0{}^\beta(\mathbf{x},0)]&=i\epsilon^{\alpha\beta\gamma}V_0{}^\gamma(\mathbf{x},0)\delta^3(\mathbf{x}-\mathbf{x}').\end{aligned}\tag{1}$$

(2) One notes that

$$\langle 0|\,\partial^\mu A_\mu{}^\alpha(0)\,|\pi^\beta(q)\rangle=-i\delta^{\alpha\beta}f_\pi\mu^2,$$

where f_π is the pion decay constant. The fact that $f_\pi\mu^2$ is nonzero allows one to use the divergence $\partial^\mu A_\mu{}^\alpha$ as an interpolating field for the pion $\phi^\alpha(x)$:

$$\partial^\mu A_\mu{}^\alpha(x)\equiv f_\pi\mu^2\phi^\alpha(x).\tag{2}$$

PCAC is an assumption that the off-shell amplitudes, in which the pion fields are defined by Eq. (2), has a smooth extrapolation from the pion momentum $q\to 0$ [at which the current commutation relations (1) make definite predictions about the amplitude] to the physical value of q, $q^2=\mu^2$.

It was Weinberg's[1] observation that a Lagrangian in which Eq. (1) is true and in which the pion field satisfies Eq. (2) must necessarily reproduce the results of current algebra in the lowest-order calculation. We now turn to an example.

Let us construct a theory[5] in which Eqs. (1) and (2) are true using only the pion field $\phi^\alpha(x)$. Such a theory may not be a bad approximation to the real world if higher mass excitations do not influence low-energy behaviors of the pionic system in an essential way. We must assign a transformation property of $\phi^\alpha(x)$ under the chiral $SU(2)\otimes SU(2)$. The simplest possibility is that $\phi^\alpha(x)$ transform like the three $I=1$ components of $(\frac{1}{2},\frac{1}{2})$. This means that

$$\sigma^2+\boldsymbol{\phi}^2=\text{invariant under } SU(2)\otimes SU(2),\tag{3}$$

where $\sigma(x)$ is an object which is isoscalar and scalar under the (improper) Poincaré group. Since there is no other field than ϕ^α in the theory, the right-hand side of Eq. (3) must be a c-number constant:

$$\sigma^2+\boldsymbol{\phi}^2=f_\pi{}^2,\tag{4}$$

which gives

$$\begin{aligned}\sigma(x)&=[f_\pi{}^2-\boldsymbol{\phi}^2(x)]^{1/2}\\ &=f_\pi-(1/2f_\pi)\boldsymbol{\phi}^2-(1/8f_\pi{}^3)(\boldsymbol{\phi}^2)^2+\cdots.\end{aligned}\tag{5}$$

In Eqs. (4) and (5), f_π is to be understood as some constant, which, as we will later see, is actually the pion decay constant. The unique Lagrangian density that is chirally symmetric and no more than quadratic in the first derivative of the pion field is

$$\mathcal{L}_{\text{inv}}=\tfrac{1}{2}[(\partial_\mu\boldsymbol{\phi})^2+(\partial_\mu\sigma)^2].\tag{6}$$

To implement the PCAC condition, Eq. (2), we add a term to Eq. (6), which breaks the chiral symmetry:

$$\begin{aligned}\mathcal{L}&=\tfrac{1}{2}[(\partial_\mu\boldsymbol{\phi})^2+(\partial_\mu\sigma)^2]+f_\pi\mu^2\sigma';\\ \sigma'&\equiv\sigma-f_\pi=-(1/2f_\pi)\boldsymbol{\phi}^2-(1/8f_\pi{}^3)(\boldsymbol{\phi}^2)^2+\cdots.\end{aligned}\tag{7}$$

The coefficient of σ' in Eq. (7) is chosen so that the mass of the pion is μ^2:

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2-\tfrac{1}{2}\mu^2\boldsymbol{\phi}^2+O(\phi^4).$$

The vector and axial-vector currents are identified by considering the infinitesimal local $SU(2)\otimes SU(2)$ gauge transformations:

$$\begin{aligned}&\delta_\omega\boldsymbol{\phi}(x)=\boldsymbol{\omega}(x)\times\boldsymbol{\phi}(x),\\ &\delta_\lambda\boldsymbol{\phi}(x)=-\boldsymbol{\lambda}(x)\sigma(x),\quad \delta_\lambda\sigma(x)=\boldsymbol{\lambda}(x)\cdot\boldsymbol{\phi}(x).\end{aligned}\tag{8}$$

In particular, the axial-vector current is given by

$$A_\mu{}^\alpha(x)=\delta L/\delta\partial^\mu\lambda^\alpha(x)=[-\sigma\partial_\mu\phi^\alpha+\phi^\alpha\partial_\mu\sigma](x)\tag{9}$$

and its divergence by

$$\partial^\mu A_\mu{}^\alpha=\delta L/\delta\lambda^\alpha(x)=f_\pi\mu^2\phi^\alpha(x),\tag{10}$$

where

$$L=\int d^4x\,\mathcal{L}(x).$$

We note firstly that the charges (vector as well as axial vector), being the generators of the $SU(2)\otimes SU(2)$ gauge transformations, Eq. (8), certainly satisfy the $SU(2)\otimes SU(2)$ algebra, Eq. (1), and, secondly, the PCAC condition is satisfied. Equation (10) justifies calling the constant on the right-hand side of Eq. (3) the pion decay constant f_π.

The term quartic in the pion field in Eq. (7) is

$$\mathcal{L}_{\phi^4}=(1/8f_\pi{}^2)[(\partial_\mu\partial^2)^2-\mu^2(\boldsymbol{\phi}^2)^2].$$

Here, one recognizes the structure of the T matrix for the pion-pion scattering $[(q_1,\alpha)+(q_2,\beta)\to(q_3,\gamma)+(q_4,\delta)]$ given by Weinberg[14]:

$$\begin{aligned}T_{\pi\pi}=&-(1/f_\pi{}^2)\\ &\times[\delta_{\alpha\beta}\delta_{\gamma\delta}(s-\mu^2)+\delta_{\alpha\beta}\delta_{\beta\delta}(t-\mu^2)+\delta_{\alpha\delta}\delta_{\beta\gamma}(u-\mu^2)],\end{aligned}$$

where

$$s=(q_1+q_2)^2,\quad u=(q_1-q_4)^2,\quad t=(q_1-q_3)^2.$$

[14] S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

The reason for the lowest-order calculation (in f_π^{-1}) yielding precisely the current-algebra result will be discussed in a slightly different context in the next section.

For the ensuing discussion it is convenient to introduce tensor notation appropriate to $SU(2)\otimes SU(2)$, at this juncture. The upper index $a(=1,2)$ denotes a cogredient spinor, and the lower index a contragredient spinor. While the unbarred indices transform under $Q^\alpha+Q_5^\alpha$, the barred ones transform under $Q^\alpha-Q_5^\alpha$, where

$$Q^\alpha=\int d^3x\, V_0^\alpha(\mathbf{x},t)\,,\qquad Q_5^\alpha=\int d^3x\, A_0^\alpha(\mathbf{x},t)\,. \tag{11}$$

Under this convention,

$$M^a{}_{\bar b}(x)=[\sigma(x)+i\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)]_{ab}\,,$$
$$M^{\bar a}{}_b(x)=[\sigma(x)-i\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)]_{ab}\,.$$

Under the parity operation $Q^\alpha+Q_5^\alpha\leftrightarrow Q^\alpha-Q_5^\alpha$,

$$M^a{}_{\bar b}(\mathbf{x},t)\leftrightarrow M^{\bar a}{}_b(-\mathbf{x},t)\,.$$

With the definition of the M fields given by Eq. (11), Eq. (6) may be written as

$$\mathfrak{L}_{\rm inv}=\tfrac14\partial^\mu M^a{}_{\bar b}(x)\partial_\mu M^{\bar b}{}_a(x)\,. \tag{6'}$$

[The model presented here is essentially the nonlinear model of Gell-Mann and Lévy, Ref. 11; ramifications of this model are discussed in Ref. 5.]

III. PHENOMENOLOGICAL LAGRANGIAN

The notion of the algebra of fields[9] is motivated by (1) the empirical observation that the vector mesons dominate the form factors associated with the currents,[15–17] and (2) the desirability of having well-defined Schwinger terms[18] in, e.g., the equal-time commutators between the time and the space components of the currents. A precise formulation of this notation is the statement that the currents are proportional to the vector (or axial-vector) fields, as discussed by Lee, Weinberg, and Zumino.[9] The only Lagrangian model so far devised to implement the field-current identity is that of the massive Yang-Mills fields.[17,19] We caution here that since the notion of field algebra involves the extrapolation of certain empirical low-energy features into the high-energy domain, field algebra may run into difficulty for processes that depend upon the high-energy behavior of the system in an essential way. As we shall see in Sec. V, this is indeed the case.

Under the infinitesimal $SU(2)\otimes SU(2)$ local gauge transformations, the Yang-Mills fields $(V_\pm)_\mu{}^\alpha$ corresponding to $Q_\pm{}^\alpha=Q^\alpha\pm Q_5{}^\alpha$ undergo the change

$$\delta\mathbf{V}_\pm{}^\mu=\partial^\mu\boldsymbol{\chi}_\pm+2g\boldsymbol{\chi}_\pm\times\mathbf{V}_\pm{}^\mu\,, \tag{12}$$

and the M fields defined by Eq. (11) undergo the change

$$\begin{aligned}\delta M^a{}_{\bar b}&=-ig[(\boldsymbol{\chi}_+\cdot\boldsymbol{\tau})^a{}_cM^c{}_{\bar b}-M^a{}_{\bar c}(\boldsymbol{\chi}_-\cdot\boldsymbol{\tau})^{\bar c}{}_{\bar b}]\,,\\ \delta M^{\bar a}{}_b&=ig[(\boldsymbol{\chi}_-\cdot\boldsymbol{\tau})^{\bar a}{}_{\bar c}M^{\bar c}{}_b-M^{\bar a}{}_c(\boldsymbol{\chi}_+\cdot\boldsymbol{\tau})^c{}_b]\,,\end{aligned} \tag{13}$$

where

$$\boldsymbol{\chi}_\pm=\boldsymbol{\omega}\pm\boldsymbol{\lambda}.$$

With the covariant derivatives $\mathfrak{D}^\mu$, which act on the M fields, defined by

$$\begin{aligned}(\mathfrak{D}^\mu M)^a{}_{\bar b}&=\partial_\mu M^a{}_{\bar b}+ig[\mathbf{V}_+{}^\mu\cdot\boldsymbol{\tau}M-M\mathbf{V}_-{}^\mu\cdot\boldsymbol{\tau}]^a{}_{\bar b}\,,\\ (\mathfrak{D}^\mu M)^{\bar a}{}_b&=\partial_\mu M^{\bar a}{}_b-ig[\mathbf{V}_+{}^\mu\cdot\boldsymbol{\tau}M-M\mathbf{V}_-{}^\mu\cdot\boldsymbol{\tau}]^{\bar a}{}_b\,,\end{aligned} \tag{14}$$

the Lagarangian density which formally satisfies the field algebra and PCAC can be immediately written down:

$$\mathfrak{L}=-\sum_{i=+,-}\left\{\frac12(\partial_\mu\mathbf{V}_{i\nu}-\partial_\nu\mathbf{V}_{i\mu}-g\mathbf{V}_{i\mu}\times\mathbf{V}_{i\nu})^2-m^2(\mathbf{V}_{i\mu})^2+\frac14\frac{1}{1-\beta^2}[(\mathfrak{D}_\mu M)^a{}_{\bar b}(\mathfrak{D}^\mu M)^{\bar b}{}_a]+f_\pi\mu^2\sigma'\right\}. \tag{15}$$

The reason for introducing the factor $(1-\beta^2)^{1/2}$ will become clear shortly; it has to do with the requirement that the kinetic-energy term of the physical pion field in the Lagrangian density be of the standard form $\frac12(\partial_\mu\boldsymbol{\phi})^2$. We define the vector and axial-vector fields $\boldsymbol{\rho}^\mu$ and $\mathbf{a}^\mu$ by

$$\begin{aligned}\boldsymbol{\rho}^\mu&=\tfrac12(\mathbf{V}_+{}^\mu+\mathbf{V}_-{}^\mu)\\ \mathbf{a}^\mu&=\tfrac12(\mathbf{V}_+{}^\mu-\mathbf{V}_-{}^\mu).\end{aligned} \tag{16}$$

Making use of Eqs. (11), (14), and (16), we may write Eq. (14) as

$$\begin{aligned}\mathfrak{L}=&-\tfrac14(\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu-g\boldsymbol{\rho}_\mu\times\boldsymbol{\rho}_\nu-g\mathbf{a}_\mu\times\mathbf{a}_\nu)^2\\&-\tfrac14(D_\mu\mathbf{a}_\nu-D_\nu\mathbf{a}_\mu)^2+\tfrac12m^2(\boldsymbol{\rho}_\mu{}^2+\mathbf{a}_\mu{}^2)\\&+\tfrac12(1-\beta^2)^{-1}\{(\partial_\mu\sigma'-g\boldsymbol{\phi}\cdot\mathbf{a}_\mu)^2\\&+[D_\mu\boldsymbol{\phi}+g(f_\pi+\sigma')\mathbf{a}_\mu]^2\}+f_\pi\mu^2\sigma'\,,\end{aligned} \tag{17}$$

where

$$D_\mu\equiv\partial_\mu-g\boldsymbol{\rho}_\mu\times.$$

The vector and axial-vector currents are obtained from the gauge principle[20]:

$$\begin{aligned}V_\mu{}^\alpha(x)&=\frac{\delta L}{\delta\partial^\mu\omega^\alpha(x)}=\frac{m^2}{g}\rho_\mu{}^\alpha(x)\,,\\ \partial^\mu V_\mu{}^\alpha(x)&=\frac{\delta L}{\delta\omega^\alpha(x)}=0\end{aligned} \tag{18}$$

[15] Y. Nambu, Phys. Rev. **106**, 1366 (1957).

[16] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).

[17] M. Gell-Mann and F. Zachariasen, Phys. Rev. **134** 953 (1961).

[18] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959); T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955).

[19] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[20] The generalized Gell-Mann–Lévy equation is of the form

$$\frac{\delta L}{\delta\lambda(x)}-\partial_\mu\frac{\delta L}{\delta\partial_\mu\lambda(x)}+\tfrac12\partial_\mu\partial_\nu\frac{\delta L}{\delta\partial_\mu\partial_\nu\lambda(x)}-\cdots=0\,,$$

which can be easily proved by using Schwinger's action principle. In our case,

$$\frac{\delta L}{\delta\partial_\mu\partial_\nu\lambda(x)}=\frac{\delta L}{\delta\partial_\mu\partial_\nu\partial_{\gamma\nu}\lambda(x)}=\cdots=0\,.$$

and

$$A_\mu{}^\alpha(x)=\frac{\delta L}{\delta\partial^\mu\lambda^\alpha(x)}=\frac{m^2}{g}a_\mu{}^\alpha(x)\,,$$
$$\partial^\mu A_\mu{}^\alpha(x)=\frac{\delta L}{\delta\lambda^\alpha(x)}=f_\pi\mu^2\phi^\alpha(x)\,. \tag{19}$$

The axial-vector field $\mathbf{a}_\mu$ is not entirely associated with the 1^+ particle (A_1?) in perturbation theory, since the Lagrangian density (17) contains the bilinear coupling term $gf_\pi\mathbf{a}_\mu\cdot\partial^\mu\boldsymbol{\phi}$. The diagonalization of the $\mathbf{a}_\mu$ and $\partial_\mu\boldsymbol{\phi}$ fields leads to

$$\mathbf{a}_\mu(x)\equiv\mathbf{a}_\mu{}^1(x)-(\beta/m)D_\mu\boldsymbol{\phi} \tag{20}$$

and

$$\begin{aligned}\mathfrak{L}=&-\tfrac{1}{4}[\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu-g\boldsymbol{\rho}_\mu\times\boldsymbol{\rho}_\nu-g\mathbf{a}_\mu{}^1\times\mathbf{a}_\nu{}^1+g(\beta/m)(\mathbf{a}_\mu{}^1\times D_\nu\boldsymbol{\phi}-\mathbf{a}_\nu{}^1\times D_\mu\boldsymbol{\phi})-g(\beta/m)^2D_\mu\boldsymbol{\phi}\times D_\nu\boldsymbol{\phi}]^2\\&+\tfrac{1}{2}m^2\boldsymbol{\rho}_\mu{}^2-\tfrac{1}{4}[D_\mu\mathbf{a}_\nu{}^1-D_\nu\mathbf{a}_\mu{}^1-(\beta/m)(D_\mu D_\nu-D_\nu D_\mu)\boldsymbol{\phi}]^2+\frac{1}{2}\left(\frac{m^2}{1-\beta^2}\right)(\mathbf{a}_\mu{}^1)^2+\tfrac{1}{2}(D_\mu\boldsymbol{\phi})^2\\&+\frac{1}{2}\frac{1}{1-\beta^2}\left[(\partial_\mu\sigma')-g\left(\boldsymbol{\phi}\cdot\mathbf{a}_\mu{}^1-\frac{\beta}{m}\boldsymbol{\phi}\cdot D_\mu\boldsymbol{\phi}\right)\right]^2+\frac{1}{2}\frac{g^2}{1-\beta^2}\sigma'^2\left(\mathbf{a}_\mu{}^1-\frac{\beta}{m}D_\mu\boldsymbol{\phi}\right)^2\\&+g\sigma'\left(\mathbf{a}_\mu{}^1-\frac{\beta}{m}D_\mu\boldsymbol{\phi}\right)\left(D_\mu\boldsymbol{\phi}+\frac{\beta m}{1-\beta^2}\mathbf{a}_\mu{}^1\right)+\mu^2\frac{\beta m}{g}\sigma'.\end{aligned} \tag{21}$$

The requirement that the pion field $\phi^\alpha(x)$ satisfies the canonical commutation relation demands that

$$\beta=gf_\pi/m\,, \tag{22}$$

the use of which has been made in writing down Eqs. (20) and (21). The mass of the axial-vector meson $\mathbf{a}_\mu{}^1$ is given by

$$m_a{}^2=m_\rho{}^2/(1-\beta^2)\,,\quad(m_\rho=m)\,, \tag{23}$$

which may be written in the more suggestive form

$$\left(\frac{m_\rho{}^2}{g}\right)^2\left(\frac{1}{m_\rho{}^2}-\frac{1}{m_a{}^2}\right)=f_\pi{}^2. \tag{23$'$}$$

This is precisely the first sum rule of Weinberg[21] in the single-particle approximation. From Eqs. (19) and (20), it follows that

$$A_\mu(x)=(m_\rho{}^2/g)\mathbf{a}^1{}_\mu-(m_\rho\beta/g)D_\mu\boldsymbol{\phi}\,, \tag{20$'$}$$

which is equivalent to the second sum rule of Weinberg.[21] The celebrated Fayyazuddin-Riazuddin-Kawarabayashi-Suzuki (FRKS) sum rule[22,23] is

$$\beta=gf_\pi/m_\rho=1/\sqrt{2}\,. \tag{22$'$}$$

In our consideration, this relation cannot follow, since g, f_π, and m_ρ are independent parameters of the theory. Indeed, we do not know of a derivation of Eq. (22′) based entirely on current algebra and PCAC, without additional assumptions.[24] However, we note that the FRKS relation is empirically well satisfied.

[21] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[22] Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).
[23] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 225 (1966).
[24] J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966).

IV. ALGORITHM

We shall now state the rules of obtaining current-field algebra (for hard pions, in the sense of Schnitzer and Weinberg[13]) from the Lagrangian (21).

Recipe: To compute n-point functions of the particle fields, apply the Feynman-Dyson rules to the Lagrangian in Eq. (21) and sum over only the tree diagrams with appropriate external lines. We call a diagram a tree if every point in the diagram is not self-connected: It is impossible to start from a point and traverse along lines to return to the same point without retracing any path. In short, a tree is a diagram without loops; or a diagram which requires no integrations after four-momentum conservation at each vertex is taken into account.

To prove that this recipe gives amplitudes which satisfy all constraints imposed upon them by the field algebra and PCAC, we give two lemmas. Let us begin with some definitions and conventions. We write the Lagrangian density (21) in the form

$$\begin{aligned}\mathfrak{L}=&-\tfrac{1}{4}(\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu)^2+\tfrac{1}{2}m_\rho{}^2\boldsymbol{\rho}_\mu{}^2\\&-\tfrac{1}{4}(\partial_\mu\mathbf{a}_\nu{}^1-\partial_\nu\mathbf{a}_\mu{}^1)^2+\tfrac{1}{2}m_a{}^2(\mathbf{a}_\mu{}^1)^2\\&+\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2-\tfrac{1}{2}\mu^2\boldsymbol{\phi}^2+\mathfrak{L}_{\rm int}(g,\beta;\,\boldsymbol{\phi},\boldsymbol{\rho}_\mu,\mathbf{a}_\mu{}^1)\,.\end{aligned} \tag{24}$$

We have written $\mathfrak{L}_{\rm int}$ as a function of g and β, since the constant f_π may be eliminated by using the relation

$$f_\pi=\beta m_\rho/g\,.$$

Denoting by ψ any of the fields ($\boldsymbol{\phi},\boldsymbol{\rho}_\mu,\mathbf{a}_\mu{}^1$), we have the following lemmas.

Lemma 1: The n-point vertex $(\psi)^n$ in $\mathfrak{L}_{\rm int}$ is of order g^{n-2}, β being fixed. This is obvious when $\mathfrak{L}_{\rm int}(g,\beta)$ is explicitly written out. Note that in the covariant derivative D_μ, the combination $g\rho_\mu{}^\alpha$ appears, and that the field σ' is now written as

$$\sigma' = -\tfrac{1}{2}g(\beta m_\rho)^{-1}\boldsymbol{\phi}^2 - \tfrac{1}{8}g^2(\beta m_\rho)^{-2}(\boldsymbol{\phi}^2)^2 + \cdots.$$

The reason for keeping β fixed is clear from Eq. (20). By doing this, $\mathbf{a}_\mu{}^1$ and $\partial_\mu\boldsymbol{\phi}$ are, so far as the power counting of the parameter g is concerned, on the same footing.

Lemma 2: Tree diagrams of any N-point function for the particle fields are all of order g^{N-2}, and conversely, diagrams of order g^{N-2} of an N-point function are all trees. Lemma 2 follows from the definition of a tree diagram, Feynman rules and Lemma 1, by induction. To prove our main contention, namely that the sum of all tree diagrams for a given process satisfies field-algebraic and PCAC constraints, we begin with the following observation. Constraints imposed by the field algebra and PCAC upon an N-point function for the particle fields can be formulated as precise mathematical statements[13] (such as generalized Ward identities, to borrow a phrase), which relate an N-point function to an $(N-1)$-point function with a proportionality constant linear in g. Symbolically, we shall write

$$\langle N\text{-point function}\rangle \propto g\langle (N-1)\text{-point function}\rangle. \quad (25)$$

There are also relations of the form

$$\lim_{q_\mu\to 0}\langle N \text{ p.f.}\rangle \propto \frac{1}{f_\pi}\langle N-1 \text{ p.f.}\rangle = g(\beta m_\rho)^{-1}\langle N-1 \text{ p.f.}\rangle$$

when q_μ refers to the momentum of a pion, and of the form

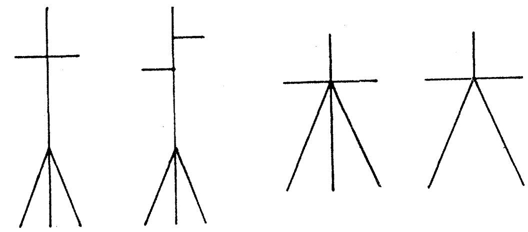
$$\lim_{q_\mu\to 0}\langle N \text{ p.f.}\rangle \propto \frac{g}{m_\rho{}^2}\langle N-1 \text{ p.f.}\rangle$$

when q_μ refers to the momentum of a vector (or an axial-vector) meson. Equations of the form of Eq. (25) are true order by order in g, and, in particular, in the lowest nonvanishing order (g^{N-2} for an N-point function) in g. This concludes the proof.

In comparing our procedure to that of, e.g., Schnitzer and Weinberg,[13] and Gerstein and Schnitzer,[25] we note that tree diagrams with branches [examples of which are shown in Fig. 1(a)] correspond to pole terms in theirs; contact terms [examples in Fig. 1(b)] correspond to their polynomial terms in momenta. For hard pions, we need only to retain derivative terms in the Lagrangian which would otherwise vanish in the soft-pion limit.

Some of the 3- and 4-point vertices, which we will

[25] I. Gerstein and H. J. Schnitzer (to be published). To be more precise, the scheme of Gerstein, Schnitzer, and Weinberg is more general than ours, in that the field-algebra constraints, such as Eqs. (20) and (23), need not be imposed, and the "anomalous magnetic moment" type couplings have been included.



(a) (b)

FIG. 1. Examples of tree diagrams. (a) Trees with branches; (b) trees corresponding to contact interactions.

need in subsequent applications, are as follows:

$$g^{-1}\mathcal{L}_{\rho\pi^2} = -\boldsymbol{\rho}_\mu\cdot(\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi}) + \tfrac{1}{2}(\beta/m_\rho)^2(\partial_\mu\boldsymbol{\rho}_\nu - \partial_\nu\boldsymbol{\rho}_\mu)\cdot\partial^\mu\boldsymbol{\phi}\times\partial^\nu\boldsymbol{\phi}, \quad (26a)$$

$$g^{-1}\mathcal{L}_{A_1,\rho\pi} = -\tfrac{1}{2}(\beta/m_\rho)(\partial_\mu\boldsymbol{\rho}_\nu - \partial_\nu\boldsymbol{\rho}_\mu)\cdot(\mathbf{a}_\mu{}^1\times\partial_\nu\boldsymbol{\phi} - \mathbf{a}_\nu{}^1\times\partial_\mu\boldsymbol{\phi}) - \tfrac{1}{2}(\beta/m_\rho)(\partial_\mu\mathbf{a}_\nu{}^1 - \partial_\nu\mathbf{a}_\mu{}^1)\cdot(\partial_\mu\boldsymbol{\rho}_\nu - \partial_\nu\boldsymbol{\rho}_\mu)\times\boldsymbol{\phi}, \quad (26b)$$

$$g^{-2}\mathcal{L}_{\rho^2\pi^2} = \tfrac{1}{2}(\boldsymbol{\rho}_\mu\times\boldsymbol{\phi})^2 - (\beta/m_\rho)^2(\partial_\mu\boldsymbol{\rho}_\nu - \partial_\nu\boldsymbol{\rho}_\mu)\cdot(\boldsymbol{\rho}^\mu\times\boldsymbol{\phi})\times\partial^\nu\boldsymbol{\phi} - \tfrac{1}{4}(\beta/m_\rho)^2[(\partial_\mu\boldsymbol{\rho}_\nu - \partial_\nu\boldsymbol{\rho}_\mu)\times\boldsymbol{\phi}]^2 - \tfrac{1}{2}(\beta/m_\rho)^2(\boldsymbol{\rho}_\mu\times\boldsymbol{\rho}_\nu)\cdot(\partial^\mu\boldsymbol{\phi}\times\rho^\mu\boldsymbol{\phi}), \quad (26c)$$

$$g^{-1}\mathcal{L}_{\rho^3} = \tfrac{1}{2}(\partial_\mu\boldsymbol{\rho}_\nu - \partial_\nu\boldsymbol{\rho}_\mu)\cdot\boldsymbol{\rho}^\mu\times\boldsymbol{\rho}^\nu, \quad (26d)$$

$$g^{-2}\mathcal{L}_{\rho^4} = -\tfrac{1}{4}(\boldsymbol{\rho}_\mu\times\boldsymbol{\rho}_\nu)^2. \quad (26e)$$

V. ELECTROMAGNETISM AND RADIATIVE CORRECTIONS

To ensure the field-current identity and electromagnetic gauge invariance in the presence of electromagnetic field, we write the Lagrangian $\mathcal{L}_{\text{EM}}$ in the presence of electromagnetism as

$$\mathcal{L}_{\text{EM}} = -\tfrac{1}{4}(F_{\mu\nu})^2 + \mathcal{L} - \tfrac{1}{2}m_\rho{}^2[(\rho_\mu{}^0)^2 - (\rho_\mu{}^0 + e\Phi_\mu/g)^2], \quad (27)$$

where $\mathcal{L}$ is given by (21), Φ_μ is the electromagnetic potential, and

$$F_{\mu\nu} = \partial_\mu\Phi_\nu - \partial_\nu\Phi_\mu.$$

As emphasized by Schwinger,[2] and Kroll, Lee, and Zumino,[26] Eq. (27) is completely gauge invariant and is canonically equivalent to the formulation of Lee and Zumino.[9]

Neither $\rho_\mu{}^0$ nor Φ_μ can be identified with the physical ρ^0 particle and photon, but rather some combinations of them. For the purpose of lowest-order radiative corrections, however, it suffices to take the additional interaction Lagrangian to be

$$(\mathcal{L}_{\text{EM}})_{\text{int}} = (em_\rho{}^2/g)\rho_\mu{}^0\Phi^\mu = eV_\mu{}^3\Phi^\mu. \quad (28)$$

[26] N. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

Thus the radiative correction to the hadronic process $\alpha \to \beta$ may be visualized as $\alpha+$(virtual ρ^0) $\to \beta$ + (virtual ρ^0) wherein two virtual ρ^0 lines are joined by a photon propagator. The process $\alpha+$(virtual ρ^0) $\to \beta$ + (virtual ρ^0) is to be described by the algorithm developed in the last section. In all radiative correction calculation, the line representing the sequence: (virtual ρ^0) $\to$ (virtual γ) $\to$ (virtual ρ^0) appears, with the effective propagator (in the Landau gauge):

$$(-i)^3\left(i\frac{m_\rho^2}{g}e\right)^2\left(g^{\mu\sigma}-\frac{q^\mu q^\sigma}{m_\rho^2}\right)\left(g_{\sigma\tau}-\frac{q_\sigma q_\tau}{q^2}\right)\times\left(g^{\tau\nu}-\frac{q^\tau q^\nu}{m_\rho^2}\right)\frac{1}{q^2}\left(\frac{1}{q^2-m_\rho^2}\right)^2$$
$$=-i\left(\frac{e}{g}\right)^2\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\frac{1}{q^2}\left(\frac{m_\rho^2}{q^2-m_\rho^2}\right)^2. \quad (29)$$

A. Pion Electromagnetic Mass

The π^+-π^0 mass difference can be calculated by considering all the tree diagrams for $\pi^++\rho^0\to\pi^++\rho^0$ and closing the ρ^0-γ-ρ^0 loop. The relevant strong vertices are contained in $\mathcal{L}_{\rho\pi^2}$, $\mathcal{L}_{A_1\rho\pi}$, and $\mathcal{L}_{\rho^2\pi^2}$, Eqs. (26a)–(26c). For our purpose, we shall group these terms into

$$\mathcal{L}'=\mathcal{L}_1+\mathcal{L}_2, \quad (30)$$

where

$$g^{-1}\mathcal{L}_1=-\boldsymbol{\rho}_\mu\cdot(\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi})+\tfrac{1}{2}(\beta/m_\rho)^2(\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu)\cdot\partial^\mu\boldsymbol{\phi}\times\partial^\nu\boldsymbol{\phi}+\tfrac{1}{2}g(\boldsymbol{\rho}_\mu\times\boldsymbol{\phi})^2, \quad (31)$$

$$g^{-1}\mathcal{L}_2=-\tfrac{1}{2}(\beta/m_\rho)(\partial_\mu\mathbf{a}_\nu{}^1-\partial_\nu\mathbf{a}_\mu{}^1)\cdot(\partial^\mu\boldsymbol{\rho}^\nu-\partial^\nu\boldsymbol{\rho}^\mu)\times\boldsymbol{\phi}-\tfrac{1}{2}(\beta/m_\rho)(\partial^\mu\boldsymbol{\rho}^\nu-\partial^\nu\boldsymbol{\rho}^\mu)\cdot(\mathbf{a}_\mu{}^1\times\partial_\nu\boldsymbol{\phi}-\mathbf{a}_\nu{}^1\times\partial_\mu\boldsymbol{\phi})-\tfrac{1}{4}g(\beta/m_\rho)^2[(\partial_\mu\boldsymbol{\rho}_\nu-\partial_\nu\boldsymbol{\rho}_\mu)\times\boldsymbol{\phi}]^2. \quad (32)$$

$\mathcal{L}_1$ and $\mathcal{L}_2$ are separately gauge invariant. Using Eq. (29) the calculation of the π^+ electromagnetic mass shift goes through in a straightforward manner. (There is no $\Delta I=2$ mass shift for the π^0.)

The contribution of $\mathcal{L}_1$ (pion pole and the related "sea gull" diagrams) to $\delta\mu^2=\mu^2(\pi^+)-\mu^2(\pi^0)$ is (with $p^2=\mu^2$)

$$(\delta\mu^2)_1=\frac{e^2}{i}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\frac{m_\rho^2}{k^2-m_\rho^2}\right)\frac{1}{(p-k)^2-\mu^2}\times\Bigg\{(2p-k)^2-\frac{[(2p-k)\cdot k]^2}{k^2}-3[(p-k)^2-\mu^2]+4(\beta/m_\rho)^2[(k\cdot p)^2-k^2p^2]+(\beta/m_\rho)^4k^2[p^2k^2-(p\cdot k)^2]\Bigg\}, \quad (33)$$

and the contribution of $\mathcal{L}_2$ (A_1 pole and the related "sea gull" diagrams) is

$$(\delta\mu^2)_2=3e^2\left(\frac{\beta}{m_\rho}\right)^2\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\frac{m_\rho^2}{k^2-m_\rho^2}\right)^2\times\left\{k^2-\frac{k^2}{(p-k)^2-m_a^2}\left[\frac{k^2-k^2p^2-(k\cdot p)^2}{3m_a^2}\right]\right\}. \quad (34)$$

To order $(\mu/m_\rho)^2$, the two terms in Eqs. (33) and (34) are, respectively,

$$(\delta\mu^2)_1\cong\frac{3\alpha}{4\pi}m_\rho^2\left\{1+\left(\frac{\mu}{m_\rho}\right)^2\left[\ln\left(\frac{m_\rho}{\mu}\right)^2+\frac{15}{16}+\frac{1}{16}\ln\left(\frac{\Lambda}{m_\rho}\right)^2\right]\right\} \quad (33')$$

and

$$(\delta\mu^2)_2\cong\frac{3\alpha}{4\pi}m_\rho^2\left\{2\ln2-1+\left(\frac{\mu}{m_\rho}\right)^2\left[\frac{19}{4}\ln2-\frac{55}{16}+\frac{1}{16}\ln\left(\frac{\Lambda}{m_\rho}\right)^2\right]\right\}, \quad (34')$$

where Λ is the ultraviolet cutoff momentum and use has been made of the FRKS relation

$$\beta=1/\sqrt{2}, \quad (22')$$

which is empirically well satisfied. The sum of the two contributions is

$$\delta\mu^2=\frac{3\alpha}{4\pi}m_\rho^2\left\{2\ln2+\frac{\mu^2}{m_\rho^2}\left[\ln\frac{m_\rho^2}{\mu^2}+\frac{19}{4}\ln2-\frac{5}{2}+\frac{1}{8}\ln\frac{\Lambda^2}{m_\rho^2}\right]\right\}. \quad (35)$$

In the limit of zero pion mass,

$$(\delta\mu^2)_{\mu^2\to0}=(3\alpha/4\pi)2\ln2, \quad (36)$$

which is exactly the soft-pion result of Das *et al.*,[27] previously derived by assuming the Weinberg sum rules and using the standard current-algebra technique. The derivation of this result using the phenomenological Lagrangian method has been independently carried out by Schwinger,[2] and Wick and Zumino.[28] Our calculation shows, however, that when the pion is not soft, namely $\mu^2\neq0$, the calculated mass shift is logarithmically divergent. The soft-pion calculation of Das *et al.* is understood only in the light of the field algebra (the Weinberg sum rules being exact according to the field algebra). Our calculation, which is based on the field

[27] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters 18, 759 (1967).

algebra, demonstrates directly that the field algebra alone does not assure a finite electromagnetic mass difference for "*real*" (as opposed to "soft") pions. This conclusion, which has been reached independently by Gerstein and Schnitzer,[25] and Wick and Zumino,[28] is consistent with the general argument of Halpern and Segrè[29] that the field algebra, in general, may give rise to logarithmically divergent electromagnetic mass differences. The numerical coefficient of the logarithmic divergence in Eq. (35) can indeed be reproduced using the method of Halpern and Segrè[29] as observed independently by Wick and Zumino.[28]

To have an idea as to the dependence of $\delta\mu^2$ on the cutoff parameter Λ, we write Eq. (35) as

$$\delta\mu = 6.0\ \text{MeV} + \frac{3\alpha}{4\pi}\frac{\mu}{8}\ln\left(\frac{\Lambda}{m_\rho}\right). \tag{35'}$$

The cutoff-dependent term contributes ~ 0.07 MeV to δ_μ for $\Lambda = 10m_\rho$ and ~ 0.14 MeV for $\Lambda = 100m_\rho$.

B. ρ-Meson Electromagnetic Mass

The ρ^+ electromagnetic mass shift can be similarly calculated by considering the tree diagrams for $\rho^+ + \rho^0 \to \rho^+ + \rho^0$ and closing the $\rho^0 - \gamma - \rho^0$ loop. The relevant strong vertices are contained in $\mathcal{L}_{\rho^3}$ and $\mathcal{L}_{\rho^4}$, Eqs. (26d) and (26e). As anticipated, the calculated ρ^+-ρ^0 mass difference is again logarithmically divergent:

$$(\delta m_\rho^2)_{\text{divergent}} = -\frac{3\alpha}{4\pi} m_\rho^2 \frac{1}{4i}\int \frac{d^4k}{(k^2+i\epsilon)^2}. \tag{37}$$

We note that in calculating the $\rho^+ - \rho^0$ mass difference, there is also the diagram $\rho^0 \to \gamma \to \rho^0$, which contributes to the $\Delta I = 2$ ρ^0-meson mass shift.

C. Radiative Corrections to the Pion β Decay

The subject of radiative corrections to β decays has recently been of active interest, mainly due to a general argument by Bjorken[30] and later elaborated by Abers, Norton, and Dicus.[31] They showed that the electromagnetic correction to the weak vector-current matrix elements is, to the order e^2, given by a universal constant

$$\frac{3\alpha}{4\pi}\frac{1}{2i}\int \frac{d^4k}{(k^2+i\epsilon)^2}, \tag{38}$$

which is logarithmically divergent and independent of the strong-interaction dynamics. The two assumptions that go into the Bjorken-Abers-Norton-Dicus theorem are the current-field algebra commutation relations and the statement concerning the violation of isospin conservation by electromagnetism:

$$(\partial_\mu \pm ie\Phi_\mu)V_\pm^\mu(x) = 0. \tag{39}$$

In the phenomenological field-algebra Lagrangian model, Eq. (27), these conditions are satisfied. Therefore, we expect that the result, Eq. (38), should come out from a direct calculation based on the phenomenological Lagrangian model, and it indeed does.

To see that the condition, Eq. (39), is satisfied by the phenomenological field-algebra Lagrangian given by Eq. (27), we observe that the local gauge transformation

$$\begin{aligned} \delta\boldsymbol{\rho}_\mu &= g\boldsymbol{\omega}\times\boldsymbol{\rho} + \partial_\mu\boldsymbol{\omega}, \\ \delta\Phi_\mu &= -(g/e)\partial_\mu\omega_3 \end{aligned} \tag{40}$$

induces the change of $\mathcal{L}$:

$$\delta\mathcal{L} = m_\rho^2[\boldsymbol{\rho}^\mu\cdot\partial_\mu\boldsymbol{\omega} - e(\boldsymbol{\omega}\times\boldsymbol{\rho}^\mu)_3\Phi_\mu],$$

which implies that

$$(\partial_\mu \pm ie\Phi_\mu)V_\pm^\mu(x) = 0, \quad \mathbf{V}_\mu = (m_\rho^2/g)\boldsymbol{\rho}_\mu.$$

The calculation of the radiative correction to the pion β decay can be carried out by considering the electromagnetic modification of the tree diagram $\pi^+ + \pi^0 \to \rho^+ \to e^+ + \nu$. In the Landau gauge, the divergence of the radiative correction comes from (1) the electromagnetic modification of the ρ^+ propagation function and (2) the modification of the $\rho^- e^+ \nu$ vertex, with a ρ^0-γ line connecting the ρ^+ and e^+. While the first one gives rise to a multiplicative factor $-\delta m_\rho^2/m_\rho^2$, the second contribution can be calculated in a straightforward manner. The sum of the two contributions is

$$\simeq -\frac{\delta m_\rho^2}{m_\rho^2} + \frac{3\alpha}{4\pi}\frac{1}{4i}\int \frac{d^4k}{(k^2+i\epsilon)^2}, \tag{41}$$

which, on account of Eq. (37), is indeed in agreement with the general result of Bjorken, and Abers, Norton, and Dicus [31] (BAND).

We end this section with a few remarks.

(1) The divergence of the calculated π^+-π^0 mass difference is hardly surprising, since the present theory, which is tailored to describe low-energy phenomena, has been unjustifiably extrapolated to a high-energy virtual process ($k \to \infty$). Thus, high-energy damping effects, which made it possible for Harari[32] to argue (plausibly) that the $\Delta I = 2$ electromagnetic mass shifts should be dominated by low-lying excitations and should therefore be computable, is in fact lacking in our consideration. In addition to this, the present experimental data seem to indicate a much faster decrease of the

[28] G. C. Wick and B. Zumino, Phys. Letters **25B**, 479 (1967).
[29] M. B. Halpern and G. Segrè, Phys. Rev. Letters **19**, 611 (1967).
[30] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).
[31] E. S. Abers, R. E. Norton, and D. A. Dicus, Phys. Rev. Letters **18**, 676 (1967).
[32] H. Harari, Phys. Rev. Letters **17**, 1303 (1967).

electromagnetic form factor than that implied by the ρ-dominance model. We feel therefore confident that when these effects are taken into account properly,[33] the effective cutoff momentum Λ will be a relatively small one, being not much more than several ρ-meson masses.

(2) A similar remark applies to the divergence of the ρ^+-ρ^0 mass difference. As for the renormalization constant Z_4 (in the notation of Abers, Norton, and Dicus[31]), we may write

$$\sqrt{Z_4}=1-\frac{\delta m_\rho^2}{m_\rho^2}+\frac{3\alpha}{4\pi}\frac{1}{4i}\int\frac{d^4k}{(k^2+i\epsilon)^2},$$

where the second term will be finite, once δm_ρ^2 is made finite, and the last term refers to the divergent integral associated with the photon exchange between the ρ^+ and e^+. Assuming that the effective cutoff for the last term comes from the faster damping of the ρ electromagnetic form factor, for example, and that it is not much different from that appearing in the ρ^+-ρ^0 mass difference, we may argue that

$$\sqrt{Z_4}\simeq 1-2\delta m_\rho^2/m_\rho^2\simeq 1-4\delta m_\rho/m_\rho.$$

Taking δm_ρ to be equal to the uncertainty in the ρ mass, ± 3 MeV (which is probably an overestimate), we find the correction factor to be not much more than 1.5%. [In a gauge-invariant model in which the electromagnetic form factors decrease faster than $(k^2-m_\rho^2)^{-1}$, the relation

$$(\partial_\mu \pm ie\Phi_\mu)V_\pm{}^\mu(x)=0,$$

which is one of the conditions required for the BAND theorem, is no longer valid, and there is no contradiction to Z_4 being finite; this model will be discussed elsewhere.]

VI. CONCLUDING REMARKS

In the phenomenological Lagrangian approach we have described, we have tacitly assumed that the low-energy, small-momentum transfer limit of a scattering amplitude is describable by the sum of tree diagrams, where the masses and coupling constants are to be regarded as the physical ones, loop diagrams, which are neglected, producing only the renormalization effects on these parameters in this limit.[34] The work of Schnitzer and Weinberg fortifies this belief, to some extent, but how one should interpret a "physical coupling constant," in order to make the above statement correct, remains to be examined (is it the value of the vertex function when every particle is on the mass shell, or when some or all momenta go to zero?, etc.; this probably is an academic question in view of the PCAC smoothness condition).

An alternative view is to regard the phenomenological Lagrangian as a device for exploiting, in phenomenological analyses, symmetries of low-energy hadronic phenomena through the action principle applied to a nonoperator, numerical Lagrange function. This is another way of understanding the algorithm developed in Sec. IV in which renormalization effects, characteristic of the operator field theory, have been consistently neglected.[35]

In discussing radiative corrections, we have devised an intuitive rule, whereby the radiative correction to the process $\alpha\to\beta$ is computed by first computing the strong process $\alpha+\rho^0\to\alpha+\rho^0$ by the algorithm and then connecting the two ρ^0's by a photon propagator. [In this case we admit a loop created by the effective photon propagator, Eq. (29).] The justification for this rule, aside from being reasonable intuitively, is that this prescription is precisely that of Cottingham,[36] in which we substitute the forward Compton scattering amplitude by the model amplitude constructed from the phenomenological Lagrangian. We have seen that this prescription reproduces the field-algebra result of Das *et al.* for the π^+-π^0 mass difference in the limit $(\mu/m_\rho)^2\to 0$.

An obvious defect of theories of this sort is that they do not satisfy the unitarity. Thus the extrapolation of this method to higher energies is dangerous and unwarranted, until and unless a reasonable means of implementing the unitarity requirements has been achieved. [The same remark applies equally well to the current algebra, of course.] With the implementation of the unitarity, the damping of the (undesirable, and clearly wrong) polynomial growth of scattering amplitudes for high energies may be presumed to occur.

In short, what we have achieved is a way of determining low-energy limits of amplitudes based on such notions as the field algebra (or an equivalent action-principle statement), PCAC, and vector-meson dominance. To this, we must add a means of incorporating the unitarity requirements, at least in some phenomenological sense, so as to allow an extrapolation to higher energies. It appears to us that the works of Schwinger,[37] and Okubo *et al.*,[38] among others, are first steps in this direction. It is a largely unexplored terrain, however.

Note added in proof. A more detailed discussion of the π^+-π^0 mass difference is given in I. S. Gerstein, B. W. Lee, H. T. Nieh and H. J. Schnitzer [Phys.

[33] J. Schwinger [Phys. Rev. Letters **19**, 154 (1967)] discusses possibility in terms of a momentum-dependent (nonlocal) $\gamma\rho^\circ$ coupling. We understand that T. M. Yan of Harvard University is considering the various implications of this model to the radiative corrections [J. Schwinger (private communication)].

[34] We wish to thank Professor H. Lehmann for stressing to us this viewpoint. This view is apparently also held by V. Weisskopf [Varenna Summer Shcool lectures, 1967 (to be published)].

[35] We believe this view comes very close to the view of J. Schwinger. See Ref. 2. See also Ref. 37 below.

[36] W. N. Cottingham, Ann. Phys. (N. Y.) **25**, 424 (1963).

[37] J. Schwinger, Phys. Rev. **152**, 1219 (1966); **158**, 1391 (1967).

[38] S. Okubo, R. E. Marshak, and V. S. Mathur, Phys. Rev. Letters **19**, 407 (1967).

Rev. Letters **19**, 1064 (1967)]. The tree diagrams we refer to in Sec. IV are precisely what Schwinger called skeletal interactions [J. Schwinger, Phys. Rev. **158**, 1391 (1967)]. We understand that S. Coleman and B. Zumino have considered the problem we dealt with in Sec. IV [S. Coleman and B. Zumino (to be published)].

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Daughter Regge Trajectories in the Van Hove Model*

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The Van Hove model of Regge poles is generalized to include propagator self-energy insertions and used to study unequal-mass daughter trajectories. The first daughter trajectory is found to have negative slope at $t=0$.

I. INTRODUCTION

RECENTLY the study of Feynman diagrams has shed new light on the origin and behavior of Regge poles in relativistic quantum mechanics. Van Hove[1] has suggested a simple model in which the amplitude for Regge exchange is given by the sum of the one-particle exchange diagrams for the set of particles lying on an infinitely rising Regge trajectory. Durand[2] has emphasized the close correspondence between the daughter trajectories found by Freedman and Wang[3] in unequal-mass scattering and the lower spin components that are carried by off-mass-shell Feynman propagators for particles with spin.

We wish in this paper to show that the Van Hove model when studied for unequal external masses and generalized to include self-energy insertions on the propagators of the exchanged particles leads to and gives information about moving daughter trajectories. Our results, while model-dependent, suggest that only in accidental cases are the daughter trajectories expected to move parallel to the parent trajectory. In particular we find the first daughter has negative slope at $t=0$ for $\alpha_D(0)>-\frac{5}{2}$.

Lest the reader get lost below in the technical details of higher spin, let us first state the plan and simple physical ideas of our work. We first consider the unequal-mass scattering $m_1+m_1\to m_2+m_2$ computed with bare Feynman propagators for the exchanged particles. We find that the singularities at $t=0$ of the leading Regge-pole contribution are cancelled by fixed daughter poles. As is well known, fixed poles in the angular momentum plane are incompatible with (t-channel) unitarity. It is natural to hope, therefore, that when the Van Hove model is unitarized, the fixed daughter poles will turn into moving daughter trajectories. Our calculations show that this is precisely what happens, and we find an expression which determines the first daughter trajectory.

II. FIXED DAUGHTER POLES

We begin by studying the unequal-mass scattering $m_1+m_1\to m_2+m_2$ as $s\to\infty$ with momenta as defined in Fig. 1. In order to avoid undue complications we have throughout confined our attention to the leading and first daughter trajectories. The amplitude for the ex-

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[1] L. Van Hove, Phys. Letters **24B**, 183 (1967). Durand [Loyal Durand III, Phys. Rev. **161**, 1610 (1967)] has studied the smoothness conditions which are required in order to obtain Regge-type behavior from an infinite set of t-channel diagrams. In particular he has pointed out that the "particles" which are exchanged need not actually occur as physical resonances. They can be poles on the second sheet with negative mass squared as would occur for trajectories which turn over at some finite value of t. Hence, the requirement of infinitely rising trajectories is not necessary for obtaining Regge behavior. The authors wish to thank Professor Durand for helpful discussions on this point.

[2] Loyal Durand III, Phys. Rev. **154**, 1537 (1967).

[3] D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1956 (1967).