Low-Energy Theorem for Graviton Scattering

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A low-energy theorem for the scattering of gravitons from spin-0 particles is derived. We use the dispersion-theoretic method, recently utilized by Abarbanel and Goldberger to derive low-energy theorems for the Compton scattering of photons, to write unsubtracted dispersion relations for physical helicity amplitudes. The scattering amplitude at fixed angle is shown to be given by the Born approximation up to fourth-order terms in the graviton energy.

I. INTRODUCTION

A CLASSIC result of quantum-field theory is the derivation of low-energy theorems for Compton scattering by Low[1] and by Gell-Mann and Goldberger.[2] To prove these theorems, tacit assumptions about the commutation relations of field and current operators were made, but the main ingredient in the proofs was gauge invariance. It should therefore be possible to derive similar theorems for graviton scattering where one also has invariance under gauge transformations, and where charge conservation is replaced by energy-momentum conservation.

The amplitude for the process, matter state $a+$ graviton $\rightarrow$ matter state b is given by $A\sim\langle b|T_{\mu\nu}|a\rangle \epsilon_{\mu\nu}(k,\lambda)$, where $T_{\mu\nu}$ is the energy-momentum tensor of the matter system and $\epsilon_{\mu\nu}(k,\lambda)$ is the polarization tensor of the graviton, whose momentum is k. Gravitons are spin-2 massless particles, and this implies that $\epsilon_{\mu\nu}$ be symmetric, traceless, and orthogonal to k_μ. Further, invariance under gauge transformations[3,4] allows us to write $\epsilon_{\mu\nu}=\epsilon_\mu\epsilon_\nu$, where $\epsilon_\mu k_\mu=0$, $\epsilon_\mu\epsilon^\mu=0$, and to require that A be invariant under a change of gauge:

$$\epsilon_\mu\epsilon_\nu\rightarrow\epsilon_\mu\epsilon_\nu+\lambda(k_\mu\epsilon_\nu+\epsilon_\mu k_\nu)\,.$$

This means that $k_\mu\langle b|T_{\mu\nu}|a\rangle=0$.[4] One can then, as in the derivation of Low,[1] evaluate the matrix elements of the 4-momentum density $\langle b|T_{0\nu}|a\rangle$. However, there is one complication which prevents a straightforward derivation, i.e., the nonvanishing of the commutators $[T_{\mu\nu},T_{\mu'\nu'}]$. In the case of Compton scattering, the presence of a Schwinger term in $[J_0,J_K]$ is inessential, in that it merely serves to cancel the seagull term.[5] However, here the commutators are nontrivial and will serve to represent the contribution of the graviton-exchange pole. Such a pole must be present because gravitons interact with themselves as well as with matter, i.e., they couple to the total energy-momentum tensor, including the gravitational part.

Recently, an elegant derivation of the low-energy theorem for Compton scattering has been given by Abarbanel and Goldberger.[6] They work with physical helicity amplitudes, gauge invariance appearing only insofar as the photon has two helicity states. Utilizing the existence of kinematical zeros and assuming reasonable high-energy behavior, they define new amplitudes which satisfy unsubtracted dispersion relations. From these one derives that, at fixed scattering angle, the Born term is exact to second order in the photon energy. It is this method that we shall apply to graviton scattering. Only one new feature is present—the pole in the t channel due to graviton exchange, which necessitates a somewhat more involved argument. The result we obtain is that the low-energy scattering amplitude as a function of the graviton energy, at fixed scattering angle, is determined by the mass of the scalar particle up to fourth-order terms in the graviton energy, and to this order the Born approximation is exact.

In Sec. II, we discuss the kinematics and the derivation of the Born term. The low-energy theorem is derived in Sec. III, and the results are summarized and discussed in Sec. IV.

II. KINEMATICS AND THE BORN TERM

The kinematics, crossing relations, and position of kinematical singularities of the scattering amplitude for gravitons are identical to those of photons, except, of course, that the graviton helicity is ± 2. Therefore most of the kinematics is identical to that given in Ref. 6, and we shall use their notation.

The elastic scattering of a graviton by a scalar particle

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[1] F. E. Low, Phys. Rev. **96**, 1428 (1954).
[2] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).
[3] S. Weinberg, Phys. Rev. **134**, B882 (1964); **135**, B1049 (1965); **138**, B988 (1965).
[4] R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963).
[5] S. G. Brown, Phys. Rev. **158**, 1444 (1967).
[6] H. D. I. Abarbanel and M. L. Goldberger, Phys. Rev. (to be published).

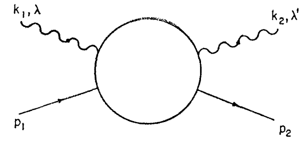

FIG. 1. Diagram representing the elastic scattering of a graviton by a scalar particle.

of mass m is described by the helicity amplitude

$$S_{\lambda'\lambda}=\delta_{\lambda'\lambda}+i\frac{(2\pi)^4\delta(p_2+k_2-p_1-k_1)}{(16p_{10}p_{20}k_{10}k_{20})^{1/2}}A_{\lambda'\lambda}(s,t). \quad (1)$$

p_1 (p_2) and k_1 (k_2) are the 4-momenta of the initial (final) massive particle and graviton, respectively, and λ (λ') is the initial (final) helicity of the graviton. See Fig. 1. The square of the center-of-mass (c.m.) energy is $s=(p_1+k_1)^2$, the momentum transfer squared is $t=(p_1-p_2)^2$, and $u=2m^2-t-s$. We also defined θ_s to be the c.m. scattering angle and $E(p)$ to be the c.m. energy (momentum) of the massive particle. Useful relations are

$$p=(s-m^2)/2\sqrt{s}\,,\quad E=(s+m^2)/2\sqrt{s}\,, \quad (2)$$

$$\cos\tfrac{1}{2}\theta_s=\frac{[(s-m^2)^2+st]^{1/2}}{(s-m^2)}=\frac{(m^4-su)^{1/2}}{(s-m^2)}\,, \quad (3a)$$

$$\sin\tfrac{1}{2}\theta_s=(-t/4p^2)^{1/2}=(-ts)^{1/2}/(s-m^2)\,, \quad (3b)$$

$$-t(t-4m^2)\sin^2\theta_t=4(m^4-su)\,, \quad (3c)$$

where θ_t is the t-channel c.m. scattering angle.

The Born approximation to the process under consideration has been discussed by Feynman,[4] using a quantization procedure for the gravitational field first developed by Gupta.[7] Although the amplitude has appeared in the literature,[12] we believe it instructive to give a derivation here.

The general relativity Lagrangian density describing the interaction of matter with gravity may be taken to be a sum of a field Lagrangian L_F and a matter Lagrangian L_M. These are

$$\mathfrak{L}_F=(1/K^2)R(-g)^{1/2}\,,\quad \mathfrak{L}_M=\tfrac{1}{2}(-g)^{1/2}(g^{\mu\nu}\varphi_\nu\varphi_\mu-m^2\varphi^2)\,,$$

$$L=\int d^3x(\mathfrak{L}_M+\mathfrak{L}_F)\,. \quad (4)$$

Here we have used φ to describe a spinless matter field of mass m. R is the Riemann curvature scalar, and $g_{\mu\nu}$ is the covariant metric tensor of the Riemannian space, with g being $\det\{g_{\mu\nu}\}$. We shall also use the Minkowski metric tensor

$$\eta_{\mu\nu}=\eta^{\mu\nu}\,,\quad \eta_{00}=\eta_{ii}=1\,,\quad \eta_{\mu\nu}=0\,,\quad \mu\neq\nu\,.$$

We shall *never* make use of contravariant quantities, except for $g^{\mu\nu}$, which is the inverse of $g_{\mu\nu}$, i.e., $g^{\alpha\beta}g_{\beta\gamma}=\delta_\gamma{}^\alpha$. However, we shall on occasion raise indices of covariant tensors (except for the metric tensor) by using the Minkowski metric. Thus for any tensor (except $g_{\mu\nu}$) we have by definition $T^{\mu\nu\cdots}=\eta^{\mu\alpha}\eta^{\nu\beta}\cdots T_{\alpha\beta\cdots}$.

The quantization procedure we adopt, following Gupta and Feynman, is to set

$$g_{\mu\nu}\equiv\eta_{\mu\nu}+Kh_{\mu\nu}$$

and expand (4) in powers of K. Terms independent of K then represent the free Lagrangian, while terms proportional to K, K^2, etc., represent the interaction Lagrangian. We therefore have

$$\mathfrak{L}_F=\mathfrak{L}_F{}^0+K\mathfrak{L}_F{}^1+K^2\mathfrak{L}_F{}^2+\cdots\,,$$
$$\mathfrak{L}_M=\mathfrak{L}_M{}^0+K\mathfrak{L}_M{}^1+K^2\mathfrak{L}_M{}^2+\cdots\,, \quad (5)$$

$$\mathfrak{L}_F{}^0=-\tfrac{1}{4}[2h^{\nu\alpha}{}_{,\rho}h_\nu{}^\rho{}_{,\alpha}-2h^\sigma{}_{\sigma,\lambda}h^{\beta\lambda}{}_{,\beta}-h^\alpha{}_{\beta,\nu}h_\alpha{}^{\beta,\nu}+h^\alpha{}_\alpha{}^{,\nu}h^\beta{}_{\beta,\nu}]\,,$$
$$\mathfrak{L}_F{}^1=\tfrac{1}{4}[(h^{\mu\nu}-\tfrac{1}{2}\eta^{\mu\nu}h^\sigma{}_\sigma)\eta^{\alpha\beta}\eta^{\beta\rho}+\eta^{\mu\nu}h^{\alpha\lambda}\eta^{\beta\rho}+\eta^{\mu\nu}\eta^{\alpha\lambda}h^{\beta\rho}]$$
$$\times[2h_{\mu\lambda,\beta}h_{\nu\beta,\alpha}-2h_{\mu\nu,\alpha}h_{\beta\lambda,\beta}-h_{\beta\lambda,\mu}h_{\alpha\beta,\nu}+h_{\beta\rho,\mu}h_{\alpha\lambda,\nu}]\,, \quad (6)$$

$$\mathfrak{L}_M{}^0=\tfrac{1}{2}(\varphi^{,\mu}\varphi_{,\mu}-m^2\varphi^2)\,,$$
$$\mathfrak{L}_M{}^1=-\tfrac{1}{2}[h^{\mu\nu}-\tfrac{1}{2}\eta^{\mu\nu}h^\alpha{}_\alpha]\varphi_{,\nu}\varphi_{,\mu}+\tfrac{1}{2}m^2h^\alpha{}_\alpha\varphi^2\,,$$
$$\mathfrak{L}_M{}^2=\tfrac{1}{2}[h^{\mu\gamma}h_\gamma{}^\nu-\tfrac{1}{2}h^{\mu\nu}h^\sigma{}_\sigma-\tfrac{1}{4}\eta^{\mu\nu}h^\gamma{}_\alpha h_\gamma{}^\alpha+\tfrac{1}{8}\eta^{\mu\nu}h^\sigma{}_\sigma h^\alpha{}_\alpha]\varphi_{,\nu}\varphi_{,\mu}$$
$$+\tfrac{1}{4}m^2[h^{\mu\nu}h_{\mu\nu}-\tfrac{1}{2}h^\mu{}_\mu h^\nu{}_\nu]\varphi^2\,.$$

[In offering the expansion of L_F, we did not expand $R\sqrt{(-g)}$, but an expression which differs from $R\sqrt{(-g)}$ by a total divergence and which contains no derivations of $g_{\mu\nu}$ higher than the first.]

In addition to the Euler-Lagrange equations which give the Einstein equations, we also impose on the field $h^{\mu\nu}$ a subsidiary condition of the form[4,7]

$$h^{\alpha\beta}{}_{,\beta}-\tfrac{1}{2}\eta^{\alpha\beta}h^\sigma{}_{\sigma,\beta}=0\,, \quad (7)$$

introducing the notation

$$\bar{h}^{\alpha\beta}\equiv h^{\alpha\beta}-\tfrac{1}{2}\eta^{\alpha\beta}h^\sigma{}_\sigma\,. \quad (8)$$

The free-field equations, which follow from the free Lagrangian, are

$$\partial^\mu\partial_\mu\bar{h}^{\alpha\beta}=0\,, \quad (9)$$

$$\bar{h}^{\alpha\beta}{}_{,\beta}=0\,, \quad (10)$$

$$(\partial^\mu\partial_\mu+m^2)\varphi=0\,. \quad (11)$$

Evidently, the fields which are to be quantized and expanded (in the interaction picture) in creation and annihilation operations are $\bar{h}^{\alpha\beta}$ and φ.

The S matrix is then given by the Feynmann-Dyson expression

$$S=T\exp\left\{i\int\mathfrak{L}_{\rm int}(x)dx\right\}\,.$$

The term proportional to K^2, which is of interest to us, is

$$S^2=iK^2\int d^4x(:\mathfrak{L}_F{}^2:+:\mathfrak{L}_M{}^2:)-\tfrac{1}{2}K^2T\int d^4xd^4y$$
$$\times[:\mathfrak{L}_F{}^1(x):+:\mathfrak{L}_M{}^1(x):][:\mathfrak{L}_F{}^1(y):+:\mathfrak{L}_M{}^1(y):]\,. \quad (12)$$

[7] S. N. Gupta, Proc. Phys. Soc. (London) A65, 161 (1952); A65, 608 (1952); Phys. Rev. 96, 1683 (1954).

The terms involving $L_F{}^2$ and $L_F{}^1(x)L_F{}^1(y)$ do not contribute to graviton scattering. Hence the matrix element of interest is given by

$$\langle p,k,\lambda|S^2|p_2k_2\lambda'\rangle=\delta_{\lambda\lambda'}+A_a+A_b+A_c\,, \qquad (13)$$

where the states are normalized according to $\langle P_\alpha|P_\beta\rangle=(2\pi)^3\delta(\mathbf{P}_\alpha-\mathbf{P}_\beta)$. The three terms on the right side of (13) correspond to the three different types of diagrams represented by Figs. 2(a), 2(b), and 2(c), respectively.

Figure 2(c) exhibits the t pole which is a unique feature of gravitation theory. It arises from the three-graviton interaction given by $L_F{}^1$ and corresponds to a long-range gravitational force between matter and gravitational radiation. Figure 2(a) is a "seagull" term, familiar from photon-meson theory.

The usual Wick contraction procedure may now be applied to evaluate (13). A straightforward but exceedingly tedious exercise yields for the individual contributions of the separate graphs to the amplitude

$$\begin{aligned}A_a=2K^2[&(\epsilon_1\cdot p_1)(\epsilon_2{}^*\cdot p_2)(\epsilon_1\cdot\epsilon_2{}^*)\\&+(\epsilon_1\cdot p_2)(\epsilon_2{}^*\cdot p_1)(\epsilon_1\cdot\epsilon_2{}^*)\\&+\tfrac{1}{2}(\epsilon_1\cdot\epsilon_2{}^*)^2(m^2-p_1\cdot p_2)]\,,\end{aligned} \qquad (14)$$

$$\begin{aligned}A_b=&-(K^2/p_1\cdot k_1)(p_1\cdot\epsilon_1)^2(p_2\cdot\epsilon_2{}^*)^2\\&+(K^2/p_1\cdot k_2)(p_1\cdot\epsilon_2{}^*)^2(p_2\cdot\epsilon_1)^2\,,\end{aligned} \qquad (15)$$

$$\begin{aligned}A_c=K^2\big[&(\epsilon_1\epsilon_2{}^*)^2(k_1\cdot k_2+(p_1\cdot k_1)(p_1\cdot k_2)/k_1\cdot k_2)\\&-(\epsilon_1\cdot\epsilon_2{}^*)\big((p_1\cdot k_1/k_2\cdot k_1)[(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot p_1)\\&+(\epsilon_1\cdot p_2)(\epsilon_2{}^*\cdot k_1)]+(p_1\cdot k_2/k_1\cdot k_2)[(\epsilon_1\cdot p_1)(\epsilon_2{}^*\cdot k_1)\\&+(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot p_2)]+(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot k_1)\\&+(\epsilon_1\cdot p_2)(\epsilon_2{}^*\cdot p_1)+(\epsilon_1\cdot p_1)(\epsilon_2{}^*\cdot p_2)\big)\\&+(1/k_1\cdot k_2)[(\epsilon_1\cdot p_2)(\epsilon_1\cdot k_2)(\epsilon_2{}^*p_1)(\epsilon_2{}^*\cdot k_1)\\&+(\epsilon_1\cdot p_1)(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot p_2)(\epsilon_2{}^*\cdot k_1)\\&+(\epsilon_1\cdot k_2)^2(\epsilon_2{}^*\cdot p_2)(\epsilon_2{}^*\cdot p_1)\\&+(\epsilon_1\cdot p_1)(\epsilon_1\cdot p_2)(\epsilon_2{}^*\cdot k_1)^2]\big]\,.\end{aligned} \qquad (16)$$

We have described the initial polarization λ by $\epsilon_1{}^\mu\epsilon_1{}^\nu$, and the final polarization λ' by $\epsilon_2{}^{\mu'*}\epsilon_2{}^{\nu'*}$. The individual terms are not gauge-invariant, though their sum is. Indeed, the fact that the sum turns out to be invariant under gauge transformations (which, in the present context, take the form that the amplitude vanishes when $\epsilon_1{}^\mu\epsilon_1{}^\nu$ is replaced by $k_1{}^\mu\epsilon_1{}^\nu+k_1{}^\nu\epsilon_1{}^\mu$) is a very strong test, which reassures one that no algebraic errors have entered into this calculation.

We now evaluate the separate c.m. helicity amplitudes $A^s{}_{++}$ and $A^s{}_{+-}$[8]:

$$A^s{}_{++}=-8\pi G(m^4-su)^2/(s-m^2)(u-m^2)t\,, \qquad (17)$$

$$A^s{}_{+-}=-8\pi Gm^4t/(s-m^2)(u-m^2)\,, \qquad (18)$$

where $K^2=16\pi G$, G is Newton's constant, and $\hbar$ and c have been set equal to one throughout. The magnitude of G in our units is $1.2\times10^{-40}m_\pi{}^{-2}$, where m_π is the pion mass.

[8] The polarization vector ϵ^μ is chosen, as for photons, to be $\epsilon_\pm{}^0(\mathbf{K})=0$, $\boldsymbol{\epsilon}_\pm(\mathbf{K})=\mp\frac{1}{2}(\hat{x}\cos\theta\pm i\hat{y}-\hat{z}\sin\theta)$, where $\mathbf{K}=|\mathbf{K}|(\hat{x}\sin\theta+\hat{z}\cos\theta)$ is the 3-momentum of the graviton.

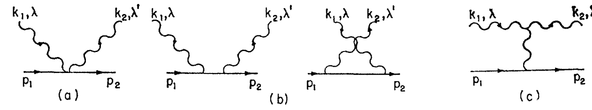

FIG. 2. The Feynman diagrams that contribute to the Born term.

III. DERIVATION OF THE LOW-ENERGY THEOREM

The Abarbanel-Goldberger method of deriving the low-energy theorem[6] consists of first locating the kinematical zeros of the helicity amplitudes. These are a simple consequence of angular momentum conservation, which forces the helicity-flip (non-helicity-flip) amplitude to vanish in the forward (backward) direction. There are two independent s-channel c.m. helicity amplitudes: $A^s{}_{2,\pm2}=A^s{}_{+\pm}$ ($A^s{}_{-\pm}$ is related by parity: $A^s{}_{-\pm}=A^s{}_{+\pm}$), which are expanded in partial waves:

$$A^s{}_{\lambda'\lambda}(s,t)=\sum_J(2J+1)A^{s,J}{}_{\lambda'\lambda}(s)d^J{}_{\lambda\lambda'}(\theta_s)\,. \qquad (19)$$

Since the rotation matrices $d^J{}_{\lambda\lambda'}(\theta_s)$ are equal to $(\sin\frac{1}{2}\theta_s)^{|\lambda'-\lambda|}(\cos\frac{1}{2}\theta_s)^{|\lambda'+\lambda|}$ times a Jacobi polynomial in $\cos\theta_s$,[9] we would expect

$$A^s{}_{++}\sim\cos^4(\tfrac{1}{2}\theta_s)=\tfrac{1}{4}(1+\cos\theta_s)^2 \quad \text{as} \quad \cos\theta_s\to-1\,, \qquad (20)$$

$$A^s{}_{+-}\sim\sin^4(\tfrac{1}{2}\theta_s)=\tfrac{1}{4}(1-\cos\theta_s)^2 \quad \text{as} \quad \cos\theta_s\to+1\,. \qquad (21)$$

However, we have to consider the effect of the graviton-exchange pole at $t=0$, i.e., in the forward direction. In fact, because of the presence of this pole, the above partial-wave expansion does not really exist. We can, however, imagine that the graviton has a small mass μ, go through the above argument, and let μ go to zero. This is justified since the forward pole is dynamical and arises from the divergence of the partial-wave expansion, whereas the kinematical zeros appear in each term of the series. For positive μ the above kinematical zeros will be present, and as we let $\mu\to0$, the pole in $A^s{}_{+\pm}$ at $\cos\theta_s=1+\mu^2/2p^2$ will approach the forward direction. Therefore the correct behavior of the helicity amplitudes is

$$\begin{aligned}&A^s{}_{++}\sim\cos^4(\tfrac{1}{2}\theta_s)=\tfrac{1}{4}(1+\cos\theta_s)^2 \quad \text{as} \quad \cos\theta_s\to-1\,,\\&A^s{}_{+-}\sim\sin^2(\tfrac{1}{2}\theta_s)=\tfrac{1}{2}(1-\cos\theta_s) \quad \text{as} \quad \cos\theta_s\to+1\,.\end{aligned} \qquad (22)$$

It is reassuring that the Born terms, derived previously, exhibit exactly this behavior.

The independent c.m. helicity amplitudes in the t channel [scalar+scalar → graviton (helicity ν)+ graviton (helicity ν')], $A^t{}_{00,2\pm2}=A^t{}_{+\pm}$ ($A^t{}_{-+}=A^t{}_{+\mp}$), satisfy

$$A^t{}_{\nu\nu'}=\sum_J(2J+1)A^{t,J}{}_{\nu\nu'}(t)d^J{}_{0,\nu-\nu'}(\theta_t)\,, \qquad (23)$$

[9] L. L. Wang, Phys. Rev. 142, 1187 (1966).

there being no problem of convergence here. Therefore we have

$$A^t_{+-}\sim(\sin\tfrac{1}{2}\theta_t)^4(\cos\tfrac{1}{2}\theta_t)^4=\tfrac{1}{16}\sin^4\theta_t \quad \text{as} \quad \theta_t\to 0,\pi\,. \tag{24}$$

The crossing relations for zero-mass particles are particularly simple,[10,11] i.e., to cross from the s channel to t channel, one merely flips the helicity of the crossed particle. Thus

$$A^s_{++}=A^t_{+-}\,,\quad A^s_{+-}=A^t_{++}\,. \tag{25}$$

The next step in the derivation is to define new amplitudes by removing the kinematical zeros in s and t, which should (as we argue below, using a Regge-pole argument) obey unsubtracted dispersion relations.

From (20), (24), and (3), we see that

$$\tilde{A}_{++}=\frac{A^s_{++}}{(m^4-su)^2}=\frac{A^s_{++}}{(s-m^2)^2\cos^4(\frac{1}{2}\theta_s)}=\frac{16A^t_{+-}}{t^2(t-4m^2)^2\sin^4\theta_t} \tag{26}$$

is free of both s- and t-kinematical singularities. Furthermore, since for fixed t

$$\tilde{A}_{++}(s,t)\sim(1/s^4)A^s_{++}(s,t)\sim s^{\alpha(t)-4}\,, \tag{27}$$

we can reasonably expect A_{++} to satisfy a fixed-t unsubtracted dispersion relation. Since we can exchange an elementary spin-2 particle (the graviton), we expect $\alpha(t)\leq 2$ for $t\leq 0$, and indeed the Born term behaves like s^2 for large s. Therefore it is quite justified to write $\tilde{A}_{++}$ as

$$\tilde{A}_{++}(s,t)=\frac{1}{\pi}\int_{m^2} ds'\,\mathrm{Im}\tilde{A}^s_{++}(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right), \tag{28}$$

where we have used the crossing symmetry of $\tilde{A}_{++}$ in order to combine the integrals over the absorptive parts in the s and u channels. The contribution of the one-particle intermediate state can be written immediately from the Born terms given in Sec. II. Alternatively, it can be derived from extended unitarity, using the form of the vertex given above. One gets

$$\mathrm{Im}\tilde{A}_{++}(s,t)\,|_{\mathrm{Born}}=-(8\pi^2G/t^2)\delta(s-m^2)\,. \tag{29}$$

Therefore,

$$\tilde{A}_{++}=\frac{-8\pi G}{(s-m^2)(u-m^2)t}+\frac{1}{\pi}\int_{s_0} ds'\,\mathrm{Im}\tilde{A}_{++}(s't)\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right), \tag{30}$$

where s_0 is the inelastic threshold $s_0>4m^2$, since we work to lowest order in G. Therefore the helicity nonflip amplitude is given by

$$A^s_{++}(s,t)=-\frac{8\pi G(m^4-su)^2}{(s-m^2)(u-m^2)t}+\frac{(m^4-su)^2}{\pi}\times\int_{s_0} ds'\frac{\mathrm{Im}A^s_{++}(s',t)}{(m^4-s'u')^2}\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right). \tag{31}$$

Since $(m^4-su)^2=4p^4s^2(1+\cos\theta_s)^2$, we see that A^s_{++}, for fixed θ_s, is given by the first term, i.e., the Born approximation, to order p^4.

We now turn to the helicity-flip amplitude $A^s_{+-}=A^t_{++}$. Since A^t_{++} is free of s-kinematical singularities, and A^s_{+-}/t is free of t-kinematical singularities, we have

$$\tilde{A}_{+-}=A^s_{+-}/t \tag{32}$$

is free of s- and t-kinematical singularities. For $\tilde{A}_{+-}$ we we expect an asymptotic behavior $t^{\alpha(s)-1}$, as $t\to\infty$. We now assume that the massive scalar particle does not have the quantum numbers of the vacuum, so that we cannot exchange the Pomeranchuk trajectory in the s or u channels. It is therefore reasonable to assume that $\alpha(s)<1$ for $s<s_1$, for some finite range $s_1>m^2$. We can therefore write a fixed-s unsubtracted dispersion relation for $\tilde{A}_{+-}$

$$\tilde{A}_{+-}(s,t)=\frac{1}{\pi}\int_0\frac{dt'}{t'-t}\,\mathrm{Im}\tilde{A}_{+-}(s,t')+\frac{1}{\pi}\int_{m^2}\frac{du'}{u'-u}\,\mathrm{Im}\tilde{A}_{+-}(s,u')\,. \tag{33}$$

$\tilde{A}_{+-}$ has no pole in the t channel and a one-particle pole in the u channel, as can be shown from extended unitarity or from the Born terms given previously. The one-particle contribution is

$$\mathrm{Im}\tilde{A}_{+-}(s,u)\,|_{\mathrm{Born}}=\frac{8\pi^2G}{(s-m^2)}\delta(u-m^2)\,. \tag{34}$$

Therefore ($u_0>m^2$, $t_0>0$, since we work to lowest order in G)

$$\tilde{A}_{+-}(s,t)=-\frac{8\pi Gm^4}{(s-m^2)(u-m^2)}+\frac{1}{\pi}\int_{t_0}\frac{dt'}{t'-t}\,\mathrm{Im}\tilde{A}_{+-}(s,t')+\frac{1}{\pi}\int_{u_0}\frac{du'}{u'-u}\,\mathrm{Im}\tilde{A}_{+-}(s,u')\,, \tag{35}$$

$$A^s_{+-}(s,t)=-\frac{8\pi Gm^4t}{(s-m^2)(u-m^2)}+\frac{t}{\pi}\int_{t_0}\frac{dt'}{t'(t'-t)}\,\mathrm{Im}A^s_{+-}(s,t')+\frac{t}{\pi}\int_{u_0}\frac{du'}{(u'-u)(2m^2-s-u')}\,\mathrm{Im}A^s_{+-}(s,u')\,. \tag{36}$$

[10] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 232 (1964).

[11] H. D. I. Abarbanel and S. Nussinov, Phys. Rev. **158**, 1462 (1967).

It follows that $A^s{}_{+-}$, for fixed $\cos\theta_s$, is given by the Born approximation up to terms of order p^2, since $t=-2p^2\times(1-\cos\theta_s)$. This is strange because we found that the helicity-non-flip amplitude was given by the Born term up to terms of order p^4, and we would expect the same for the helicity-flip amplitude. The problem is due to the graviton-exchange term, which gives rise to a pole at $t=0$ and reduces the double kinematical zero of $A^s{}_{+-}$ at $t=0$ to a single zero. One could argue that since the rest of the amplitude, i.e., after subtracting the Born term, has no pole at $t=0$, it must vanish like t^2 at $t=0$. Then one could divide by t^2, introducing a pole into the Born term—whose residue is calculable—and write an unsubtracted dispersion relation for $A^s{}_{+-}/t^2$. We would then derive

$$A^s{}_{+-}=-\frac{8\pi Gm^4t}{(s-m^2)(u-m^2)}+\frac{t^2}{\pi}\int_{t_0}\frac{dt'}{t'-t}\frac{\mathrm{Im}A^s{}_{+-}(s,t')}{t'^2}+\frac{t^2}{\pi}\int_{u_0}\frac{du'}{u'-u}\frac{\mathrm{Im}A^s{}_{+-}(s,u')}{(2m^2-s-su')^2},\quad(37)$$

showing that the Born term, for fixed θ_s, is exact to order p^4. However, this argument is somewhat circular. It can be made rigorous by the following line of reasoning. We write down the most general form that a gauge-invariant graviton-scalar scattering amplitude can take and prove that, if there are no singularities at threshold, the helicity-flip amplitude vanishes, for fixed scattering angle, like p^4. Since the second term in (36) satisfies these requirements (no poles at threshold and no cuts to lowest order in G), it vanishes like p^4. Therefore (37) is true, together with a superconvergence relation which we shall give below.

It is easy to write the general gauge-invariant form of the graviton-scalar scattering amplitude $A=\epsilon_1{}^{\mu\nu}\epsilon_2{}^{*\mu'\nu'}A_{\mu\nu,\mu'\nu'}$ if one remembers that $\epsilon_1{}^{\mu\nu}=\epsilon_1{}^\mu\epsilon_1{}^\nu$, where $\epsilon_1{}^\mu$ is essentially a photon-polarization 4-vector. One simply takes bilinear combinations of the two independent gauge-invariant tensors that one can form for photon-scalar scattering. These are $g_{\mu\nu}(k^1\cdot k^2)-k_\mu{}^1k_\nu{}^2$ and

$$P_\mu P_\nu(k_1\cdot k_2)-P\cdot K(P_\mu k_\nu{}^2+P_\nu k_\mu{}^1+g_{\mu\nu}(P\cdot K)^2),$$

where $P=\frac{1}{2}(p_1+p_2)$, $K=\frac{1}{2}(k_1+k_2)$. One additional requirement, which reduces the number of independent invariant amplitudes to two, is that $A_{\mu\nu,\mu'\nu'}$ be traceless $A_{\mu\mu,\mu'\nu}=A_{\mu\nu,\mu'\mu'}=0$. The most general amplitude can then be written as

$$\begin{aligned}A=&F(s,t)[(\epsilon_1\cdot\epsilon_2{}^*)(k_1\cdot k_2)-(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot k_1)]^2\\&+G(s,t)[(\epsilon_1\cdot P)(\epsilon_2{}^*\cdot P)(k_1\cdot k_2)\\&-(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot P)(P\cdot K)-(\epsilon_1\cdot P)(\epsilon_2{}^*\cdot k_1)(P\cdot K)\\&+(\epsilon_1\cdot\epsilon_2{}^*)(P\cdot K)^2][(\epsilon_1\cdot P)(\epsilon_2{}^*\cdot P)(k_1\cdot k_2)\\&-(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot P)(P\cdot K)-(\epsilon_1\cdot P)(\epsilon_2{}^*\cdot k_1)(P\cdot K)\\&+(\epsilon_1\cdot\epsilon_2{}^*)(P\cdot K)^2-(\epsilon_1\cdot\epsilon_2{}^*)(k_1\cdot k_2)P^2\\&+(\epsilon_1\cdot k_2)(\epsilon_2{}^*\cdot k_1)P^2].\end{aligned}\quad(38)$$

It is now trivial, but lengthy, to evaluate the c.m. helicity-flip amplitude in terms of the scalar amplitudes F and G. We find

$$A^s{}_{+-}=\tfrac{1}{4}t^2F(s,t)-[t^2(4m^2-t)^2/256]G(s,t).\quad(39)$$

Therefore, since F and G have only dynamical singularities, once we have removed the Born term (in a gauge-invariant fashion), $A^s{}_{+-}$ must vanish like t^2 in the forward direction, and for fixed θ_s, it must vanish like p^4 at threshold. We have thus shown that (37) holds for $A^s{}_{+-}$, and that therefore, for both helicity amplitudes, the Born term is, at fixed θ_s, exact up to terms of order p^4.

Let us examine whether superconvergence relations follow from (31). We expect that $A^s{}_{++}\sim s^2$, as $s\to\infty$ for fixed t. The Born term has this behavior, and the integral in (31) also goes like s^2 because of the crossing properties of $A^s{}_{++}$. Since we expect that $A^s{}_{+-}\sim t^{\alpha(s)}$ for $t\to\infty$ and for $s<m^2,\alpha(s)<1$ (since we have excluded the possibility of Pomeranchuk exchange), one relation will follow by setting to zero the coefficients of t^1 in an asymptotic expansion of (37) for large t. We derive

$$-\int_{t_0}\frac{dt'}{t'^2}\mathrm{Im}A^s{}_{+-}(s,t')+\int_{u_0}\frac{du'}{(2m^2-s-u')^2}\mathrm{Im}A^s{}_{+-}(s,u')=0.\quad(40)$$

There is, of course, little possibility of checking this relation, certainly not experimentally, while theoretically we cannot saturate the integrals with resonances, since we know little about the coupling of gravitons to particles of different mass.

IV. DISCUSSION AND SUMMARY

We have shown that the discussion of the low-energy limit of graviton scattering follows closely that of photon scattering. The graviton pole did not afford any undue complication. There are two aspects of the present result which may be contrasted with the low-energy Compton-scattering result. First, for graviton scattering the Born term is exact, at fixed angle, to order p^4, as opposed to p^2 for Compton scattering. This energy behavior was seen to follow from the fact that the graviton spin is 2, and therefore the kinematical zeros of the amplitude spin are raised to twice the power of the kinematical zeros of a spin-1 amplitude. One could conjecture that the Born amplitude for the scattering of a massless spin-s particle is exact at fixed angles to terms of order p^{2s}. However, arguments exist to the effect that such particles cannot interact in a Lorentz-invariant fashion at zero energy.[3]

The second part that we wish to call attention to is that the low-energy theorem we have derived is determined solely by the mass of the scalar scatterer. The general form for the graviton-scalar vertex is

$$G\langle p_1|T_{\mu\nu}(x)|p_2\rangle=\frac{Ge^{i(p_1-p_2)\cdot x}}{2(4p_{10}p_{20})^{1/2}}\times[F_1(q^2)(P_\mu P_\nu+q^2\eta_{\mu\nu}-q_\mu q_\nu)+F_2(q^2)(q^2\eta_{\mu\nu}-q_\mu q_\nu)];$$
$$P=p_1+p_2,\quad q=p_1-p_2. \tag{41}$$

This form follows from Lorentz and gauge invariance. From the properties of the energy spectrum, $F_1(0)$ is determined by the mass of the particle and in the above form is one.

In the Born approximation, $F_1(q^2)=1$ and $F_2(q^2)=0$. We might expect that $F_2(0)$ would appear in the amplitude, expanded through order p^3. The surprising fact is that it does not enter into this order.

The interaction of a graviton with spin-$\frac{1}{2}$ particles involves a vertex with three form factors. We expect that a low-energy theorem can be derived as above, and to order p^4 will involve two of these form factors (at zero argument).

In conclusion, we discuss the unpolarized differential cross section, which in our notation is

$$d\sigma/d\Omega=(1/64\pi^2 s)[|A^s{}_{++}|^2+|A^s{}_{+-}|^2].$$

The zero-energy limit is then

$$\frac{d\sigma}{d\Omega}=\tfrac{1}{2}G^2m^2\left[\frac{1+6\cos^2\theta_s+\cos^4\theta_s}{(1-\cos\theta_s)^2}\right]. \tag{42}$$

Since $G^2m_\pi{}^2\approx10^{-80}$ barn, the process under consideration is significant only when the graviton scatters off a heavy planet.

The above expression exhibits a pole in the forward direction, familiar in Rutherford scattering and reflecting the graviton-exchange pole. In Rutherford scattering, this pole is not physically interesting, since there is always some screening effect to remove the infinite tail of the Coulomb potential. Since gravitation cannot be screened, the present pole must be eliminated in a different fashion. Evidently, the usual description of a scattering event in terms of asymptotic waves which extend to infinity is not a good approximation for the description of gravitational interactions near the forward direction. To calculate the forward cross section, one must therefore account for the finite extent of the initial wave packet of the graviton.

We conclude with a word about the assumption of unsubtracted dispersion relations for the kinematic-singularity-free helicity amplitudes. The usual[12] derivations of low-energy theorems seem independent of the high-energy behavior of the amplitude. Subtractions, however, do not necessarily invalidate our derivation. Suppose that $\tilde{A}_{++}$, for example, behaves like

$$\sum_{m=0}^{n}F_m(t)s^m+O(1/s^{1+\epsilon})\quad\text{as}\quad s\to\infty\,,\quad\text{with}\quad\epsilon>0.$$

Then, as long as the $F_m(t)$ are nonsingular at threshold, for fixed z_s, the above derivation of the low-energy theorem can still be carried out. One can therefore speculate that the tacit assumptions made in the conventional derivations of low-energy theorems are such that possible subtractions must be of the above form.

[12] The explicit form of the Born approximation to graviton scattering off scalar and spin-$\frac{1}{2}$ targets appears in a paper by A. Chester, Phys. Rev. **143**, 1275 (1966).