Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Effective Stress Tensor*

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The high-frequency expansion of a vacuum gravitational field in powers of its small wavelength is continued. We go beyond the previously discussed linearization of the field equations to consider the lowest-order nonlinearities. These are shown to provide a natural, gauge-invariant, averaged stress tensor for the effective energy localized in the high-frequency gravitational waves. Under the assumption of the WKB form for the field, this stress tensor is found to have the same algebraic structure as that for an electromagnetic null field. A Poynting vector is used to investigate the flow of energy and momentum by gravitational waves, and it is seen that high-frequency waves propagate along null hypersurfaces and are not backscattered by the lowest-order nonlinearities. Expressions for the total energy and momentum carried by the field to flat null infinity are given in terms of coordinate-independent hypersurface integrals valid within regions of high field strength. The formalism is applied to the case of spherical gravitational waves where a news function is obtained and where the source is found to lose exactly the energy and momentum contained in the radiation field. Second-order terms in the metric are found to be finite and free of divergences of the $(\ln r)/r$ variety.

1. INTRODUCTION

THE principle of equivalence tells us that it is possible to transform away any uniform gravitational fields by simply changing coordinates, but is this to hold for gravitational waves as well? The energy and momentum carried by an electromagnetic field is measured by its stress tensor, but a gravitational field is usually described by a pseudotensor which can be locally annihilated by a suitable coordinate transformation. While the pseudotensor is satisfactory enough for calculating the *total* energy or momentum of an isolated system, there is no way of localizing their distribution if we keep using it.

In a preceding paper (denoted as I),[1] we showed how the assumption that the gravitational field is of high frequency led to a gauge-invariant approximation procedure to first order. When this was combined with the assumption of a WKB form $A_{\mu\nu}e^{i\phi}$ for the wave, we found that the gravitational field was remarkably similar to the electromagnetic field in the behavior of its amplitude, frequency, and polarization. In the present paper, we extend the results of the linear approximation to incorporate some of the essential nonlinear features of the Einstein equations, and in so doing further extend the analogy between light and gravitation. We shall find that in the high-frequency limit, the gravitational field has a natural gauge-invariant stress tensor. Since we now have a true tensor, it cannot be made to vanish by a simple coordinate transformation. Like the Maxwell stress tensor, the effective stress tensor for gravitational waves involves only first derivatives of the field, allows us to introduce a Poynting vector to describe the flow of energy and momentum, and acts as a source generating curvature of space-time.

While the hypothesis of high frequency (i.e., the wavelength is much smaller than the radius of curvature of the background geometry) will be used throughout this paper, we will sometimes combine it with either or both of two additional and logically independent assumptions. The first of these is to assume the WKB form for the radiation field (see I for elaboration). The second of our working tools will be an averaging procedure whereby the fine detail of some property of gravitational waves is replaced by its space-time average over a region containing many wavelengths. This method is familiar from electromagnetism or statistical mechanics, and we will call it the "BH assumption" after Brill and Hartle,[2] who applied this technique to the analysis of gravitational geons. When either of these two assumptions is used to derive important results, we will indicate it in the resultant formula by placing the letters WKB or BH after the appropriate equations.

2. EFFECTIVE STRESS TENSOR FOR GRAVITATIONAL RADIATION

In I, we expanded the vacuum field equations in powers of the wavelength of the gravitational wave. To lowest order, the field equations become $R_{\mu\nu}^{(1)}=0$, or, choosing our gauge with the reservations discussed in I, this was shown to reduce to

$$h_{\mu\nu}{}^{;\beta}{}_{;\beta}+2R_{\sigma\nu\mu\beta}{}^{(0)}h^{\beta\sigma}+R_{\sigma\mu}{}^{(0)}h_{\nu}{}^{\sigma}+R_{\sigma\nu}{}^{(0)}h_{\mu}{}^{\sigma}=0\,, \quad (2.1a)$$

$$h^{\mu\nu}{}_{;\nu}=0\,, \quad (2.1b)$$

$$h\equiv\gamma^{\alpha\beta}h_{\alpha\beta}=0\,. \quad (2.1c)$$

These equations determine the gravitational wave $h_{\mu\nu}$ once the background geometry $\gamma_{\mu\nu}$ is given. The second-

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[1] R. A. Isaacson, preceding paper, Phys. Rev. 166, 1263 (1968).

[2] D. R. Brill and J. B. Hartle, Phys. Rev. 135, B271 (1964).

order terms in the Einstein equations can be written

$$R_{\mu\nu}^{(0)} = -\epsilon^2 R_{\mu\nu}^{(2)}. \tag{2.2}$$

[see I, Eqs. (2.5)–(2.8) for explicit expressions.] Equation (2.2) shows us how the wave apparently acts as a source for the curvature of the background. Notice that (2.1) and (2.2) cannot be solved individually, but rather only in a self-consistent field scheme, as BH[2] have emphasized. We now observe that the Einstein field equations may be solved to an error of order $\epsilon \ll 1$ by simultaneously solving

$$R_{\mu\nu}^{(1)} = 0, \tag{2.3a}$$

$$R_{\mu\nu}^{(0)} - \tfrac{1}{2}\gamma_{\mu\nu} R^{(0)} = -8\pi T_{\mu\nu}^{\text{eff}}, \tag{2.3b}$$

where the effective stress tensor for the high-frequency field is given (in a completely general choice of gauge) by

$$\begin{aligned} T_{\mu\nu}^{\text{eff}} &\equiv (\epsilon^2/8\pi)(R_{\mu\nu}^{(2)} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(2)}) \\ &= (\epsilon^2/16\pi)(Q_{\mu\nu} - S_{\mu\nu}{}^{\rho}{}_{;\rho}), \end{aligned} \tag{2.4}$$

where $R^{(2)} = \gamma^{\alpha\beta} R_{\alpha\beta}^{(2)}$,

$$\begin{aligned} Q_{\mu\nu} \equiv{}& \tfrac{1}{2} h^{\rho\tau}{}_{;\mu} h_{\rho\tau;\nu} - h_{\nu}{}^{\tau;\rho}(h_{\tau\mu;\rho} - h_{\rho\mu;\tau}) \\ &- \tfrac{1}{2} h^{;\tau}(h_{\tau\mu;\nu} + h_{\tau\nu;\mu} - h_{\mu\nu;\tau}) \\ &+ \tfrac{1}{2}\gamma_{\mu\nu}[\tfrac{1}{2} h^{\rho\tau;\alpha} h_{\rho\tau;\alpha} - h^{\alpha\tau;\rho} h_{\alpha\rho;\tau} \\ &+ h^{;\tau}(h_{\tau\alpha}{}^{;\alpha} - \tfrac{1}{2} h_{;\tau})], \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} S_{\mu\nu}{}^{\rho} \equiv{}& \delta_{\nu}{}^{\rho} h^{\alpha\tau} h_{\alpha\tau;\mu} + h^{\rho\tau}(h_{\mu\nu;\tau} - h_{\tau\mu;\nu} - h_{\tau\nu;\mu}) \\ &+ \gamma_{\mu\nu}[h^{\rho\tau}(h_{\tau\alpha}{}^{;\alpha} - \tfrac{1}{2} h_{;\tau}) - \tfrac{1}{2} h_{\alpha\tau} h^{\alpha\tau;\rho}]. \end{aligned} \tag{2.6}$$

Here $Q_{\mu\nu}$ is a tensor quadratic in ∂h, whereas $S_{\mu\nu}{}^{\rho}$ is of the form $h\partial h$. Since only the divergence of $S_{\mu\nu}{}^{\rho}$ appears in (2.4), this piece of the effective stress tensor does not contribute under integral averages (see the Appendix).

From (2.3b), we find that, in the high-frequency approximation, gravitational radiation fields are uncoupled from their sources and are endowed with a vitality and independent existence of their own. They are just as good as any other source of energy when it comes to curving space, and later on we will see that they behave like any other conceivable field as far as energy and momentum transport and conservation are concerned.

3. BRILL-HARTLE AVERAGING SCHEME

The high-frequency oscillations of the gravitational waves are seen to produce the background curvature, but we are not really interested in all the fine details of the latter's fluctuations. The situation is somewhat analogous to the problem of finding electric fields in macroscopic dielectrics. While it is in principle possible to take into account all the atomic charge distributions in a dielectric to find the local electric field at any interior point, it is scarcely interesting to arrive at electric fields which fluctuate over a huge range as we move the observation point by 10^{-13} cm, and which require an exact description of the precise location of 10^{23} atoms. This sort of detail is totally irrelevant to the answering of any reasonable question about bulk matter. Rather, we take the field equation $\nabla \cdot \mathbf{E} = 4\pi\rho$ and average it over a region of space which is large compared to the scale of charge fluctuation, but small compared to the dimensions of the material of interest. Then we say that the average field is given as a solution to $\nabla \cdot \mathbf{E}_{\text{av}} = 4\pi\langle\rho\rangle$, where $\langle\rho\rangle$ denotes the space-averaged charge distribution.

Returning to the problem of gravitation, whenever regions of interest are large enough to contain many wavelengths, it is natural and advantageous to introduce a similar averaging process. Time-averaging has been done in the past by Tolman[3] for a radiation-filled universe, by Wheeler[4] for electromagnetic geons, and by BH[2] for gravitational geons. Here we wish to average over space-time as Arnowitt, Deser, and Misner[5] have done, and so we let the symbol $\langle\cdots\rangle$ denote an average over a region whose characteristic dimension is small compared to the scale over which the background changes, but independent of ϵ [i.e., $O(1)$], and therefore large compared to the wavelength of the radiation in the limit $\epsilon \to 0$. Then the averaged approximate field equations can be cast into the final form as given by BH:

$$R_{\mu\nu}^{(1)} = 0, \tag{3.1a}$$

$$R_{\mu\nu}^{(0)} - \tfrac{1}{2}\gamma_{\mu\nu} R^{(0)} = -8\pi T_{\mu\nu}^{\text{BH}}, \tag{3.1b}$$

where the BH-averaged effective stress tensor is

$$T_{\mu\nu}^{\text{BH}} = (\epsilon^2/16\pi)\langle Q_{\mu\nu} - S_{\mu\nu}{}^{\rho}{}_{;\rho}\rangle. \tag{3.2}$$

The oscillatory terms neglected by averaging (2.4) serve as a source for *higher-order* corrections to the metric (see Sec. 8).

4. BH AND WKB SIMPLIFICATION OF THE EFFECTIVE STRESS TENSOR

The general expressions (2.4)–(2.6) defining the effective stress tensor are rather unwieldy, and even if we specialize to the "Lorentz gauge" [i.e., the class of gauges satisfying (2.1)], $T_{\mu\nu}^{\text{eff}}$ is still clumsy. When we perform the BH averaging indicated in (3.2) we obtain a neat result. The rules we follow to do this are (see the Appendix for justification)

(1) Under integrals, divergences become reduced by a factor of ϵ. We may therefore drop $S_{\mu\nu}{}^{\rho}{}_{;\rho}$ and similar terms.

(2) Under integrals we may "integrate by parts,"

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[3] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, London, 1958).

[4] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).

[5] The space-time averaging used by these authors insured coordinate invariance of their expression for the Poynting vector for gravitational radiation. See R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **121**, 1556 (1961).

e.g.,

$$\langle h_\nu{}^{\tau;\rho} h_{\rho\mu;\tau}\rangle = -\langle h_\nu{}^{\tau;\rho}{}_{;\tau} h_{\rho\mu}\rangle\,,$$

if we ignore terms down by a factor ϵ.

(3) Covariant derivatives commute on high-frequency waves as $\epsilon\to 0$, e.g.,

$$h_{\mu\nu;[\rho\tau]} = \tfrac{1}{2}R_{\mu\sigma\rho\tau}{}^{(0)} h^\sigma{}_\nu + \tfrac{1}{2}R_{\nu\sigma\rho\tau}{}^{(0)} h_\mu{}^\sigma$$

or

$$h_{\mu\nu;[\rho\tau]} \doteq \epsilon^2\,,$$

where the symbol $\doteq$ is discussed in detail in I, Sec. 3.

Using these rules, we find that in the Lorentz gauge $T_{\mu\nu}{}^{\rm BH}$ is simply

$$T_{\mu\nu}{}^{\rm BH} = (\epsilon^2/32\pi)\langle h^{\rho\tau}{}_{;\mu} h_{\rho\tau;\nu}\rangle + O(\epsilon)\,. \quad \text{(BH)} \quad (4.1)$$

If we fix the gauge so that $h_{0\mu}=0$, we see that $T_{00}{}^{\rm BH}$ is positive definite.

To see if (4.1) has any invariant significance, we must investigate how $T_{\mu\nu}{}^{\rm BH}$ behaves under a change of gauge (see I, Sec. 4). Since we have from (2.5) $Q_{\mu\nu}\sim(\partial h)(\partial h)$, under the change of gauge $h_{\mu\nu}\to\bar{h}_{\mu\nu}\sim h+\partial\xi$, we obtain

$$Q_{\mu\nu}\to\bar{Q}_{\mu\nu}\sim(\partial h)(\partial h)+(\partial h)(\partial^2\xi)+(\partial^2\xi)(\partial^2\xi)\,.$$

The first group of terms of the form $(\partial h)(\partial h)$ are just the old $Q_{\mu\nu}$. The additional terms induced by the gauge transformation can be roughly divided up into either high or low frequency. For low-frequency waves we have

$$\xi_\mu = O(1)\,,\quad \xi_{\mu;\nu}=O(1)\,,\quad \xi_{\mu;\nu\tau}=O(1)\,,$$

and therefore the second and third group of terms in $\bar{Q}$ are negligible compared to the first group. On the other hand, for high-frequency waves we assume

$$\xi_\mu = O(\epsilon)\,,\quad \xi_{\mu;\nu}=O(1)\,,\quad \xi_{\mu;\nu\tau}=O(\epsilon^{-1})\,.$$

In this case, $\xi_{\mu;[\nu\tau]}\doteq\epsilon^2$, and so covariant derivatives on high-frequency coordinate waves commute. We find that the terms in $\bar{Q}$ of the form $(\partial h)(\partial^2\xi)$ can be converted to a divergence by integrating by parts, commuting derivatives, and using the general wave equation [I, Eq. (5.7)] for h. Similarly, by integrating by parts and commuting derivatives, we may reduce the terms like $(\partial^2\xi)(\partial^2\xi)$ to a divergence. Putting this all together, we obtain the behavior of the effective stress tensor under a general gauge transformation

$$T_{\mu\nu}{}^{\rm eff}\to\bar{T}_{\mu\nu}{}^{\rm eff} = T_{\mu\nu}{}^{\rm eff} + U_{\mu\nu}{}^{\rho}{}_{;\rho} + O(\epsilon)\,. \qquad (4.2)$$

While $U_{\mu\nu}{}^{\rho}{}_{;\rho}=O(\epsilon)$ for low-frequency coordinate waves, it is of order unity for high-frequency coordinate transformations, and so $T_{\mu\nu}{}^{\rm eff}$ is not gauge-invariant in general and hence not a physical observable. However, since $U_{\mu\nu}{}^{\rho}{}_{;\rho}$ is reduced under integrals,

$$T_{\mu\nu}{}^{\rm BH}\to\bar{T}_{\mu\nu}{}^{\rm BH} = T_{\mu\nu}{}^{\rm BH} + O(\epsilon)\,, \qquad (4.3)$$

and the BH-averaged stress tensor is gauge-invariant and thus given by (4.1) for *all* choices of gauge.

A corresponding simplification of the stress tensor occurs if we assume the WKB form for $h_{\mu\nu}$, but not the BH assumption. Thus we let (see I, Sec. 6)

$$h_{\mu\nu} = \mathrm{Re}\{\mathcal{A} e_{\mu\nu} e^{i\phi}\}\,, \qquad (4.4a)$$

$$k_\nu = \phi_{,\nu} \quad k^\nu k_\nu = 0\,, \qquad (4.4b)$$

$$e_{\mu\nu;\alpha}k^\alpha = 0\,, \qquad (4.4c)$$

$$(\mathcal{A}^2 k^\beta)_{;\beta} = 0\,, \qquad (4.4d)$$

$$e_{\mu\nu}e^{\mu\nu} = 1\,, \qquad (4.4e)$$

$$k_\mu e^{\mu\nu} = 0\,, \qquad (4.4f)$$

$$\gamma_{\mu\nu}e^{\mu\nu} = 0\,. \qquad (4.4g)$$

We find that the effective WKB stress tensor is

$$T_{\mu\nu}{}^{\rm WKB} = (\epsilon^2/32\pi)\mathcal{A}^2 k_\mu k_\nu \sin^2\phi + O(\epsilon)\,, \quad \text{(WKB)} \quad (4.5)$$

where $T_{00}{}^{\rm WKB}$ is positive definite as expected.

Finally, we combine the BH and WKB approximations to obtain the effective averaged high-frequency wave stress tensor in the geometrical-optics form (to lowest order)

$$T_{\mu\nu}{}^{\rm HF} = q^2 k_\mu k_\nu\,,\quad q^2 = \epsilon^2\mathcal{A}^2/64\pi\,. \quad \text{(BH-WKB)} \quad (4.6)$$

Equations (4.1), (4.5), and (4.6) may also be derived independently using the Einstein equations rewritten in the Landau and Lifshitz[6] form

$$H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 16\pi(-g)(T^{\mu\nu}+t^{\mu\nu})\,,$$

where $T^{\mu\nu}$ is the stress tensor of material sources which are present in the general case, $t^{\mu\nu}$ is quadratic in the first derivatives of the metric, and

$$H^{\mu\nu\alpha\beta} = (-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\nu})\,.$$

If we insert $g_{\mu\nu}=\gamma_{\mu\nu}+\epsilon h_{\mu\nu}$ into the field equations and either average them, use the WKB assumption, or do both, we find

$$t^{\mu\nu}\to T^{\mu\nu}{}_{\rm BH}\,,\quad T^{\mu\nu}{}_{\rm WKB}\,,\quad T^{\mu\nu}{}_{\rm HF}\,,$$

respectively.

From now on we will use (4.6) as the final form for the stress tensor for gravitational waves. This is precisely the form for electromagnetic null radiation fields, further extending the analogy between light and gravitation. Several other authors have obtained results similar to this in different contexts. Thus Trautman[7] has shown that (4.6) arises from weak-field linearized theory with Sommerfeld radiation conditions, while Brill[8] finds that in the limit of short wavelength, the development of a universe closed by the presence of gravitational waves is the same as for the Tolman electromagnetically filled universe.

From (4.6) we see that the background geometry is

[6] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962), p. 100.

[7] A. Trautman, Lectures on General Relativity, King's College, London, 1958 (unpublished).

[8] D. R. Brill, Nuovo Cimento Suppl. **II**, No. 1 (1964).

required to have the form

$$R_{\mu\nu}^{(0)}=-\sigma^2 k_\mu k_\nu\,,\quad \sigma^2=\tfrac{1}{8}\epsilon^2\mathscr{A}^2.\quad \text{(BH-WKB)}\quad (4.7)$$

Solutions to these equations have been found which exhibit, for example, spherical, axial, or plane symmetry.[9]

From (4.4b) and (4.4d) we obtain

$$T^{\mu\nu}{}_{\mathrm{HF};\nu}=0.\quad \text{(BH-WKB)}\quad (4.8)$$

These conservation laws tell us that the effective stress tensor for the high-frequency gravitational field is conserved on an equal footing with other stress tensors, and will later allow us to calculate the energy and momentum carried by the radiation field. Analogously to electromagnetic theory, we may define a spacelike gravitational Poynting vector S^α describing the flow of gravitational energy measured by an observer with timelike four-velocity v^μ. This is given by

$$S^\alpha\equiv(\delta^\alpha{}_\mu-v^\alpha v_\mu)v_\nu T^{\mu\nu}{}_{\mathrm{HF}}=\omega q^2(k^\alpha-\omega v^\alpha)\,,\quad \text{(BH-WKB)}\quad (4.9)$$

where $\omega\equiv k^\alpha v_\alpha$ is the frequency of the wave as measured in the rest frame of the observer. We see that $S^\alpha v_\alpha=0$, $S^\alpha S_\alpha=-\omega^4 q^4$, and that S^α only vanishes in the limit that the observer moves with the speed of light. In the rest frame of the observer, $k^\alpha=(\omega,\mathbf{k})$ and $S^\alpha=(0,\omega q^2\mathbf{k})$, and we find that gravitational energy flows along the rays of the field and is not scattered off of null hypersurfaces (in this approximation) by the background curvature of space-time.

5. ENERGY AND MOMENTUM OF THE GRAVITATIONAL RADIATION FIELD

In special relativity, the homogeneity of space-time leads us to the laws of conservation of energy and momentum; however, in general relativity, space-time is curved and does not in general have any symmetries. Without some method of introducing the special relativistic reference for comparison, it seems hopeless to try to extend these concepts to curved space. We must therefore content ourselves with being restricted to asymptotically flat spaces where we require energy and momentum to behave as the components of a four-vector under Lorentz transformations. For the usual problem of outgoing radiation emitted by bounded sources in a vacuum, it is reasonable to expect asymptotically flat (but radiative) space at large distances from the source, and we shall later show this to be the case for a specific example. For the rest of this section, we will assume that there exists a coordinate system $\tilde{x}^\mu$ which asymptotically becomes Lorentzian as space becomes flat, and we will use (4.8) along with Stokes's theorem to obtain formulas for computing the energy and momentum which arrive at flat infinity after being radiated from an isolated source. These expressions will be in the form of integrals over general hypersurfaces where gravitational fields may be large, and in which *arbitrary* coordinate systems may be used.

By Stokes's law, we may write

$$\int_V P^\mu{}_{;\mu}(-\gamma)^{1/2}d^4x=\int_{\partial V}P^\mu dS_\mu\,,\quad (5.1)$$

where, in the notation of exterior forms,

$$dS_\mu=(1/3!)(-\gamma)^{1/2}\epsilon_{\mu\alpha\beta\gamma}dx^\alpha\wedge dx^\beta\wedge dx^\gamma\,,\quad (5.2)$$

V is a four-dimensional region, ∂V its boundary. For the following, we choose V to be the region bounded by two nonintersecting null cones Σ_1 and Σ_2 and by two three-dimensional hypersurfaces σ_1 and σ_2 which cut both null cones. The hypersurface σ_2 will always be assumed spacelike, but σ_1 may be either spacelike or timelike as shown in Figs. 1 and 2. Σ_1 and Σ_2 should be thought of as future light cones emanating from the world line z^μ of a gravitational wave source, while σ_1 and σ_2 are chosen to make V lie entirely in a region where the high-frequency, WKB, and BH assumptions are simultaneously valid. Let us define

$$I(\sigma)\equiv\int_\sigma P^\mu dS_\mu\,,$$

where σ is an open three-surface. If P^μ has vanishing divergence ($P^\mu{}_{;\mu}=0$), then (5.1) becomes

$$0=I(\Sigma_1)+I(\Sigma_2)+I(\sigma_1)+I(\sigma_2)\,.\quad (5.3)$$

If in addition $I(\Sigma_1)=I(\Sigma_2)=0$ (as is usually the case, since radiation travels outward along null cones), we

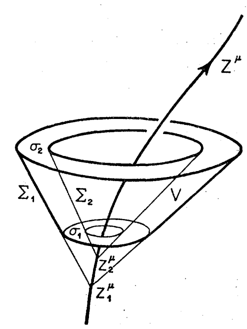

FIG. 1. The light cones Σ_1 and Σ_2 enclose an outgoing pulse of radiation from a source traveling along world line z^μ. They are cut by the three-dimensional spacelike hypersurfaces σ_1 and σ_2.

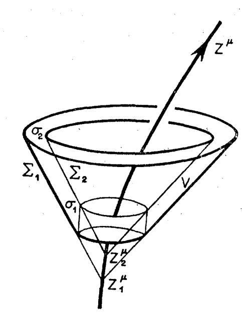

FIG. 2. The light cones Σ_1 and Σ_2 enclose an outgoing pulse of radiation from a source traveling along world line z^μ. They are cut by the spacelike hypersurface σ_2 and the timelike hypersurface σ_1.

[9] See, e.g., P. C. Vaidya, Nature **171**, 260 (1953); P. C. Vaidya and I. M. Pandya, Proc. Natl. Inst. Sci. India, **A26**, 459 (1960); R. Penrose, Rev. Mod. Phys. **37**, 215 (1965).

have an integral conservation law $I(\sigma_2)=-I(\sigma_1)$, where the minus sign just reflects the fact that the outward normals to V at σ_1 and σ_2 are oppositely directed.

For our first example let us choose $P^\mu=\mathcal{A}^2k^\mu$. Then by (4.4d) we have the required condition that $P^\mu{}_{;\mu}=0$. To evaluate $I(\Sigma)$ where $\Sigma=\Sigma_1$ or Σ_2, choose retarded time coordinates $x^\mu=(x^0=u, x^1, x^2, x^3)$. Then $du=0$ on the light cones Σ, and

$$I(\Sigma)=\int_\Sigma P^0(-\gamma)^{1/2}dx^1\wedge dx^2\wedge dx^3.$$

The phase ϕ in (4.4a) must be a function of retarded time only, and since k^α is a null vector, this in turn implies $P^0=0$ and $I(\Sigma)=0$. Then $N\equiv I(\sigma_2)=-I(\sigma_1)$ is a conserved quantity, independent of the (spacelike) hypersurface used to evaluate it. If a source radiates only while it is between points $z_1{}^\mu$ and $z_2{}^\mu$ on its world line, P^μ vanishes outside the region included between light cones Σ_1 and Σ_2, since radiation travels outward only along null surfaces and is not scattered by background curvature to lowest order in ϵ. We may therefore extend σ_2 to include an entire spacelike surface S oriented the same as σ_2 to obtain

$$N=\int_S \mathcal{A}^2k^\mu(-\gamma)^{1/2}dS_\mu, \tag{5.4}$$

and N is our conserved graviton number.

In order to develop expressions for the total energy and momentum arriving at null infinity, we must first define a tetrad $e^{(\alpha)}{}_\nu$ (α selects one vector out of the family of four, ν gives its vector components). We do this by requiring the tetrad vectors to be parallel-propagated along the rays k^α and to point along the asymptotically Lorentzian coordinates $\tilde{x}^\mu$ at null infinity. Then the tetrad is the unique solution to the differential equation

$$e^{(\alpha)}{}_{\nu;\mu}k^\mu=0, \tag{5.5}$$

with the boundary condition $e^{(\alpha)}{}_\nu\to\delta^\alpha{}_\nu$ in $\tilde{x}^\mu$ coordinates at null infinity. We define a four-momentum density for the gravitational radiation as

$$P^{(\alpha)\mu}=e^{(\alpha)}{}_\nu T^{\nu\mu}{}_{\mathrm{HF}}.$$

Then, by (4.6) and (5.5) we find $P^{(\alpha)\mu}{}_{;\mu}=0$, and we may use (5.3). As before, introducing retarded-time coordinates on Σ_1 and Σ_2, we find $I(\Sigma_1)=I(\Sigma_2)=0$, and the total four-momentum arriving at infinity is

$$\tilde{P}^{(\alpha)}\equiv I(\sigma_2)=-I(\sigma_1).$$

To show that this agrees with the usual special relativistic result, we evaluate $I(\sigma_2)$ with coordinates $\tilde{x}^\mu$ and choose σ_2 as the flat spacelike region at null infinity given by $\tilde{x}^0=$ const. Then we find

$$\tilde{P}^{(\alpha)}=\int_{\sigma_2}T^{\alpha0}{}_{\mathrm{HF}}d^3\tilde{x},$$

which is the expected result. However, the utility of all this lies in the fact that we may also evaluate $\tilde{P}^{(\alpha)}$ on the general hypersurface σ_1 where

$$\tilde{P}^{(\tau)}=-\frac{1}{3!}\int_{\sigma_1}q^2e^{(\tau)}{}_\nu k^\nu k^\mu\epsilon_{\mu\alpha\beta\gamma}(-\gamma)^{1/2}dx^\alpha\wedge dx^\beta\wedge dx^\gamma. \tag{5.6}$$

Here σ_1 may be taken inside a region of strong gravitational fields, so the tetrad field $e^{(\alpha)}{}_\nu$ has explored space to the extent that (5.6) automatically separates off the part of the radiation which will escape the binding of the source and incorporates all red shifts which this radiation suffers as it climbs out of the strong gravitational potential near its point of creation. The four integrals (5.6) are all scalars under coordinate transformations on σ_1 once the asymptotic coordinates $\tilde{x}^\mu$ are specified. This means that (5.6) may be evaluated in any coordinate system on σ_1, not necessarily in Lorentzian coordinates, as is required for expressions dependent on the Landau and Lifschitz pseudotensor.[6] It should be remembered, however, that σ_1 may not be taken arbitrarily close to an intense source, since if space becomes highly curved, the ray congruence k^α develops caustics as light rays intersect and the geometrical-optics limit breaks down. This is precisely the same limitation as in the Bondi-Sachs multipole analysis.[10]

Now let us investigate the effect of changing our choice of asymptotic coordinates $\tilde{x}^\mu$ by a Lorentz transformation corresponding to a rotation of four-space at infinity. Then new asymptotically Lorentzian coordinates are given by

$$\hat{x}^\mu=L^\mu{}_\nu\tilde{x}^\nu. \tag{5.7}$$

We require a new tetrad of vectors denoted by $f^{(\alpha)}{}_\nu$ which point along the $\hat{x}^\nu$ axis at null infinity. This is given by

$$f^{(\alpha)}{}_\nu=L^\alpha{}_\mu e^{(\mu)}{}_\nu. \tag{5.8}$$

In (5.6), the net result of changing our asymptotic Lorentzian coordinates is the effect induced by the change of tetrad. Then (5.4) and (5.6) transform as a scalar and four-vector, namely,

$$\hat{N}=\tilde{N},\quad \hat{P}^{(\alpha)}=L^\alpha{}_\mu\tilde{P}^{(\mu)}, \tag{5.9}$$

confirming their special relativistic character.

6. GENERALIZATION TO SEVERAL WAVE FRONTS

So far we have been concerned with the presence of only one monochromatic wavefront at a point, as is implicit in the WKB form (4.4a). We may extend this to the case where several sources are present, or where a source (or sources) produces a polychromatic spectral

[10] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) **A269**, 21 (1962); R. K. Sachs, *ibid.* **270**, 103 (1962).

distribution. In either case, we assume the existence of an asymptotic expansion as ϵ approaches zero in the form

$$h_{\mu\nu}=\sum_m h_{\mu\nu}(m)\,, \quad \text{(WKB)} \tag{6.1}$$

where each component $h_{\mu\nu}(m)=A_{\mu\nu}(m)e^{i\phi(m)}$ is the (approximate) solution to the wave equation (2.1). Since the wave equation is linear, (6.1) is also a solution. Combining the WKB and BH approximations, we substitute (6.1) in (4.1) and drop corrections of order ϵ. Then, since k_μ and $A^{\rho\tau}$ are slowly varying functions over the region of integration, we have

$$\langle h^{\rho\tau}{}_{,\mu}h_{\rho\tau,\nu}\rangle=\sum_{m,n}k_\mu(m)k_\nu(n)A^{\rho\tau}(m)A_{\rho\tau}(n)\times\langle\sin\phi(m)\sin\phi(n)\rangle\,.$$

For the usual case of incoherent sources of waves, we may drop all terms where $m\neq n$ and use $\langle\sin^2\phi(m)\rangle=\frac{1}{2}$. In this manner, we find that for incoherent sources

$$T_{\mu\nu}{}^{\text{HF}}=\sum_m T_{\mu\nu}{}^{\text{HF}}(m)\,, \qquad T_{\mu\nu}{}^{\text{HF}}(m)=q^2(m)k_\mu(m)k_\nu(m)\,, \tag{6.2}$$

and so we have a superposition principle for the stress tensors as well as for the amplitudes of the various components of the radiation field.

7. SPHERICAL GRAVITATIONAL WAVES

To give concreteness to the formalism which has been developed, in this section we apply both the BH and WKB approximations to a specific example. In order to simplify the algebra, a highly symmetric situation is needed, so we will investigate radiation from a spherical body of mass m. This may be interpreted as a star or gravitational geon leaking away high-frequency gravitational waves. Of course, if the star were truly spherically symmetric in its motion, no radiation could be emitted, so symmetry is to be interpreted as holding after some sort of average in time or over many independent modes of oscillation. Radiation emitted by some perturbation on the average symmetry will propagate into space, still feeling the influence of the gravitational field of the star. The result is a spherical shell of radiation expanding in a spherically symmetric background geometry determined from (4.7). Vaidya[11,12] has found an exact solution to these Einstein equations for the background in the form

$$dS^2=[1-2m(u)/r]du^2+2du\,dr-r^2(d\theta^2+\sin^2\theta d\phi^2)\,. \tag{7.1}$$

Here $m(u)$ is a nonincreasing but otherwise arbitrary function of the retarded time u and may be interpreted as the mass of the star measured by an observer at infinity. If $m(u)$ is constant, then the substitution

$$u=t-r-2m\ln(r-2m)$$

brings us to the Schwarzschild form of the metric.

Now that we have found the background geometry created by and consistent with the radiation field, we may introduce observers and see what properties of the gravitational waves they may measure. We assume, following Lindquist, Schwartz, and Misner,[13] that our observer has four-velocity v^μ but only moves radially. Then, defining

$$U=dr/d\tau\equiv r_{;\mu}v^\mu$$

and using $v^\mu v_\mu=1$, we find

$$v^\mu=((\gamma+U)^{-1},U,0,0)\,, \tag{7.2}$$

where

$$\gamma\equiv(1+U^2-2m/r)^{1/2}.$$

Any spherical generator of gravitational radiation must have its boundary radius outside the physically unaccessible region bounded by $r=2m(u)$ and must emit waves whose phase can only depend upon the retarded time u. Hence, in (4.4), $\phi=\phi(u)$, and so $k_\mu=(\dot\phi,0,0,0)$, where $\dot\phi=\phi_{,u}$. The moving observer measures a frequency in his rest frame given by

$$\omega(u)=k_\mu v^\mu=(\gamma+U)^{-1}\dot\phi\,. \tag{7.3}$$

For an observer at rest at infinity, the measured frequency is $\omega_\infty(u)=\dot\phi(u)$. Thus, in general, we have the frequency-shift formula

$$\omega(\gamma+U)=\omega_\infty\,, \tag{7.4}$$

relating the frequency ω measured (or emitted) by an observer with radial velocity U to that measured by an observer at rest at infinity, including all gravitational and Doppler effects.

The modifications which the background imposes on the radiation are given by (4.4) in the Vaidya geometry. From these we find

$$h_{\mu\nu}=r^{-1}A(u)e_{\mu\nu}e^{i\phi(u)}\,, \tag{7.5}$$

allowing for AM or FM transmission of information via the functions $A(u)$ and $\phi(u)$, which must satisfy Eq. (6.4) of I but which are otherwise arbitrary. There are precisely two polarizations representing true gravitational effects rather than just coordinate waves. These are the only two modes which give a nonvanishing contribution to the dominant part of the Riemann tensor $R_{\alpha\beta\gamma\delta}{}^{(1)}\simeq 2k_{[\alpha}h_{\beta][\gamma}k_{\delta]}$ (see I). They are transverse traceless modes with explicit form

$$e_{\mu\nu}{}^{(1)}=\frac{r^2\sin^2\theta}{\sqrt{2}}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \tag{7.6}$$

[11] P. C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951).
[12] P. C. Vaidya, Nature **171**, 260 (1953).
[13] R. W. Lindquist, R. A. Schwartz, and C. W. Misner, Phys. Rev. **137**, B1364 (1965).

and

$$e_{\mu\nu}^{(2)}=\frac{r^2}{\sqrt{2}}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-\sin^2\theta\end{bmatrix},\tag{7.7}$$

where μ, $\nu=u, r, \theta, \phi$. The only nonvanishing component of the Ricci tensor of the background is

$$R_{00}^{(0)}=(2/r^2)dm/du\,,$$

which gives, from (4.7),

$$\begin{aligned}dm(u)/du&=-\tfrac{1}{16}\epsilon^2[A(u)\omega_\infty(u)]^2\\&=-\tfrac{1}{16}\epsilon^2[A(u)\omega_0(u)]^2[1-2m(u)/r_0(u)],\end{aligned}\tag{7.8}$$

where ω_0 is the frequency of the radiation measured at the surface of the star where the radius is $r=r_0$. If the star continuously emits radiation with frequency $\omega_0(u)$ and specific amplitude $A(u)$ as it collapses toward $r_0=2m(u)$, then the observed mass at infinity tends toward a constant value, and ω_∞ tends toward zero, i.e., all the radiation is red-shifted away as r_0 approaches $2m(u)$.

From (7.8), we see that there is no tail to the radiation; A and ω are nonzero only when the driving mass of the source is changing. The radiation travels outward along the light cones $u=\text{const}$, without being scattered off these null hypersurfaces. Conversely, from (7.8) we see that the $A^2\omega_\infty^2$ is a "news" function of the kind found by Bondi,[10] and a radiating star must lose mass whenever information is carried away by radiation.

The existence of radiation is compatible with having space asymptotically flat, for we may introduce coordinates

$$\begin{aligned}\tilde{x}^0&=u+r\,,\\\tilde{x}^1&=r\sin\theta\cos\phi\,,\\\tilde{x}^2&=r\sin\theta\sin\phi\,,\\\tilde{x}^3&=r\cos\theta\,,\end{aligned}\tag{7.9}$$

in terms of which the metric becomes

$$\begin{aligned}g^{00}&=1+2m/r\,,\\g^{0i}&=2mx^i/r^2\,,\\g^{ij}&=2mx^ix^j/r^3-\delta_j{}^i\,,\end{aligned}\tag{7.10}$$

which tends to the Lorentz form as r approaches infinity.

If we evaluate the conserved graviton number N by (5.4) over a general hypersurface $r=\text{const.}$, we find that it satisfies the differential law

$$\frac{dN}{du}=-\frac{64\pi}{\epsilon^0}\frac{1}{\omega_\infty}\frac{dm}{du}.\tag{7.11}$$

In order to calculate the energy and momentum in the radiation fields, we express the tetrad components in the (u,r,θ,ϕ) coordinate system which displays the symmetry of the problem. Then, using (5.6) on the $r=\text{constant}$ hypersurface, we find

$$P^{(\mu)}=(\Delta m,0,0,0)\,,$$

where Δm is the change in mass of the source during the time it radiates gravitationally, as measured by an observer at infinity. Because of spherical symmetry, the source cannot lose momentum, so we see that the energy and momentum carried by the radiation is just balanced by the energy and momentum lost by the source, and over-all total conservation is maintained.

8. HIGHER-ORDER CORRECTIONS TO THE METRIC

Having explored many features of the high-frequency approximation, we now will see if it is possible to extend it to give a systematic method for generating an exact solution. Trautman[7,14] and Fock[15] have shown that the weak-field approximation already runs into divergences when pushed to the second order, and therefore we check the analogous case for the high-frequency approximation. We consequently assume that to second order the exact metric is of the form

$$g_{\mu\nu}(x)=\gamma_{\mu\nu}(x)+\epsilon h_{\mu\nu}^{(1)}(x,\epsilon)+\epsilon^2h_{\mu\nu}^{(2)}(x,\epsilon)\,,\tag{8.1}$$

where γ is the slowly changing background, and $h^{(1)}$, $h^{(2)}$ are high-frequency wave components. The Ricci tensor may now be expanded to give

$$\begin{aligned}R_{\mu\nu}(g)=R_{\mu\nu}^{(0)}(\gamma)+\epsilon R_{\mu\nu}^{(1)}(h^{(1)})\\+\epsilon^2[R_{\mu\nu}^{(2)}(h^{(1)})+R_{\mu\nu}^{(1)}(h^{(2)})]+\epsilon^3R_{\mu\nu}^{(3+)}\,,\end{aligned}\tag{8.2}$$

where $R^{(0)}$, $R^{(1)}$, and $R^{(2)}$ are defined as in I, Eqs. (2.6)–(2.8), with either $h^{(1)}$ or $h^{(2)}$ replacing the argument h given there, and $R^{(3+)}$ is again a remainder term. We next decompose (8.2) into a system of equations in which the various frequency components are separately made to vanish. Thus, if $h^{(1)}$ is a wave of frequency ω, $R^{(1)}(h^{(1)})$ will still be of frequency ω, while $R^{(0)}(\gamma)$ has frequency 0. Since $R^{(2)}(h^{(1)})$ contains terms typically of the form $\partial h^{(1)}\partial h^{(1)}$, it will have frequencies 0 and 2ω. To satisfy the field equations, $R^{(1)}(h^{(2)})$ must then have frequencies ω and 2ω, which requires that

$$h_{\mu\nu}^{(2)}=p_{\mu\nu}+m_{\mu\nu}\,,\tag{8.3}$$

where $p_{\mu\nu}$ is of frequency ω, while $m_{\mu\nu}$ is of frequency 2ω. Now we group terms of the same frequency into separate equations to get an approximate solution to the Einstein equations as

$$R_{\mu\nu}^{(1)}(h^{(1)}+\epsilon p)=0\,,\tag{8.4a}$$

$$R_{\mu\nu}^{(0)}(\gamma)=-\epsilon^2\langle R_{\mu\nu}^{(2)}(h^{(1)})\rangle\,,\tag{8.4b}$$

$$R_{\mu\nu}^{(1)}(m)=\langle R_{\mu\nu}^{(2)}\rangle-R_{\mu\nu}^{(2)}.\tag{8.4c}$$

[14] A. Trautman, in Proceedings of the International Conference on Relativistic Theories of Gravitation, London, 1965 (unpublished).

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As an example of how this may be solved, we now add the WKB assumption and calculate $m_{\mu\nu}$ according to (8.4). Choosing $h^{(1)}$ to have the WKB form

$$h_{\mu\nu}^{(1)}=A_{\mu\nu}\cos\phi\,, \qquad (8.5)$$

where $\phi_{,\nu}\equiv k_\nu$ and $A_{\mu\nu}=\mathcal{A}e_{\mu\nu}$, we find [see Eq. (4.5)] that to lowest order

$$R_{\mu\nu}^{(2)}(h^{(1)})=\tfrac{1}{4}\mathcal{A}^2\sin^2\phi=\tfrac{1}{8}\mathcal{A}^2(1-\cos2\phi)\,,$$

where the 1 represents the zero-frequency part and the 2ϕ gives the double-frequency component. Now γ and $h^{(1)}$ are found from (4.7) and (8.5), while m is determined from

$$m_{\mu\nu}{}^{;\beta}{}_{;\beta}+m_{;\mu\nu}-m_\mu{}^\beta{}_{;\nu\beta}-m_\nu{}^\beta{}_{;\mu\beta}=\tfrac{1}{4}\mathcal{A}^2\,k_\mu k_\nu\cos2\phi\,. \qquad (8.6)$$

We assume m has the usual WKB form

$$m_{\mu\nu}=B_{\mu\nu}\cos\psi\,,\quad \psi_{,\nu}\equiv q_\nu\,.$$

Substitute this into (8.6), retain only the dominant terms for a consistent approximation, and thereby get

$$-B_{\mu\nu}q_\beta q^\beta\cos\psi-B^\alpha{}_\alpha q_\mu q_\nu\cos\psi+B_\mu{}^\beta q_\nu q_\beta\cos\psi$$
$$+B_\nu{}^\beta q_\mu q_\beta\cos\psi=\tfrac{1}{4}\mathcal{A}^2k_\mu k_\nu\cos2\phi\,. \qquad (8.7)$$

Let $\psi=2\phi$; then $q_\nu=2k_\mu$, and (8.7) becomes

$$-B^\alpha{}_\alpha k_\mu k_\nu+B_\mu{}^\beta k_\beta k_\nu+B_\nu{}^\beta k_\beta k_\mu=\tfrac{1}{16}\mathcal{A}^2k_\mu k_\nu\,. \qquad (8.8)$$

Then to solve (8.8) we need only require

$$B^\alpha{}_\alpha=-\tfrac{1}{16}\mathcal{A}^2\,, \qquad (8.9a)$$

$$B^{\mu\nu}k_\nu=0\,. \qquad (8.9b)$$

Now we try to determine $p_{\mu\nu}$ in (8.4a) in order to improve our approximate WKB solution to the wave equation. With proper choice of gauge, this reduces to (2.1), into which we insert

$$h_{\mu\nu}=h_{\mu\nu}^{(1)}+\epsilon p_{\mu\nu}\,,$$

with $p_{\mu\nu}$ in the form

$$p_{\mu\nu}=iC_{\mu\nu}e^{i\phi}=C_{\mu\nu}e^{i(\phi+\pi/2)}\,.$$

This converts the wave equation into an ordinary differential equation along the rays, with solution given in terms of integrals along the null geodesics $x^\mu(l)$ as

$$C_{\mu\nu}(l)=e^{-\alpha(l)}\left\{\epsilon^{-1}\int_{l_0}^{l}e^{\alpha(q)}[A_{\mu\nu}{}^{;\alpha}{}_{;\alpha}-2R^{(0)}{}_\mu{}^\alpha{}_\nu{}^\beta A_{\alpha\beta}]dq+C_{\mu\nu}(l_0)\right\}\,, \qquad (8.10)$$

$$\alpha(l)\equiv\int_{l_0}^{l}k^\beta{}_{;\beta}dq\,.$$

If $R^{(0)}$ is bounded and no caustics develop, then $C_{\mu\nu}$ remains finite, and the metric is given to second order by

$$g_{\mu\nu}=\gamma_{\mu\nu}+\epsilon A_{\mu\nu}e^{i\phi}+\epsilon^2B_{\mu\nu}e^{2i\phi}+\epsilon^2C_{\mu\nu}e^{i(\phi+\pi/2)}\,. \qquad (8.11)$$

Note that no divergences arise to second order, and so the second-order terms really are a factor of ϵ smaller than the first-order ones.

The success uncovered so far makes it seem likely that the high-frequency expansion (8.11) may be pushed to still higher orders, and the results to be uncovered thereby should be an interesting subject for future research.

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APPENDIX: COMPUTATION OF AVERAGES

In this Appendix, we show how to construct the average of a rapidly oscillating tensor field, and justify integration by parts within such an average.

The result of integrating a tensor field does not give a tensor in a curved space, because tensors at different points have different transformation properties. Since it is permissible to add tensors at the same point, we must go about constructing an average by somehow carrying the tensors back to a common point and adding them there. To do this in a unique manner, we introduce the *bivector of geodesic parallel displacement*, denoted by $g_\alpha{}^{\beta'}(x,x')$ (DeWitt and Brehme[16]; Synge[17]). This transforms as a vector with respect to coordinate transformations at either x or x', and, assuming that x and x' are sufficiently close together to insure the existence of a unique geodesic of the metric $\gamma_{\alpha\beta}$ between them, given the vector $A_{\beta'}$ at x', then $A_\alpha=g_\alpha{}^{\beta'}A_{\beta'}$ is the unique vector at x which can be obtained by parallel-transporting $A_{\beta'}$ from x' back to x along the geodesic.

Given a tensor $T_{\mu\nu}$ which is assumed to have high-frequency components of wavelength λ and a background geometry $\gamma_{\mu\nu}$ containing only low-frequency components of wavelength $L\gg\lambda$, then we define the average of $T_{\mu\nu}$ to be the tensor

$$\langle T_{\mu\nu}(x)\rangle\equiv\int_{\text{all space}}g_\mu{}^{\alpha'}(x,x')g_\nu{}^{\beta'}(x,x')\times T_{\alpha'\beta'}(x')f(x,x')d^4x'\,, \qquad (A1)$$

where $f(x,x')$ is a weighting function which falls smoothly to zero when x and x' differ by a distance d ($\lambda\ll d\ll L$), and where

$$\int_{\text{all space}}f(x,x')d^4x=1\,.$$

Since f vanishes well within the region where the background remains approximately flat, there is no problem about the global existence of unique geodesics from x needed in the definition of $g_\mu{}^{\alpha'}$. Also, $\partial f\sim f/d=O(1)$.

[16] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N. Y.) **9**, 220 (1960).

[17] J. L. Synge, *Relativity, the General Theory* (North-Holland Publishing Co., Amsterdam, 1960), p. 57.

Since $g_\mu{}^{\alpha'}$ depends on the background geometry, it clearly changes only over a distance L, and $\partial g_\mu{}^{\alpha'} \sim g_\mu{}^{\alpha'}/L = O(1)$.

The only rapidly varying element in the construction is $T_{\mu\nu}$, since $\partial T \sim T/\lambda = O(\epsilon^{-1})$.

Let us now ask what happens to tensors of the form $T_{\mu\nu} = S_{\mu\nu}{}^{\rho}{}_{;\rho}$ under averages. Inserting this into (A1), we obtain

$$\langle S_{\mu\nu}{}^{\rho}{}_{;\rho}\rangle = \int g_\mu{}^{\alpha'} g_\nu{}^{\beta'} S_{\alpha'\beta'}{}^{\rho'}{}_{;\rho'} f d^4x'$$

$$= \int [(ggSf)_{;\rho'} - (g_{;\rho'}gSf) - (gg_{;\rho'}Sf) - (ggSf_{;\rho'})] d^4x'. \quad \text{(A2)}$$

Since the first term may be converted to a surface integral taken in the region where $f \to 0$, we see that the right side of (A2) contains no contributions of the form ∂S. Since we assume $\partial S = O(\epsilon^{-1})$, this implies

$$\langle S_{\mu\nu}{}^{\rho}{}_{;\rho}\rangle \doteq \epsilon,$$

proving our earlier assertion that divergence may be neglected in averages as $\epsilon \to 0$.

As for justifying integration by parts, this is a trivial corollary, since

$$\langle h_\nu{}^{\tau;\rho} h_{\rho\mu;\tau}\rangle = -\langle h_\nu{}^{\tau;\rho}{}_{;\tau} h_{\rho\mu}\rangle + \langle S_{\mu\nu}{}^{\tau}{}_{;\tau}\rangle,$$

where

$$S_{\mu\nu}{}^{\tau} = h_\nu{}^{\tau;\rho} h_{\rho\mu}.$$

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Search for Quarks in Cosmic Rays at Sea Level*

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A search for quarks has been conducted at sea level. No events have been found with energy losses by ionization in the range 0.06–0.165 and 0.25–0.65 that of singly charged minimum-ionizing particles. The experiment sets an upper limit on the fluxes as 6.6×10^{-11} and 8.8×10^{-11} cm^{-2} sr^{-1} sec^{-1} for $\frac{1}{3}e$ and $\frac{2}{3}e$ minimum-ionizing quarks, respectively.

THEORETICAL concepts based on $SU(3)$ symmetry have led to speculations concerning the existence of fractionally charged ($\pm\frac{1}{3}e$ and $\pm\frac{2}{3}e$) particles, or quarks.[1,2] Prompted by the possibility that such particles might have a sufficiently small mass to be produced in cosmic-ray interactions, a number of groups have carried out experiments in search of quarks.[3–10] An experiment has been running at the Massachusetts Institute of Technology which provides an upper limit to the flux of quarks at sea level of

$$\pm\tfrac{1}{3}e: 6.6\times10^{-11}\ \text{cm}^{-2}\ \text{sr}^{-1}\ \text{sec}^{-1},$$

$$\pm\tfrac{2}{3}e: 8.8\times10^{-11}\ \text{cm}^{-2}\ \text{sr}^{-1}\ \text{sec}^{-1},$$

with a 95% confidence. Although these limits are not appreciably lower than those obtained from the other experiments,[7,10] the conditions under which they are obtained are considerably different, and hence the biases will be different. A new signature technique using proportional counters has been investigated that shows promise of reducing background effects by a factor of ten for each counter in the detector array.[11]

The detector array and electronics block diagram are shown in Fig. 1. The plastic scintillators are disks 40 in. in diameter and 6 in. thick inside a metal can that provides a 6-in. air gap between the photomultiplier photocathodes and the scintillator to improve uniformity. The liquid scintillator is identical except it is 50 in. in diameter to eliminate edge effects. Each scintillator is viewed by four 5-in. photomultipliers (Dumont 6364) providing a uniform response over the entire sensitive area of each detector. The pulse-

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11 An additional signature can be obtained from the proportional counters by using the rise-time characteristics of the output pulses. In the proportional counters used here, the rise time for a minimum-ionizing particle passing through the counter is between 1 and 2 μsec. The pulse produced by a Compton-scattered electron in the gas or counter wall corresponding to $\frac{1}{9}$ or 4/9 minimum ionization is found to have a rise time of 0.2–0.5 μsec. By using the rise-time signature a gain of about 10 in background suppression has been realized over simple pulse-height analysis, in a pilot experiment using the detector array with a single proportional counter and a short delay. It was not possible to use this technique with both proportional counters, because of equipment limitations. Instead, long delay lines had to be used, which lost the rise-time character of the pulse.