the same range of excitation and for the same optical potential, however, the (p,t) cross sections are found to be only in marginally better agreement with experiment. The agreement in these $(p,{}^3\text{He})$ cross sections is certainly acceptable—the present theory predicting fairly well those states which are strongly or weakly excited.

It would appear from the above results that a comparison of experimental *relative* cross sections with theory in a $(p,{}^3\text{He})$ [or $({}^3\text{He},p)$] reaction on a $T=\frac{1}{2}$ target does not clarify the discussion presented earlier[52] which indicated (1) a necessity for some spin-dependent nucleon-nucleon interaction in the two-nucleon transfer theory, and (2) the probable necessity for including spin-orbit coupling in the optical potential. DWBA calculations that reliably predicted *absolute* cross sections for these two-nucleon transfer reactions and could incorporate these effects would certainly resolve the problem. Insofar as the first effect is considered, a comparison of experimental and theoretical relative cross sections for $(p,{}^3\text{He})$ [or $({}^3\text{He},p)$] transitions on $T=0$ targets would be expected to be much more sensitive to the presence of a spin-dependent nucleon-nucleon force, since here the neutron-proton pair is transferred in unique ${}^3S$, $T=0$ or ${}^1S$, $T=1$ states. Calculations by Hardy and Towner[59] of the states populated both in the ${}^{12}\text{C}({}^3\text{He},p){}^{14}\text{N}$ reaction at 20 MeV[8] and the ${}^{16}\text{O}(p,{}^3\text{He}){}^{14}\text{N}$ reaction at 40 MeV[60] show that a spin dependence, consistent with that used in theoretical calculation and compatible with that used previously by us,[52] is required in order to account for the experimental cross sections. We will further discuss the necessity for including a spin dependence in the two-nucleon DWBA calculation in a forthcoming report on the ${}^{16}\text{O}(p,{}^3\text{He}){}^{14}\text{N}$ and ${}^{16}\text{O}(p,t){}^{14}\text{O}$ reactions.

ACKNOWLEDGMENT

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[60] R. E. Brown, N. M. Hintz, C. G. Hoot, J. R. Maxwell, and A. Scott, University of Minnesota Annual Progress Report, 1966, p. 70 (unpublished).

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Pology and (d,p) Reactions

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Cross sections for (d,p) reactions calculated in the physical region are extrapolated to the Butler pole in order to investigate the possibility of extracting reduced widths. The cross sections are calculated by distorted-wave Born approximation (DWBA), using a method of evaluation yielding cross sections with the correct analytic behavior near the pole. The cross sections in the unphysical region between the pole and the start of the physical region are calculated and compared with the extrapolation of cross sections from the physical region. Detailed calculations are made for two reactions in which the neutron is captured into an s state: $\text{Si}^{28}(d,p)\text{Si}^{29}$ to the ground state, and $\text{C}^{12}(d,p)\text{C}^{13}$ (3.09 MeV). It is found that an effect of the Coulomb interaction prevents accurate extrapolation to the pole for heavy nuclei or low energies. The method of evaluating the DWBA cross section has an advantage over the usual method in that fewer partial waves need be summed.

I. INTRODUCTION

THE amplitude for the stripping reaction $A+d\rightarrow B+p$ for fixed deuteron energy E_d is a function of the variable

$$Q^2=[\mathbf{k}_d-(m_A/m_B)\mathbf{k}_p]^2, \qquad (1.1)$$

with a pole,[1,2] the Butler pole, at $Q^2=-\kappa_n^2$. $\mathbf{k}_d$ and $\mathbf{k}_p$ are the deuteron and proton momenta in the c.m. system, and

$$\kappa_n^2=-2m_{nA}B/\hbar^2, \qquad (1.2)$$

where B is the binding energy of the captured neutron. The notation

$$m_{xy}=m_xm_y/(m_x+m_y)$$

is used for reduced masses. The residue at the Butler pole is proportional to the reduced width for the reaction $A+n\rightarrow B$. Amado[1] pointed out the possibility of obtaining this reduced width by extrapolating the stripping cross section to the pole.

Very accurate experimental results are required for this extrapolation, and the reliability of the reduced widths obtained cannot be checked. It thus seems desirable to perform the extrapolation when the reduced width is already known in order to check the reliability of the extrapolation. A way of doing this is by means of

[1] R. D. Amado, Phys. Rev. Letters 2, 399 (1959).
[2] H. J. Schnitzer, Rev. Mod. Phys. 37, 666 (1965).

a computer experiment.[3,4] Cross sections are calculated in the physical region and then extrapolated to the pole to yield a reduced width which is then compared with the known reduced width of the calculation. This paper describes such a computer experiment.

A difficulty with these computer experiments is that the stripping cross section cannot be calculated exactly, but some approximation must be used. The results of this paper show that the approximation must be chosen with some care.

Extensive computer experiments on stripping reactions have been performed by Dullemond and Schnitzer.[3] They concluded that cross sections calculated by the distorted-wave Born approximation (DWBA) cannot be extrapolated reliably to the Butler pole. However, in the DWBA method, the partial-wave expansion of the initial and final wave functions is used, and in the DWBA calculation used in DS (and in most, if not all, other DWBA calculations) only a finite number of partial waves are summed, so that the amplitudes obtained are polynomials in $\cos\theta$ and do not have poles. Thus a sufficiently accurate extrapolation of such a DWBA cross section should yield zero for the reduced width, and it is interesting to note that this result was obtained for one case in DS.

Because in the usual calculations, the DWBA amplitude does not have a pole, the problem of extrapolating stripping cross sections to the Butler pole cannot be investigated reliably by examining the extrapolations of these DWBA cross sections.

This paper describes a computer experiment similar to that in DS. However, the cross sections are obtained from a DWBA calculation which was modified to include the summation over all the high partial waves by using a closed form for this infinite sum. The amplitudes obtained from this calculation have the correct analytic behavior in a neighborhood of the Butler pole. The effect of the Coulomb interaction between the incoming deuteron and the target nucleus and between the outgoing proton and the residual nucleus is considered. As well as adding a cut from the Butler pole to $Q^2=-\infty$, the Coulomb field also modifies the relation between the reduced width and the residue at the pole. This latter effect of the Coulomb field was neglected in DS.

With the modified DWBA, cross sections are also calculated in the unphysical region between the Butler pole and the start of the physical region, and compared with those obtained by extrapolation from the physical region. It can then be seen that the effects of the Coulomb interactions make the extrapolation of stripping cross sections to the Butler pole almost impossible in some cases.

II. IN A NEIGHBORHOOD OF THE POLE

The differential cross section for the stripping reaction $A+d\rightarrow B+p$ can be written[4,5]

$$\sigma(\theta)=\frac{m_{dA}m_{pB}}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}\frac{1}{3(2J_A+1)}\sum_{\substack{M_A\mu_d\\M_B\mu_p}}|I_{\mu_pM_B\mu_dM_A}|^2. \quad (2.1)$$

If the neutron is captured into a state of definite angular momenta j_n and l_n, then

$$I_{\mu_pM_B\mu_dM_A}=\sum_{M_nm_n}\langle J_A,M_A,j_n,M_n|J_B,M_B\rangle$$
$$\times\langle l_n,m_n,\tfrac{1}{2},\mu_n|j_n,M_n\rangle\langle\tfrac{1}{2},\mu_p,\tfrac{1}{2},\mu_n|1,\mu_d\rangle M_{j_nl_nm_n}. \quad (2.2)$$

Since the singularities introduced by the short-range nuclear potentials between the deuteron and the target nucleus and between the proton and the residual nucleus do not coincide with the Butler pole,[1] the DWBA amplitude in a neighborhood of the pole is given correctly by the Coulomb-wave Born approximation. It has been shown[6] that, in a neighborhood of the Butler pole, the zero-range approximation can be used for the neutron-proton force and the bound-state radial wave function $u_l(r)$ of the captured neutron can be replaced by its asymptotic form

$$u_l(r)=N_lr^{-1}\exp(-\kappa_nr). \quad (2.3)$$

N_l can be related to the usual reduced width. However, DS have suggested N_l as a definition of the reduced width because it is a model-independent quantity. The DWBA amplitude in a neighborhood of the Butler pole is then given by (2.2), with $M_{j_nl_nm_n}$ replaced by

$$M_{j_nl_nm_n}{}^c=N_{l_n}Ke^{-\frac{1}{2}\pi(\eta_1+\eta_2)}\Gamma(1+i\eta_1)\Gamma(1+i\eta_2)$$
$$\times\int(e^{-\kappa_nr}/r)Y_{l_nm_n}{}^*(\mathbf{r}/r)e^{i\mathbf{Q}\cdot\mathbf{r}}F(1)F(2)d\mathbf{r}, \quad (2.4)$$

with

$$F(1)=F[-i\eta_1;1;i(k_1r-\mathbf{k}_1\cdot\mathbf{r})],$$
$$F(2)=F[-i\eta_2;1;i(k_2r+\mathbf{k}_2\cdot\mathbf{r})],$$
$$K=(\hbar^2/2m_{np})(8\pi\kappa_d),\quad \kappa_d{}^2=-(2m_{np}/\hbar^2)B_d,$$

where B_d is the binding energy of the deuteron.

$$\eta_1=Ze^2/\hbar v_{dA},\quad \eta_2=Ze^2/\hbar v_{pB},$$

where v_{dA} and v_{pB} are the velocities of d relative to A and of p relative to B, respectively, and Z is the charge of the target nucleus,

$$\mathbf{k}_1=\mathbf{k}_d,\quad \mathbf{k}_2=(m_A/m_B)\mathbf{k}_p,$$

and

$$\mathbf{Q}=\mathbf{k}_1-\mathbf{k}_2.$$

For the case where $l_n=0$, the integral in (2.4) can be

[3] C. Dullemond and H. J. Schnitzer, Phys. Rev. **129**, 821 (1963), hereafter referred to as DS.

[4] N. K. Glendenning, Ann. Rev. Nucl. Sci. **13**, 191 (1963).

[5] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961), Chap. 1.

[6] M. Andrews, W. K. Bertram, and L. J. Tassie (to be published).

evaluated, yielding[7,8]

$$M_{\frac{1}{2}00}{}^c=4\pi N_0 K e^{-\frac{1}{2}\pi(\eta_1+\eta_2)}\Gamma(1+i\eta_1)\Gamma(1+i\eta_2)$$
$$\times\left[\frac{(\kappa_n-ik_1)^2+k_2{}^2}{Q^2+\kappa_n{}^2}\right]^{i\eta_1}\left[\frac{(\kappa_n-ik_2)^2+k_1{}^2}{Q^2+\kappa_n{}^2}\right]^{i\eta_2}\frac{1}{Q^2+\kappa_n{}^2}$$
$$\times{}_2F_1\left[-i\eta_1,-i\eta_2;1;1-\frac{Q^2+\kappa_n{}^2}{\kappa_n{}^2+(k_1-k_2)^2}\right]. \quad (2.5)$$

Writing the differential cross section as

$$\sigma(Q^2)=(Q^2+\kappa_n{}^2)^{-2}P(Q^2), \quad (2.6)$$

the residue of $\sigma(Q^2)$ at the Butler pole is

$$\lim_{Q^2\to-\kappa_n{}^2} P(Q^2)=P^c(-\kappa_n{}^2), \quad (2.7)$$

where P^c is evaluated using $M_{j_nl_nm_n}{}^c$ in Eqs. (2.1), (2.2), and (2.6). For $l_n=0$, using (2.5) in this way, we obtain

$$P^c(-\kappa_n{}^2)=\frac{2J_A+1}{2J_B+1}\frac{m_{dA}m_{pB}}{m_{np}{}^2}\frac{k_p}{k_d}\kappa_d N_0{}^2$$
$$\times\frac{2\pi(\eta_1+\eta_2)\exp[2\pi(\eta_1+\eta_2)]}{\exp[2\pi(\eta_1+\eta_2)]-1}$$
$$\times\exp[-2(\eta_1\phi_1+\eta_2\phi_2)], \quad (2.8)$$

where

$$\tan\phi_1=2\kappa_n k_1/(k_1{}^2-k_2{}^2-\kappa_n{}^2),\quad 0<\phi_1\leqq\pi$$
$$\tan\phi_2=2\kappa_n k_2/(k_2{}^2-k_1{}^2-\kappa_n{}^2),\quad 0<\phi_2\leqq\pi.$$

When Coulomb interactions are neglected, the amplitude in a neighborhood of the pole is given by the plane-wave Born approximation using the asymptotic form (2.3) of the neutron bound-state wave function. Then

$$M_{j_nl_nm_n}{}^{\rm pw}=4\pi N_{l_n}K(Q^2+\kappa_n{}^2)^{-1}, \quad (2.9)$$

and the residue at the pole is

$$P^{\rm pw}(-\kappa_n{}^2)=\frac{2J_A+1}{2J_B+1}\frac{m_{dA}m_{pB}}{(m_{np})^2}\frac{k_p}{k_d}\kappa_d|N_{l_n}|^2. \quad (2.10)$$

Comparing Eqs. (2.8) and (2.10) it is seen that the Coulomb interaction affects the relation between the residue at the Butler pole and N_{l_n}.

The ratio of the non-Coulomb to the Coulomb wave residues at the pole R is a function of the bombarding energy of the incoming deuteron E_d:

$$R(E_d)=\frac{P^{\rm pw}(-\kappa_n{}^2)}{P^c(-\kappa_n{}^2)}=\frac{e^{2\pi(\eta_1+\eta_2)}-1}{2\pi(\eta_1+\eta_2)}$$
$$\times\exp[-2\eta_1(\pi-\phi_1)-2\eta_2(\pi-\phi_2)]. \quad (2.11)$$

[7] K. A. Ter-Martirosian, Zh. Eksperim. i Teor. Fiz. **29**, 713 (1956) [English transl.: Soviet Phys.—JETP **2**, 620 (1956)].
[8] F. B. Morinigo, Nucl. Phys. **50**, 136 (1964).

III. CALCULATION OF CROSS SECTIONS

In order to investigate the extrapolation to the Butler pole we have used cross sections obtained by the DWBA with the zero-range approximation for the neutron-proton force, using the method of evaluation outlined below, which yields amplitudes with the correct analytic behavior in the neighborhood of the pole.

The differential cross section for the stripping reaction is then given by Eqs. (2.1) and (2.2), with

$$M_{j_nl_nm_n}=K\int\psi_p{}^{(-)*}\left(\frac{m_A}{m_B}\mathbf{r}\right)\phi_n{}^*(\mathbf{r})\psi_d{}^{(+)}(\mathbf{r})d\mathbf{r}, \quad (3.1)$$

where $\psi_p{}^{(-)}$ and $\psi_d{}^{(+)}$ are the elastic-scattering wave functions of the proton and the deuteron in the exit and incident channels, respectively, and $\phi_n(r)$ is the bound-state wave function of the captured neutron.

Substituting (3.1) and (2.2), and using the partial-wave expansion of the deuteron and proton wave functions, yields

$$I_{\mu_pM_B\mu_dM_A}=K\sum_{\substack{L_pJ_pM_p\\L_dJ_d}}X(L_dJ_dL_pJ_pM_p;\mu_pM_B\mu_dM_A)$$
$$\times Y_{L_pM_p}(\mathbf{k}_p)Y_{L_dM_d}{}^*(\mathbf{k}_d)R_{L_dJ_dL_pJ_pj_nl_n}, \quad (3.2)$$

where $X(L_dJ_dL_pJ_pM_p;\mu_pM_B\mu_dM_A)$ is a factor which contains all the relevant coupling coefficients[9] and

$$R_{L_dJ_dL_pJ_pj_nl_n}=\exp[i(\delta_{L_pJ_p}+\delta_{L_dJ_d}+\sigma_{L_p}+\sigma_{L_p})]$$
$$\times\int_0^\infty\psi_{L_pJ_p}{}^{(-)*}(k_2r)u_{j_nl_n}(r)\psi_{L_dJ_d}{}^{(+)}(k_1r)r^2dr. \quad (3.3)$$

$\psi_{L_dJ_d}{}^{(+)}(k_1r)$, $\psi_{L_pJ_p}{}^{(-)}(k_2r)$, and $u_{j_nl_n}(r)$ are the radia wave functions of the deuteron, proton, and the neutron; δ and σ are the nuclear and Coulomb phase shifts.

In a neighborhood of the Butler pole, $I_{\mu_pM_B\mu_dM_A}$ is given by $I_{\mu_pM_B\mu_dM_A}{}^c$, which is obtained from Eq. (2.2) by replacing $M_{j_nl_nm_n}$ by $M_{j_nl_nm_n}{}^c$ given by Eq. (2.4) or, for $l_n=0$, by Eq. (2.5). Alternatively, $I_{\mu_pM_B\mu_dM_A}{}^c$ can be obtained from (3.2) by replacing $R_{L_dJ_dL_pJ_pj_nl_n}$ by

$$R_{L_dJ_dL_pJ_p}{}^c\equiv R_{L_dJ_dL_pJ_p\frac{1}{2}0}{}^c=\exp[i(\sigma_{L_d}+\sigma_{L_p})]\frac{N_{l_n}}{k_1k_2}$$
$$\times\int_0^\infty r^{-1}e^{-\kappa_nr}F_{L_d}(\eta_1,k_1r)F_{L_p}(\eta_2,k_2r)dr, \quad (3.4)$$

where $F_l(\eta,r)$ are the regular Coulomb wave functions. The DWBA amplitude (3.2) can now be rewritten as

$$I_{\mu_pM_B\mu_dM_A}=I_{\mu_pM_B\mu_dM_A}{}^c$$
$$+\sum_{L_dL_pJ_dJ_pM_p}X(L_dJ_dL_pJ_pM_p;\mu_pM_B\mu_dM_A)$$
$$\times Y_{L_dM_d}{}^*(\mathbf{k}_d)Y_{L_pM_p}(\mathbf{k}_p)$$
$$\times(R_{L_dJ_dL_pJ_p}-R_{L_dJ_dL_pJ_p}{}^c). \quad (3.5)$$

[9] D. Robson, Nucl. Phys. **22**, 34 (1961).

In a neighborhood of the Butler pole, $I_{\mu_p M_B \mu_d M_A} - I_{\mu_p M_B \mu_d M_A}{}^c$ is finite, so that $I_{\mu_p M_B \mu_d M_A}$ as calculated from Eq. (3.5) has the correct analytic behavior. Furthermore,[10]

$$R_{L_d J_d L_p J_p} \to R_{L_d L_d J_p J_p}{}^c$$

rapidly, as L_p and L_d increase, so that the summation over L_p and L_d in Eq. (3.5) converges very much faster than the one in (3.2). When Coulomb interactions are neglected, then in Eq. (3.5) $I_{\mu_p M_B \mu_d M_A}{}^c$ and $R_{L_d J_d L_p J_p}{}^c$ are replaced by $I_{\mu_p M_B \mu_d M_A}{}^{nc}$ and $R_{L_d J_d L_p J_p}{}^{nc}$, where

$$I_{\mu_p M_B \mu_d M_A}{}^{nc} = \sum_{\mu_n} \langle J_A, M_A, \tfrac{1}{2}, \mu_n | J_B, M_B \rangle \langle \tfrac{1}{2}, \mu_p, \tfrac{1}{2}, \mu_n | 1, \mu_d \rangle \times 4\pi K N_0 (Q^2 + \kappa_n^2)^{-1}, \quad (3.6)$$

$$R_{L_d J_d L_p J_p}{}^{nc} = N_0 \int_0^\infty r^{-1} e^{-\kappa_n r} j_{L_d}(k_1 r) j_{L_p}(k_2 r) r^2 dr. \quad (3.7)$$

Calculations have been performed on an IBM-360 computer for stripping reactions in which the neutron is captured into an s state ($l_n = 0$). The program was written by modifying a conventional DWBA program of Robson[11] to include the calculation of $I_{\mu_p M_B \mu_d M_A}{}^c$ and $R_{L_d J_d L_p J_p}{}^c$ and the summation of Eq. (3.5), and to extend the calculation of the amplitude to the unphysical region for real Q^2 between the Butler pole and the beginning of the physical region. The cross sections for the case with no Coulomb interactions were calculated by using small values of η_1 and η_2 rather than writing a separate program using (3.6) and (3.7).

The deuteron and proton are scattered by optical-model potentials of the Woods-Saxon type with spin-orbit coupling

$$\mathcal{V}(r) = [V + iW + V_s K' k r^{-1}(d/dr)] \times [\exp((r-R)/d) + 1]^{-1}, \quad (3.8)$$

and K' are eigenvalues of the operator $\mathbf{s} \cdot \mathbf{L}$. The Coulomb interactions are of the form

$$C(r) = \eta(3 - r^2/r_0^2)/r_0, \quad r < r_0$$
$$2\eta/r, \quad r \geqq r_0.$$

The neutron is captured into a state with wave function corresponding to a potential

$$\mathcal{V}_n(r_n) = V_n[\exp((r_n - R_n)/d_n) + 1]^{-1}, \quad (3.9)$$

where V_n is adjusted during the calculation to yield the required binding energy of the neutron. The parameters used in these potentials are listed in Table I.

IV. EXTRAPOLATION TO THE POLE

The method used for extrapolating cross sections to the pole was essentially the same as that described by DS. Polynomials of degree M were fitted to $P(Q^2)$, defined in (2.6), *in the physical region* using a standard curve-fitting program on a CDC-3600 computer. This extrapolation procedure was first checked to be satisfactory in extrapolating $P(Q^2)$ as calculated by the plane-wave Butler theory.[12]

Calculations have been performed for $Si^{28}(d,p)Si^{29}$ to the ground state (Q value = 6.25 MeV), the same reaction used by DS. Cross sections were calculated for cases with Coulomb interactions (WC) and without Coulomb interactions (NC). Cross sections calculated

TABLE I. Values of the parameters of the optical-model potentials used in the DWBA calculation for the reactions $Si^{28}(d,p)Si^{29}$ (denoted by Si^{28}) and $C^{12}(d,p)C^{13*}$ (3.09 MeV) (denoted by C^{12}). The strengths of potentials, V, W, and V_s are in units of MeV and the radius and diffuseness parameters in fermis. The parameters used in the potential of the captured neutron are $R_n = 4$ F and $d_n = 0.6$ F.

	Deuteron parameters						Proton parameters					
Case	V	W	V_s	R	d	r_0	V	W	V_s	R	d	r_0
Si^{28}												
1	−80	−15	0	4.56	0.6	3.95	−42	−8	0	4.0	0.6	4.0
2	−40	−7	0	4.56	0.6	3.95	−20	−4	0	4.0	0.6	4.0
3	−80	−15	+10	4.56	0.6	3.95	−42	−8	+5	4.0	0.6	4.0
4	−80	−15	−10	4.56	0.6	3.95	−42	−8	−5	4.0	0.6	4.0
5	−80	−15	0	8.0	0.6	6.0	−42	−8	0	6.0	0.6	6.0
6	−80	−15	0	2.0	0.6	1.0	−42	−8	0	1.0	0.6	1.0
C^{12}												
1	−55	−10	9	3.66	0.5	2.7	−45	−10	0	2.76	0.5	2.76
2	−70	−10	0	3.66	0.5	2.7	−60	−10	0	2.76	0.5	2.76
3	−30	−10	0	3.66	0.5	2.7	−20	−10	0	2.76	0.5	2.76
4	−55	−10	0	5.0	0.5	4.8	−45	−10	0	4.5	0.5	4.5
5	−55	−10	0	1.5	0.5	1.5	−45	−10	0	1.0	0.5	1.0
6	−55	−10	0	3.66	1.0	2.7	−45	−10	0	2.76	1.0	2.76
7[a]												

[a] Deuteron and proton optical-model potentials as in C^{12} (1) but with $R_n = 2.5$ F and $d_n = 0.8$ F.

[10] C. F. Clement, Nucl. Phys. **66**, 241 (1965).
[11] B. Robson, Australian National University Report, 1965 (unpublished).
[12] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951).

TABLE II. Results of extrapolations for $Si^{28}(d,p)Si^{29}$. The value of the reduced width used in the DWBA calculations is $N_0=10.16$.

	DWBA calculation				Extrapolation	
Type	Case	E_d (MeV)	$P(-\kappa_n^2)$	M	$P(-\kappa_n^2)$	N_0
WC WP	1	7	3.77	13	42.3	34.1
		12	13.9	12	127	31.0
		15	22.5	14	155	26.8
		30	70.9	15	95.3	11.8
	2	15	22.5	12	163	27.5
	3	15	22.5	14	155	26.8
	4	15	22.5	14	155	26.8
	5	15	22.5	14	15.0	8.3
	6	15	22.5	14	55.0	15.9
NC WP	1	7	322	13	326	10.3
		12	290	14	284	10.1
		15	280	14	239	9.4
		30	260	15	86.7	5.9

using Eq. (3.5), and thus having a pole at $Q^2=-\kappa_n^2$, are designated by WP. For comparison cross sections were also calculated using Eq. (3.2) and are designated by NP because they have no poles. The results of extrapolation are summarized in Table II. The WC angular distributions in the physical region for $E_d=7$ MeV are shown in Fig. 1. There is little difference between the WP and NP angular distributions.

As well as in the physical region, the angular distributions were also calculated in the unphysical region for real Q^2 between the pole and the start of the physical region. In Figs. 2(a) and 2(b), $P(Q^2)$ calculated from (3.5) (i.e., WP) is compared with $P(Q^2)$ obtained by extrapolation from the physical region, i.e., with the polynomial fitted to $P(Q^2)$ in the physical region.

$P(Q^2)$ calculated from (3.2) (i.e., NP) for the case with Coulomb field and the case without Coulomb field

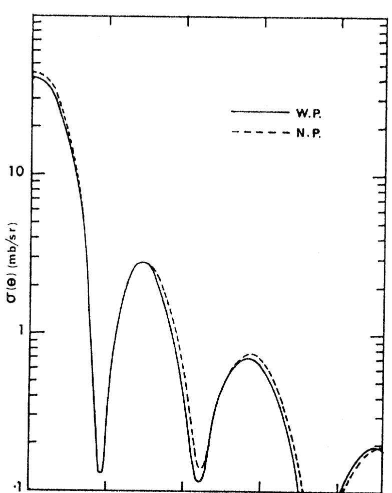

FIG. 1. Comparison of differential cross sections for the reaction $Si^{28}(d,p)Si^{29}$ to the ground state for $E_d=7$ MeV, calculated using DWBA for WP and NP. Coulomb interactions are included.

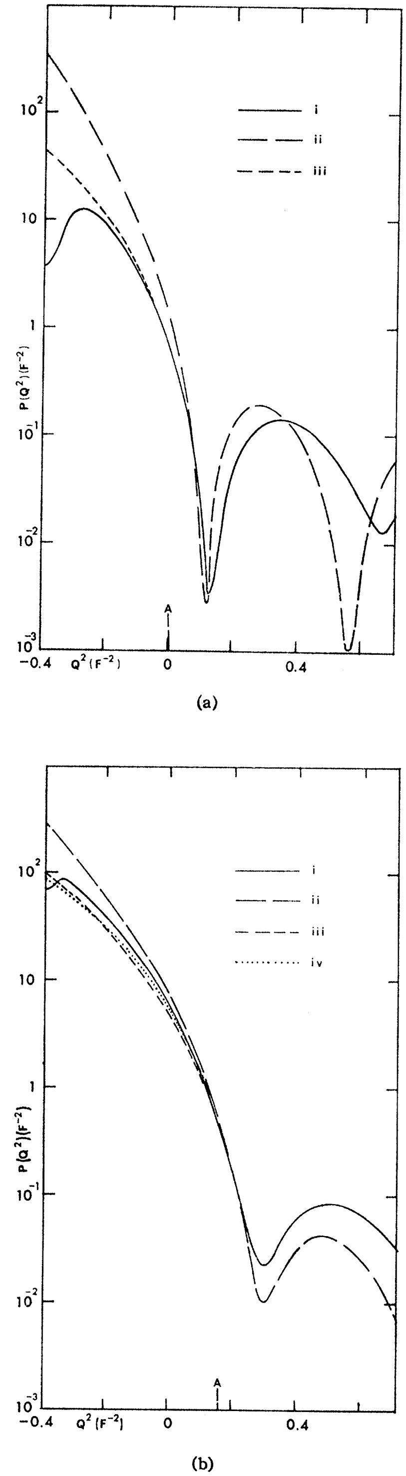

FIG. 2. The functions $P(Q^2)$ for $Si^{28}(d,p)Si^{29}$ in the unphysical region and part of the physical region for (a) $E_d=7$ MeV and (b) $E_d=30$ MeV. (i) $P_{WC}(Q^2)$ obtained by direct calculation using DWBA WP including Coulomb interactions; (ii) extrapolation from the physical region of $P_{WC}(Q^2)$; (iii) $P_{NC}(Q^2)$ obtained by direct calculation using DWBA WP without Coulomb interactions; (iv) extrapolation from physical region of $P_{NC}(Q^2)$. The boundary of the physical region is denoted by "A" on the Q^2 axis.

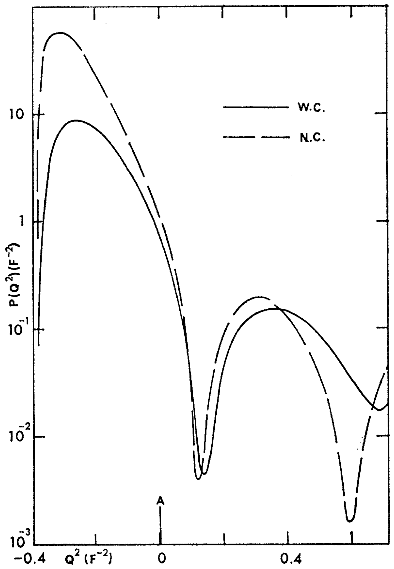

FIG. 3. The functions $P_{WC}(Q^2)$ and $P_{NC}(Q^2)$ for $Si^{28}(d,p)Si^{29}$, $E_d=7$ MeV, calculated in both the physical region and the unphysical region using DWBA NP. "A" marks the boundary of the physical region.

are shown in Fig. 3 for $E_d=7$ MeV. These are the functions which DS were attempting to extrapolate. Although there is good agreement in the physical region between the angular distributions calculated from (3.2) (NP) and from (3.5) (WP) as shown in Fig. 1, comparison of Fig. 3 with Fig. 2 shows that there is a large difference between the two calculations near the pole. Because of this large difference, the extrapolation cannot be satisfactorily investigated using NP cross sections.

The results in Table II show that, at low energies without Coulomb interactions, the values of the reduced width obtained by extrapolation are in good agreement with the correct value. However, at the higher energies the difference between the extrapolated and the true values of N_0 become larger. A simple explanation for this can be given. Whereas the position of the pole at $Q^2=-\kappa_n^2$ is independent of the energy E_d of the incident deuteron, the forward boundary of the physical region moves away from the pole as E_d is increased. As a result, at high energies, the extrapolation, which must be made over a larger distance, has less chance of succeeding than for the lower energies.

From Table II it is seen that when the Coulomb field is included in the calculations, the extrapolation fails to yield the correct value of N_0. The reason for this failure is seen by comparing the polynomial used for extrapolation with the correct $P(Q^2)$ in the unphysical region near the pole, as shown for $E_d=7$, 30 MeV in Figs. 2(a) and 2(b). $P(Q^2)$ has a turning point fairly close to the pole, and it is too difficult to fit the polynomial to $P(Q^2)$ in the physical region with sufficient accuracy to reproduce this turning point. Near the pole, there is a tendency for the extrapolation polynomial to follow $P_{NC}(Q^2)$, which does not have a turning point, and so the value of $P(-\kappa_n^2)$ obtained by extrapolation then lies somewhere between $P_{NC}(-\kappa_n^2)$ and $P_{WC}(-\kappa_n^2)$. This suggests that for sufficiently high energy E_d such that $R(E_d)<\epsilon$, ϵ being some prescribed fixed constant greater than unity, it should be possible to obtain N_0^2 to within a factor ϵ. For instance, choosing $\epsilon=2$, Fig. 4 shows $E(R=2)$, the value of E_d for which $R=2$, as a function of Z for various Q values. Then, for $E_d<E(R=2)$, extrapolation to the pole will not be reliable.

TABLE III. Results of extrapolation for $C^{12}(d,p)C^{13*}$ (3.09 MeV). The reduced width used in the DWBA calculations is $N_0=2.08$ for cases 1–6 and $N_0=1.90$ for case 7.

Type	DWBA calculation Case	E_d (MeV)	$P(-\kappa_n^2)$	M	Extrapolation $P(-\kappa_n^2)$	N_0
WC WP	1	7	2.44	14	4.40	2.80
		15	5.28	15	8.43	2.63
	2	15	5.28	15	8.26	2.60
	3	15	5.28	14	7.95	2.55
	4	15	5.28	14	8.60	2.66
	5	15	5.28	14	7.09	2.41
	6	15	5.28	14	9.17	2.74
	7	7	2.04	13	4.04	2.68
		12	3.72	14	6.43	2.50
		15	4.40	15	7.13	2.42

For the $Si^{28}(d,p)Si^{29}$ (Q value=6.25 MeV) reaction, $E(R=2)$ is about 60 MeV. But for $E_d=60$ MeV the boundary of the physical region will be too far removed from the pole to make extrapolation feasible.

We therefore look for a reaction for which $E(R=2)$ is lower, so that the physical region is closer to the pole. From Fig. 4 we see that we require a light target nucleus, a suitable nucleus being C^{12} for which $E(R=2)\approx 15$ MeV. Calculations have been carried out for $C^{12}(d,p)C^{13}$ (3.09 MeV) (Q value$=-0.38$ MeV), at $E_d=7$, 12, and

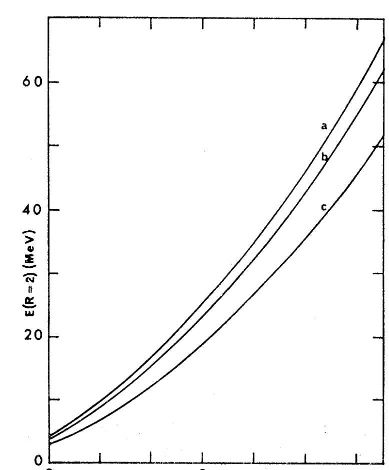

FIG. 4. The dependence of $E(R=2)$ on the charge of the target nucleus when the Q value of the reaction is (a) $Q=6$, (b) $Q=3$, and (c) $Q=-1$ MeV. For incident deuteron energies below the curves. extrapolations cannot be performed reliably.

15 MeV, and the results are summarized in Table III. Angular distributions in the physical region are shown in Fig. 5 for 7 MeV. The extrapolation polynomial and $P(Q^2)$ are compared in Fig. 6. Examination of Table III shows that the values of $P(-\kappa_n^2)$ obtained by extrapolation are correct to within a factor of 2, i.e., the reduced widths are correct to within a factor $\sqrt{2}$. Changes in the parameters of the optical-model potentials do not affect the results for reduced widths.

As shown in Fig. 5, there is an appreciable difference between the angular distribution calculated using Eq. (3.5) (WP) and that calculated using Eq. (3.2) (NP). This suggests that WP angular distributions are more reliable unless an extremely large number of

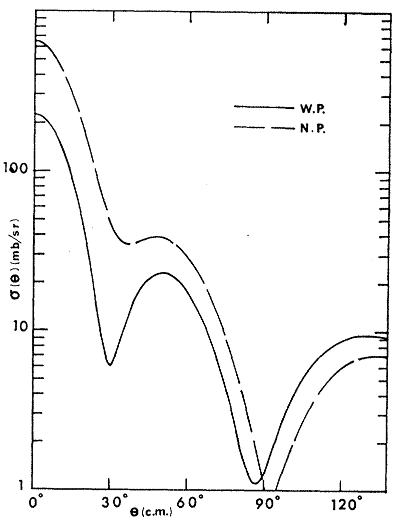

FIG. 5. Comparison of differential cross sections calculated by DWBA WP and NP for the reaction $C^{12}(d,p)C^{13*}$ (3.09 MeV) at $E_d=7$ MeV.

partial waves are used in calculating the NP angular distributions.

V. CONCLUSIONS AND DISCUSSION

The extraction of reduced widths by extrapolating $P(Q^2)$ to the Butler pole fails for heavy nuclei or low energies because of Coulomb effects, an estimate of the energy below which the extrapolation is unreliable being given by $E(R=2)$ as shown in Fig. 4. At high energies, the extrapolation is unreliable because the physical region is too far removed from the pole. Thus the extrapolation can only be done, if at all, for a rather narrow range of incident energies.

Because of the high degree of polynomials required,

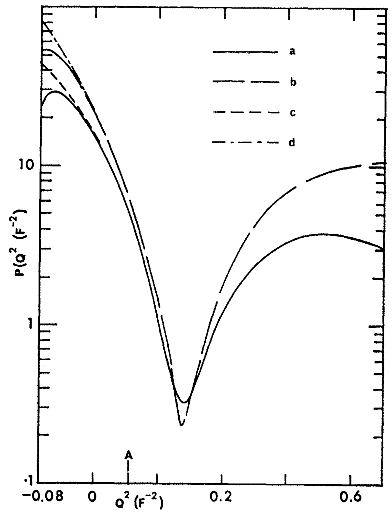

FIG. 6. The function $P_{WC}(Q^2)$ for $C^{12}(d,p)C^{13*}$ (3.09 MeV). $P_{WC}(Q^2)$ obtained by direct calculation using DWBA WP for (a) $E_d=7$ and (b) $E_d=15$ MeV. Extrapolation from the physical region of $P_{WC}(Q^2)$ for (c) $E_d=7$ and (d) $E_d=15$ MeV.

the application of this method to the analysis of experimental results does not seem feasible even in the most favorable cases, since the accuracy of the experimental results would need to be very high. For instance, for $C^{12}(d,p)C^{13*}$ (3.09 MeV) the experimental cross sections should be accurate to better than 0.1%.

For heavy nuclei for incident energies below the Coulomb barrier, the turning point in $P(Q^2)$ occurs in the physical region (or beyond the physical region) yielding the backward-peaked angular distributions typical of stripping below the Coulomb barrier. It may then be possible to extrapolate $\sigma(Q^2)/\sigma^c(Q^2)$ to the pole, where $\sigma^c(Q^2)$ is the cross section using $M_{j_n l_n m_n}{}^c$. While this possibility is worth investigating further, there is the serious difficulty that the cross section in the forward direction is very small and thus difficult to measure accurately.

In conclusion, the calculation of stripping cross sections for cases where the pole is near the physical region may be greatly facilitated by using the summation method of Eq. (3.5). In this way, one can expect to obtain an accurate description of the forward peak, which can only be obtained using the conventional summation method of Eq. (3.2) by including an inordinately large number of partial waves.

ACKNOWLEDGMENT

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