Properties of an Abrikosov Fluxoid in Type-II Superconductors*

R. LEADON† AND H. SUHL
University of California, San Diego, La Jolla, California
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An Abrikosov fluxoid in a pure, bulk, type-II superconductor has been studied by the Green's-function technique for superconductors, as developed by Gor'kov. An approximate Green's function has been obtained for the system, and the local densities of states have been evaluated numerically as a function of energy and position in the fluxoid. In agreement with the results of Caroli, de Gennes, and Matricon, the calculated density of states near the Fermi surface is comparable to that of a cylinder of normal metal with radius ξ_0, the coherence distance. The effect of fluxoids on the nuclear spin relaxation rate of a superconductor has been estimated from these results. For fairly high temperatures, the relaxation rate of a superconductor with fluxoids is less than the rate for the same superconductor in zero field, while for very low temperatures, the rate is increased. These trends are in agreement with the experimental data of Silbernagel.

I. INTRODUCTION

IN the Abrikosov description of type-II superconductors,[1] an external magnetic field penetrates the metal in quantized units of flux, called fluxoids or vorticies. In the region of the fluxoid, the order parameter is considerably reduced from its zero-field value, going to zero at the fluxoid centerline. This reduction in the order parameter affects various physical properties of the superconductor, for example, the density of states near the Fermi energy E_F and the nuclear spin relaxation rate $R \equiv 1/T_1$.

Caroli, de Gennes, and Matricon[2] have calculated the eigenfunctions of the bound-electron states localized near a single fluxoid in a pure, bulk superconductor. Except for a small energy gap at the Fermi surface of order Δ^2/E_F, the total density of states which they calculate is finite and comparable to the density of states for a cylinder of normal metal of radius ξ_0, the coherence distance. For certain purposes, what is required is not the total density of states but the spatial distribution of the local densities of states. Since this would be difficult or tedious to obtain from the results of Caroli *et al.*, we have considered the same problem in the Green's-function formulation as developed by Gor'kov.[3] By a slight modification of Werthamer's theory of local superconductivity,[4] we have obtained an approximate Green's function for the system which leads to a total density of states near the fluxoid comparable to the results of Caroli *et al.* and reduces to the usual expression for a uniform superconductor far from the vortex. The imaginary parts of the retarded Green's functions, $G_\omega^R(\mathbf{r}, \mathbf{r})$ and $F_\omega^R(\mathbf{r}, \mathbf{r})$, have been evaluated numerically and these results have been used to estimate the effect of fluxoids on the nuclear spin relaxation rate of the superconductor.

II. DERIVATION OF THE GREEN'S FUNCTION OF THE SYSTEM

Gor'kov's equations[3] for the thermodynamic Green's functions, $G_\omega(\mathbf{r}_1, \mathbf{r}_2)$ and $F_\omega^\dagger(\mathbf{r}_1, \mathbf{r}_2)$, of a pure superconductor can be written (with $\hbar = 1$)

$$\{i\omega + (1/2m)[\nabla_1 - (ie/c)\mathbf{A}(\mathbf{r}_1)]^2 + \mu\} G_\omega(\mathbf{r}_1, \mathbf{r}_2) + \Delta(\mathbf{r}_1) F_\omega^\dagger(\mathbf{r}_1, \mathbf{r}_2) = \delta^3(\mathbf{r}_1 - \mathbf{r}_2),$$

$$\{-i\omega + (1/2m)[\nabla_1 + (ie/c)\mathbf{A}(\mathbf{r}_1)]^2 + \mu\} F_\omega^\dagger(\mathbf{r}_1, \mathbf{r}_2) - \Delta^*(\mathbf{r}_1) G_\omega(\mathbf{r}_1, \mathbf{r}_2) = 0. \quad (1)$$

ω is the discrete frequency $(2n+1)\pi T$ with n an integer; $\mathbf{A}$ is the vector potential of the system; μ is the chemical potential; $\Delta(\mathbf{r})$ is the order parameter, related to $F_\omega^\dagger$ by the equation

$$\Delta^*(\mathbf{r}) = VT \sum_n F_\omega^\dagger(\mathbf{r}, \mathbf{r}); \quad (2)$$

and V is the strength of the electron–electron interaction which is assumed to produce superconductivity.

In his theory of local superconductivity, Werthamer[4] developed a procedure for obtaining successive approximations to these Green's functions for systems where $\mathbf{A}$ and $\Delta(\mathbf{r})$ are slowly varying in space. His solution is essentially an expansion in gradients of the order parameter. He points out that, at low temperatures, his criterion of slowly varying $\Delta(\mathbf{r})$ is valid only when $\Delta(\mathbf{r})$ is not too different from its zero-field value. Consequently, since $\Delta(\mathbf{r})$ is zero at the centerline of a vortex, his procedure is not directly usable for a fluxoid. If one does try to apply it to a fluxoid and assumes $\mathbf{A}$ is essentially zero near the centerline of the vortex, each term of his successive approximation to G_ω indicates a gap in the local density of states at the Fermi surface equal to $\pm|\Delta(\mathbf{r})|$. Evidently, to obtain a

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† Present address: General Atomic, San Diego, Calif.
[1] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1959)].
[2] C. Caroli, P. G. de Gennes, and J. Matricon, Phys. Letters **9**, 307 (1964).
[3] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].
[4] N. R. Werthamer, Phys. Rev. **132**, 663 (1963).

density of states comparable to Caroli *et al.*, one would have to sum an infinite series of such terms.

In general, Werthamer's series cannot be summed completely. However, by using a slightly different procedure from Werthamer, a Green's function can be obtained which amounts to a partial summation of his infinite series. The lowest-order approximation to this Green's function gives a density of states comparable to the result of Caroli *et al.*, and reduces to the Green's function for a uniform superconductor far from the fluxoid. The details of this procedure are outlined below.

Because the vector potential **A** of an isolated vortex drops off rather slowly at large radii,[5] it is convenient to define two new Green's functions by the equations

$$G_\omega(\mathbf{r}_1,\mathbf{r}_2)=G_\omega'(\mathbf{r}_1,\mathbf{r}_2)\exp\left(\frac{ie}{c}\int_{\mathbf{r}_2}^{\mathbf{r}_1}\mathbf{A}(\mathbf{s})\cdot d\mathbf{s}\right),$$

$$F_\omega^\dagger(\mathbf{r}_1,\mathbf{r}_2)=F_\omega'^\dagger(\mathbf{r}_1,\mathbf{r}_2)\times\exp\left[-\frac{ie}{c}\left(\int_0^{\mathbf{r}_1}+\int_0^{\mathbf{r}_2}\right)\mathbf{A}(\mathbf{s})\cdot d\mathbf{s}\right]. \quad (3)$$

The contours of integration are arbitrary and the lower limits of the integrals in $F_\omega^\dagger$ can be any position independent of $\mathbf{r}_1$ and $\mathbf{r}_2$.

We restrict ourselves to conditions where the fluxoids are widely separated (fluxoid spacing$\gg\lambda$, the penetration depth) and the magnetic field ($H\approx\Phi_0/\lambda^2$) is small. Φ_0 is the quantum of flux in a fluxoid$=|hc/2e|$. The fluxoids can then be treated individually. The vector potential **A** will be small and slowly varying in space, and its derivatives can be neglected. With these limitations, the equations which the new Green's functions [$G_\omega'(\mathbf{r}_1,\mathbf{r}_2)$ and $F_\omega'^\dagger(\mathbf{r}_1,\mathbf{r}_2)$] satisfy are approximately

$$[i\omega+(1/2m)\nabla_1^2+\mu]G_\omega'(\mathbf{r}_1,\mathbf{r}_2)+\Delta'(\mathbf{r}_1)\times F_\omega'^\dagger(\mathbf{r}_1,\mathbf{r}_2)=\delta^3(\mathbf{r}_1-\mathbf{r}_2),$$

$$[-i\omega+(1/2m)\nabla_1^2+\mu]F_\omega'^\dagger(\mathbf{r}_1,\mathbf{r}_2)-\Delta'^*(\mathbf{r}_1)\times G_\omega'(\mathbf{r}_1,\mathbf{r}_2)=0, \quad (4)$$

where

$$\Delta'(\mathbf{r}_1)=\Delta(\mathbf{r}_1)\exp\left[-\frac{2ie}{c}\int_0^{\mathbf{r}_1}\mathbf{A}(\mathbf{s})\cdot d\mathbf{s}\right]. \quad (5)$$

Following Werthamer, the integral equations corresponding to Eq. (4) can be written

$$\int d^3r_3\,K_\omega(\mathbf{r}_1,\mathbf{r}_3)G_\omega'(\mathbf{r}_3,\mathbf{r}_2)=\delta^3(\mathbf{r}_1-\mathbf{r}_2),$$

$$\int d^3r_3\,K_{-\omega}(\mathbf{r}_3,\mathbf{r}_1)F_\omega'^\dagger(\mathbf{r}_3,\mathbf{r}_2)=\Delta'^*(\mathbf{r}_1)G_\omega^0(\mathbf{r}_1-\mathbf{r}_2), \quad (6)$$

where

$$K_\omega(\mathbf{r}_1,\mathbf{r}_3)=\delta^3(\mathbf{r}_1-\mathbf{r}_3)[i\omega+(1/2m)\nabla_3^2+\mu]+\Delta'(\mathbf{r}_1)G_{-\omega}^0(\mathbf{r}_3-\mathbf{r}_1)\Delta'^*(\mathbf{r}_3). \quad (7)$$

$G_\omega^0(r)$ is the usual Green's function for a uniform normal metal. Changing variables to the sum and difference coordinates, expanding the quantities in Taylor series about the sum coordinates, and taking Fourier transforms relative to the difference coordinates, one obtains equations similar to Werthamer's Eq. (7):

$$\theta[K_\omega(\mathbf{p},\mathbf{R})G_\omega'(\mathbf{p}',\mathbf{R}')]=1,$$

$$\theta[K_{-\omega}(-\mathbf{p},\mathbf{R})F_\omega'^\dagger(\mathbf{p}',\mathbf{R}')]=\theta[\Delta'^*(\mathbf{R})G_\omega^0(\mathbf{p})], \quad (8)$$

where θ is the differential operator

$$\theta\equiv\lim_{\mathbf{R}'\to\mathbf{R},\mathbf{p}'\to\mathbf{p}}\exp[\tfrac{1}{2}i(\nabla_R\cdot\nabla_{p'}-\nabla_{R'}\cdot\nabla_p)], \quad (9)$$

$$K_\omega(\mathbf{p},\mathbf{R})=i\omega-\xi_p+\int\Delta'(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\Delta'^*(\mathbf{R}-\tfrac{1}{2}\mathbf{r})\times G_\omega^0(-\mathbf{r})\exp(-i\mathbf{p}\cdot\mathbf{r})\,d^3r, \quad (10)$$

and

$$\xi_p=(p^2/2m)-\mu. \quad (11)$$

If Werthamer's procedure were followed again, the order parameters in Eq. (10) would be expanded in Taylor series about the position **R**. The four quantities, θ, $K_\omega(\mathbf{p},\mathbf{R})$, $G_\omega'(\mathbf{p}',\mathbf{R}')$, and $F_\omega'^\dagger(\mathbf{p}',\mathbf{R}')$, would then be expanded in the degree of their inhomogeneity and substituted into Eq. (8). The successive approximations to G_ω' and $F_\omega'^\dagger$ would be obtained by equating equal degrees of inhomogeneity on the two sides of the equations. The approximations to G_ω' obtained in this way are those which show a gap in the local density of states equal to the local order parameter $\pm|\Delta(\mathbf{r})|$.

The modification which is used herein consists in *not* expanding the order parameters in Eq. (10). The other three quantities, θ, G_ω', and $F_\omega'^\dagger$, are still expanded in their degree of inhomogeneity and substituted into Eq. (8), but the complete form for $K_\omega(\mathbf{p},\mathbf{R})$ [Eq. (10)] is retained. This modification obviously amounts to summing Werthamer's expansion for K_ω and is therefore a partial summation of his complete series. In this manner, the lowest-order approximations to $G_\omega'(\mathbf{p},\mathbf{R})$ and $F_\omega'^\dagger(\mathbf{p},\mathbf{R})$ are obtained as

$$[G_\omega'(\mathbf{p},\mathbf{R})]_0=[K_\omega(\mathbf{p},\mathbf{R})]^{-1}, \quad (12)$$

$$[F_\omega'^\dagger(\mathbf{p},\mathbf{R})]_0=\Delta'^*(\mathbf{R})G_\omega^0(p)/K_{-\omega}(-\mathbf{p},\mathbf{R}). \quad (13)$$

For simplicity, only these lowest-order approximations have been used to calculate the densities of states. For one condition, the effect of the next higher-order nonzero term on the density of states was estimated and was found to be quite small compared to the contribution from the zero-order term. The convergence of this series should be examined further to see if the

[5] R. E. Leadon, Doctoral thesis, University of California, San Diego, Calif., 1967 (unpublished).

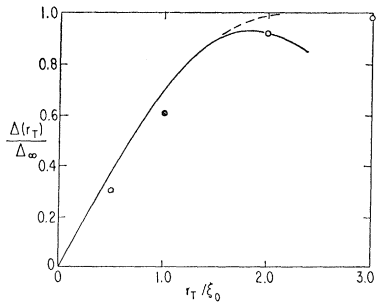

FIG. 1. Order parameter versus distance from vortex centerline. Solid curve is $1.6J_1(r_T/\xi_0)$, Ref. 5. Dashed curve is arbitrary fairing into $\Delta(r_T/\xi_0)/\Delta_\infty=1.0$. ○ are points calculated by self-consistency equation (Sec. III B).

smallness of the second term indicates rapid convergence of the series or is accidental. This has not been done as yet.

In a bulk superconductor, the vortex will be essentially two dimensional, so $\Delta(\mathbf{R})$ will be independent of z, the distance parallel to the vortex centerline. The z integration in Eq. (10) can then be done exactly, giving

$$K_\omega(\mathbf{p},\mathbf{R})=i\omega-\xi_p+\tfrac{1}{2}(im)\int\Delta'(\mathbf{R}_T+\tfrac{1}{2}\mathbf{r}_T)\Delta'^*(\mathbf{R}_T-\tfrac{1}{2}\mathbf{r}_T)\times\exp(-i\mathbf{p}_T\cdot\mathbf{r}_T)H_0^{(2)}(p_1r_T)r_Tdr_Td\theta_r, \quad (14)$$

where $p_1\equiv(p_0^2-p_z^2)^{1/2}$ and $p_0^2\equiv[2m(\mu-i\omega)]$. $\mathbf{R}_T$, $\mathbf{r}_T$, and $\mathbf{p}_T$ are the components of $\mathbf{R}$, $\mathbf{r}$, and $\mathbf{p}$ in the transverse plane, that is, perpendicular to z. θ_r is the angle of $\mathbf{r}_T$ in the transverse plane, measured from $\mathbf{R}_T$, and $H_0^{(2)}$ is the zero-order Hankel function of the second kind. The region of phase space with $p_z^2>p_0^2$ contributes little to the density of states[5] so only the condition $p_0^2>p_z^2$ has been considered in deriving Eq. (14).

For simplicity, the vectors $\mathbf{R}_+$ and $\mathbf{R}_-$ are defined as

$$\mathbf{R}_+=\mathbf{R}_T+\tfrac{1}{2}\mathbf{r}_T,$$
$$\mathbf{R}_-=\mathbf{R}_T-\tfrac{1}{2}\mathbf{r}_T. \quad (15)$$

The reference angles of these vectors in the transverse plane are denoted as θ_+ and θ_-, respectively.

Since the phase of the order parameter $\Delta(\mathbf{r})$ changes by 2π in going around a fluxoid centerline,[1] it can be written

$$\Delta(\mathbf{r})=\Delta(r_T)\exp(-i\theta_r), \quad (16)$$

where $\Delta(r_T)$ is a real function of the magnitude of r_T. Thus, the order parameters in Eq. (14) can be written

$$\Delta'(\mathbf{R}_+)\Delta'^*(\mathbf{R}_-)=\Delta(R_+)\Delta(R_-)\exp[-i(\theta_+-\theta_-)]\times\exp\left(-\frac{2ie}{c}\int_{\mathbf{R}_-}^{\mathbf{R}_+}\mathbf{A}(\mathbf{s})\cdot d\mathbf{s}\right)$$
$$\equiv[L_1(\mathbf{R}_T,\mathbf{r}_T)-iL_2(\mathbf{R}_T,\mathbf{r}_T)\sin\theta_r]\times\exp\left(-\frac{2ie}{c}\int_{\mathbf{R}_-}^{\mathbf{R}_+}\mathbf{A}(\mathbf{s})\cdot d\mathbf{s}\right), \quad (17)$$

where

$$L_1(\mathbf{R}_T,\mathbf{r}_T)\equiv[\Delta(R_+)\Delta(R_-)/(R_+)(R_-)][R_T^2-(\tfrac{1}{2}r_T)^2],$$
$$L_2(\mathbf{R}_T,\mathbf{r}_T)\equiv[\Delta(R_+)\Delta(R_-)/(R_+)(R_-)][r_TR_T]. \quad (18)$$

For vortices where $\lambda\gg\xi_0$, the effect of $\mathbf{A}$ is small and can be ignored.[5] At small radii where the order parameter is changing rapidly, $\mathbf{A}$ is very small and the exponential of $\mathbf{A}$ is practically unity. For larger radii where $\Delta(R)$ is nearly constant, the effect of $\mathbf{A}$ produces only a very slow change in the integrand of Eq. (14), and has a negligible effect on the Green's functions.

To numerically evaluate the densities of states for a fluxoid, the functions $L_1(\mathbf{R}_T,\mathbf{r}_T)$ and $L_2(\mathbf{R}_T,\mathbf{r}_T)$ are approximated in such a manner that the space integrations in Eq. (14) can be done in closed form. Based on an approximate solution of the first Ginzburg-Landau equation, a variation of $\Delta(r_T)$ with r_T was assumed for all temperatures (see Fig. 1). The reference length ξ_0 was assumed to be related to the zero-field order parameter Δ_∞ by the equation[6]

$$\xi_0=\hbar v_F/\pi\Delta_\infty, \quad (19)$$

where v_F is the Fermi velocity. The functions $L_1(\mathbf{R}_T,\mathbf{r}_T)$ and $L_2(\mathbf{R}_T,\mathbf{r}_T)$ were then calculated and plotted versus r_T for several values of R_T and θ_r. In the region of main interest[5] ($R_T\lesssim\xi_0$) these curves are practically independent of θ_r, so it is adequate to replace them with their averages over θ_r, that is, by $\bar{L}_1(R_T,r_T)$ and $\bar{L}_2(R_T,r_T)$. The integrals over θ_r in Eq. (14) can then be done, giving

$$K_\omega(\mathbf{p},\mathbf{R})=i\omega-\xi_p+i\pi m\int_0^\infty[\bar{L}_1(R_T,r_T)J_0(p_Tr_T)-\bar{L}_2(\mathbf{R}_T,\mathbf{r}_T)J_1(p_Tr_T)\sin\theta_p]H_0^{(2)}(p_1r_T)r_Tdr_T. \quad (20)$$

J_0 and J_1 are Bessel functions of orders zero and one, θ_p is the angle of $\mathbf{p}_T$ measured from $\mathbf{R}_T$.

To obtain the density of states, the imaginary part of the retarded Green's function $G_\omega'^R(\mathbf{p},\mathbf{R})$ must be integrated over d^3p. In this integration, the important range of the variables is $p_T\sim p_1$, and both of these momenta are a sizeable fraction of the Fermi momentum p_F. It is therefore adequate to replace J_0, J_1, and $H_0^{(2)}$ by their asymptotic expressions for large arguments. The product of these exponentials gives terms of the form $\exp[-i(p_1\pm p_T)r_T]$. Those terms with (p_T+p_1) will oscillate very rapidly with r_T and the integral over r_T in Eq. (20) will be nearly zero. Therefore, these terms can be neglected and only the terms with (p_T-p_1) are retained.

It is convenient to integrate by parts relative to r_T in Eq. (20). At the lower limit ($r_T=0$), $\bar{L}_1(R_T,r_T)=\Delta^2(R_T)$ and $\bar{L}_2(R_T,r_T)=0$. The upper limit ($r_T=\infty$) gives zero because $\exp[i(p_T-p_1)r_T]$ decays exponentially due to $i\omega$ in p_1. Since p_T is nearly equal to p_1 in

[6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

the important part of its integration range, it is permissible to replace p_T by p_1 everywhere except in the difference (p_T-p_1). With these approximations, Eq. (20) becomes

$$K_\omega(\mathbf{p},\mathbf{R})=i\omega-\xi_p-\frac{\Delta^2(R_T)}{i\omega+\xi_p}-\int_0^\infty \frac{\exp[i(p_T-p_1)r_T]}{(i\omega+\xi_p)}\frac{d}{dr_T}[\bar{L}_1(R_T,r_T)+i\bar{L}_2(R_T,\mathbf{r}_T)\sin\theta_p]dr_T. \quad (21)$$

The lowest-order approximations to the Green's functions are then, from Eqs. (12) and (13),

$$[G_\omega'(\mathbf{p},\mathbf{R})]_0=-(i\omega+\xi_p)\left(\omega^2+\xi_p^2+\Delta^2(R_T)+\int_0^\infty \exp[i(p_T-p_1)r_T]\frac{d}{dr_T}[\bar{L}_1(R_T,r_T)+i\bar{L}_2(R_T,r_T)\sin\theta_p]dr_T\right)^{-1}, \quad (22)$$

$$[F_\omega'^\dagger(\mathbf{p},\mathbf{R})]_0=\Delta'^*(\mathbf{R}_T)\left(\omega^2+\xi_p^2+\Delta^2(R_T)+\int_0^\infty \exp[i(p_T-p_1)r_T]\frac{d}{dr_T}[\bar{L}_1(R_T,r_T)+i\bar{L}_2(R_T,r_T)\sin\theta_p]dr_T\right)^{-1}. \quad (23)$$

When $\bar{L}_1(R_T,r_T)$ and $\bar{L}_2(R_T,r_T)$ vary extremely slowly in space, these equations reduce to the lowest-order terms of Werthamer's local theory. The difference between the present method and Werthamer's approach results from considering the finite gradients of $\bar{L}_1$ and $\bar{L}_2$.

III. DISCUSSION OF RESULTS

A. Density of States

To obtain the density of states, the functions $\bar{L}_1(R_T,r_T)$ and $\bar{L}_2(R_T,r_T)$ were approximated by analytic functions which permitted the integrations over r_T in Eq. (22) to be done in closed form. The thermal Green's function $G_\omega'(\mathbf{p},\mathbf{R})$ was then converted to the retarded Green's function $G_\omega'^R(\mathbf{p},\mathbf{R})$ by replacing $i\omega$ by $(\omega+i\delta)$, and the imaginary part of $G_\omega'^R(p,R)$ was integrated over d^3p.

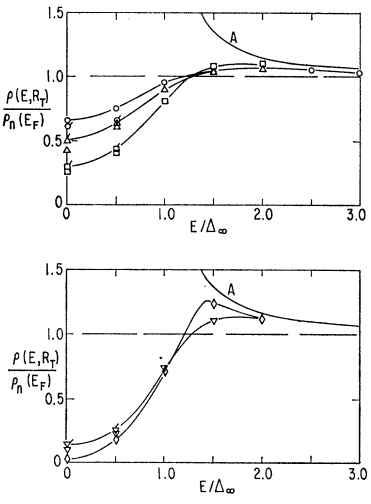

FIG. 2. Density of states for five radial positions.
Symbol: ⊙ △ □ ▽ ◇
R_T/ξ_0: 0.0 0.5 1.0 2.0 3.0
Flagged symbols from digital integration method. Unflagged symbols from integration by residues. Curve A is BCS density of states for uniform superconductor. $\rho_n(E_F)$ is density of states for uniform normal metal at the Fermi surface.

Two different methods were used to perform these integrations. In the first method, $\bar{L}_1$ and $\bar{L}_2$ were approximated very well (for each R_T) by a single Gaussian and the derivative of a single Gaussian, respectively. The integration over p_T was done step-by-step on the digital computer for fixed values of p_z and θ_p. The final integrations over p_z and θ_p were done graphically. This method is good for energies $\omega\ll\Delta(R_T)$, for then the integrand is never too sharply peaked.

The second method was used for energies ω of the order $\Delta(R_T)$ and larger, where the integrand sometimes approaches a δ function. In this method, the functions $\bar{L}_1$ and $\bar{L}_2$ were approximated somewhat more roughly than in the first method, $\bar{L}_1$ by a simple exponential and $\bar{L}_2$ by r_T times a simple exponential. The advantage of these approximations is that the subsequent integration over p_T could be done by a contour integral in the complex p_T plane, using the theory of residues. The integrations over p_z and θ_p were again done graphically.

The resulting local densities of states, normalized to the density of states for a uniform normal metal at the Fermi energy $[\rho_n(E_F)]$, are plotted in Fig. 2 versus energy for five radial distances from the fluxoid centerline. The flagged symbols at low energies are from the step-by-step digital integrations, while the unflagged points are from the integrations by residues. In spite of the rather different accuracy in approximating $\bar{L}_1$ and $\bar{L}_2$ in the two methods, the two sets of results agree quite well. For comparison, the BCS result for a uniform superconductor is also plotted in Fig. 2.

Far from the fluxoid centerline, the effect of the fluxoid should vanish and the calculated density of states should approach the BCS curve. The curve for $R_T/\xi_0=3.0$ (the largest value of R_T considered) confirms this trend. As the fluxoid centerline is approached, the calculated curves depart from the BCS result and approach the value for a uniform *normal* metal.

When the densities of states at the Fermi energy are plotted versus R_T and integrated over the cross section of the fluxoid, the result is $\pi(1.15\xi_0)^2\rho_n(E_F)$. Thus, for energies near the Fermi surface, the total density of

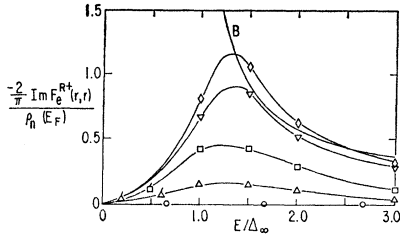

FIG. 3. Im $F_E^{R\dagger}(r,r)$ for five radial positions.
Symbol: ○ △ □ ▽ ◇
R_T/ξ_0 : 0.0 0.5 1.0 2.0 3.0
Flagged symbols for digital integration method. Unflagged symbols for integration by residues. Curve B is BCS result for uniform superconductor. $\rho_n(E_F)$ is density of states for uniform normal metal at the Fermi surface.

states due to one fluxoid is comparable to the density of states for a cylinder of normal metal with radius $1.15\xi_0$. This agrees well with the result of Caroli *et al.*[2]

The reader is reminded that these results are based on just the zero-order approximations of the Green's functions. The one calculation which was made to estimate the effect of the next nonzero term was done at the vortex centerline and at the Fermi energy ($\omega=0$). The increment in the density of states due to this term was about $0.03\rho_n(E_F)$, which is quite small compared to the result from the zero-order term at the same condition, about $0.64\rho_n(E_F)$. Of course, this one point is not a proof that the series is really convergent.

B. Self-Consistent Order Parameter

The order-parameter curve in Fig. 1 is basically an assumption, based on an approximate solution of the first Ginzburg-Landau equation for a fluxoid.[5] It is theoretically possible to substitute the calculated Green's function [Eq. (23)] into Eq. (2), determine $\Delta(\mathbf{r})$, and repeat the computations until a self-consistent curve for $\Delta(\mathbf{r})$ is obtained. Such an iteration has not been performed. However, the first step in this process was done, namely, $\Delta(\mathbf{r})$ was calculated from Eq. (2) and compared to the assumed curve in Fig. 1.

By converting the sum over n in Eq. (2) to a contour integral in the usual way, warping the contour to enclose the poles of $F_\omega^\dagger(\mathbf{r},\mathbf{r})$, and dividing by the same equation for a uniform superconductor, one obtains[5]

$$\Delta(R_T)/\Delta_\infty=\left(\int_{\Delta_\infty}^{\omega_D} d\omega\,\tanh\tfrac{1}{2}(\beta\omega)\,\frac{\Delta_\infty}{(\omega^2-\Delta_\infty^2)^{1/2}}\right)^{-1}$$

$$\times\int_0^{\omega_D} d\omega\,\tanh\tfrac{1}{2}(\beta\omega)\left(\frac{-2}{\pi\rho_n(E_F)}\,\mathrm{Im}F_\omega^{R\dagger}(R_T,R_T)\right). \quad (24)$$

The upper limits of the integrals have been cut off at the Debye energy ω_D to avoid the usual logarithmic divergence of these integrals. By $\mathrm{Im}F_\omega^{R\dagger}(\mathbf{r},\mathbf{r})$, we mean one-half the jump across the cut of the retarded Green's function $F_\omega^{R\dagger}(\mathbf{r},\mathbf{r})$ in the ω plane.

Curves of $\mathrm{Im}F_\omega^{R\dagger}(\mathbf{r},\mathbf{r})$ are given in Fig. 3 versus energy for the five distances from the vortex centerline. These curves were obtained by the same procedure used to obtain $\mathrm{Im}G_\omega'^R(\mathbf{r},\mathbf{r})$ for Fig. 2. The BCS result for a uniform superconductor is also shown for comparison. This curve is used in the denominator of Eq. (24).

For evaluating Eq. (24), two further items are needed: (1) a value for ω_D/Δ_∞ and (2) a prescription for estimating the portion of the upper integral in Eq. (24) from the maximum energy in Fig. 3 ($\omega/\Delta_\infty=3.0$) to ω_D/Δ_∞. Since ω_D enters only logarithmically into both the numerator and the denominator, its exact value is not critical, so a typical value of $\omega_D/\Delta_\infty=40$ was selected. From Eq. (23), the calculated curves in Fig. 3 should approach $\Delta(R_T)/\Delta_\infty$ times the BCS curve, for each R_T, at large energies. Therefore, it has been assumed that the area under each calculated curve from $\omega/\Delta_\infty=3.0$ to ω_D/Δ_∞ is $\Delta(R_T)/\Delta_\infty$ times the area under the BCS curve over the same range. The integrals from $\omega=0$ to $\omega/\Delta_\infty=3.0$ were done graphically.

In this way, Eq. (24) was evaluated for three temperature ratios, $T/T_c=0$, 0.5, and 0.8. The BCS relation for Δ_∞ versus temperature was used. In this temperature range, the effect of temperature was almost undetectable in the computed results. The average points for the three temperatures are plotted in Fig. 1 for comparison with the assumed curve for $\Delta(R_T)$. The maximum difference between these points and the assumed curve is about 15%. However, a curve through the points would be very similar in shape to the original curve. The difference in slopes probably indicates that the reference length ξ_0 used for the curve in Fig. 1 and given by Eq. (19) is not quite right and should be adjusted. However, the dimensionless curve of $\Delta(R_T)/\Delta_\infty$ versus R_T/ξ_0 is apparently almost independent of temperature over a wide temperature range. To the extent that it is

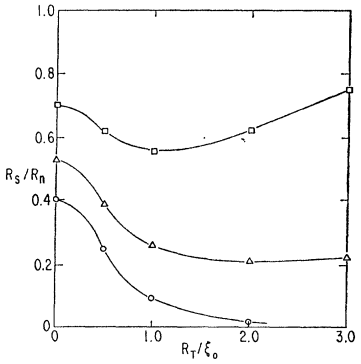

FIG. 4. R_s/R_n versus radial position. R_s is relaxation rate in superconductor. R_n is relaxation rate in normal metal.
Symbol: ○ △ □
T/T_c : 0.0 0.5 0.8
Theory for uniform superconductor (see Ref. 7):
T/T_c : 0 0.5 0.8
R_s/R_n : 0 0.6–1.0 2.0–3.0.

independent of temperature, the dimensionless curves in Figs. 2 and 3 also apply to all temperatures.

C. Nuclear Spin Relaxation Rate

The nuclear spin relaxation rate, $R \equiv 1/T_1$, is proportional to the probability $W_{n\to m}$ of flipping the nuclear spin from the state $|n\rangle$ to the state $|m\rangle$. For the usual spin-flip interaction between the spins of the nuclei and the conduction electrons,[7] this probability can be written in terms of the single-particle thermal Green's functions, G_ω and $F_\omega^\dagger$, of the unperturbed electron system as[5]

$$W_{n\to m}=\frac{C^2}{2\hbar}\sum_{j,j'}\frac{[\langle n|\mathbf{I}_j|m\rangle\cdot\langle m|\mathbf{I}_{j'}|n\rangle+\text{c.c.}]}{[1-\exp(\beta\omega)]}\,\mathrm{Im}O[i\omega(q)\to\omega+i\delta]$$
$$\times\{T\sum_s[G_{\omega(s)}(\mathbf{R}_j,\mathbf{R}_{j'})G_{\omega(s)-\omega(q)}(\mathbf{R}_{j'},\mathbf{R}_j)+F_{\omega(s)}(\mathbf{R}_j,\mathbf{R}_{j'})F_{\omega(q)-\omega(s)}{}^\dagger(\mathbf{R}_j,\mathbf{R}_{j'})]\exp[i\omega(s)\eta]\}. \quad (25)$$

$C=(-8\pi/3)\gamma_e\gamma_n\hbar^2$; γ_e and γ_n are the electron and nuclear gyromagnetic ratios; T is the temperature in energy units; I_j is the spin of the nucleus at the position R_j; $O[i\omega(q)\to\omega+i\delta]$ is an operator which changes $i\omega(q)$ to $(\omega+i\delta)$; $\omega(q)$ and $\omega(s)$ are discrete frequencies, $\omega(q)=2q\pi T$ and $\omega(s)=(2s+1)\pi T$, with q and s integers; δ and η are small positive quantities; and ω is the difference between the Zeeman energies of the nuclear states $|n\rangle$ and $|m\rangle$.

Equation (25) can also be written in the form[5]

$$W_{n\to m}=(C^2/2\pi\hbar)\sum_{jj'}[\langle n|\mathbf{I}_j|m\rangle\cdot\langle m|\mathbf{I}_{j'}|n\rangle+\text{c.c.}]\int dE f(E)[1-f(E-\omega)]$$
$$\times[\mathrm{Im}G_E{}^R(\mathbf{R}_j,\mathbf{R}_{j'})\ \mathrm{Im}G_{E-\omega}{}^R(\mathbf{R}_{j'},\mathbf{R}_j)+\mathrm{Im}F_E{}^R(\mathbf{R}_j,\mathbf{R}_{j'})\ \mathrm{Im}F_{\omega-E}{}^{R\dagger}(\mathbf{R}_j,\mathbf{R}_{j'})], \quad (26)$$

where $f(E)$ is the Fermi function and the superscript R indicates the retarded Green's functions.

The terms with $j\neq j'$ are usually much less important than those with $j=j'$, so they have been neglected.[7] Therefore, the ratio of the relaxation rate R_s to the rate for the corresponding normal metal R_n is just proportional to the ratio of the energy integrals in Eq. (26) for the superconducting and normal systems. The difference in the Zeeman energies ω is usually quite small and can be set equal to zero when the imaginary parts of the Green's functions are not too sharply peaked. The calculated curves in Figs. 2 and 3 meet this criteria.

The ratio R_s/R_n was computed for three temperature ratios ($T/T_c=0$, 0.5, and 0.8) by performing the energy integrations in Eq. (26) graphically for each temperature. The BCS relation for Δ_∞ as a function of temperature was again used. The results are plotted versus radial distance from the fluxoid centerline in Fig. 4. At large distances from the fluxoid, the curves should approach the results for a uniform superconductor. Hebel and Slichter[7] have calculated and measured this ratio for uniform superconductors. The probable range of their values for each temperature is indicated in Fig. 4. The curves in Fig. 4 tend in the proper direction to approach these limits. At fairly high temperatures, the effect of the fluxoids is to reduce the ratio R_s/R_n compared to the zero-field value, while, at very low temperatures, the fluxoids increase this ratio. Both of these tendencies are in agreement with the experimental data of Silbernagel.[8]

[7] L. C. Hebel and C. P. Slichter, Phys. Rev. **113**, 1504 (1959).

[8] B. G. Silbernagel, Doctoral thesis, University of California, San Diego, Calif., 1966 (unpublished).