Solution of the Boltzmann Equation for Electrons Interacting with Acoustic Waves in Strong Electric Fields

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We have solved the Boltzmann equation to obtain the conductivity tensor for electrons interacting with acoustic waves in the presence of strong electric fields. The presence of the dc electric field leads to two new effects: the introduction of a drifted distribution function for the electrons, and of a complex electron temperature which depends on both the electric field and the acoustic wavelength. It is shown that it is the drifted distribution function which leads to the amplification of acoustic waves in the short-wavelength limit $ql\gg 1$, while in the long-wavelength limit $ql\ll 1$, it is the complex temperature which gives rise to the amplification.

I. INTRODUCTION

In recent years there has been quite a bit of interest, both theoretical[1,2] and experimental,[3,4] in the amplification of acoustic waves through their interaction with conduction electrons in the presence of dc electric fields. Most of the experimental work up to now has been done in the region where the acoustic wavelength is longer than the electron mean free path. In this frequency region the interaction can be viewed as that between the acoustic wave and a space charge wave. However, Nill[5,6] has recently done experiments involving both attenuation and amplification in InSb in the 10-gigacycle region, where the acoustic wavelength is shorter than the mean free path. Here the amplification process can be viewed as a phonon maser with the electric field acting to invert the electron population so that emission of phonons exceeds their absorption. In addition, Solymar[7] has proposed a new mechanism for amplification of acoustic waves which should occur in strong fields for which the electron drift velocity exceeds the mean thermal or Fermi velocity of the electrons. Theoretically, the treatment of this problem requires the solution of the Boltzmann equation for the electrons interacting with the acoustic waves in the presence of the dc electric field. We previously solved this problem to terms linear in the dc electric field.[8] In our present work, we have solved the Boltzmann equation for arbitrary dc electric field, assuming of course that we can still use the relaxation-time ansatz[2,9] for the collision term in the Boltzmann equation. We have also made use of a drifted distribution for the electrons. We then use our solution of the Boltzmann equation to calculate the relevant components of the conductivity tensor. We used nondegenerate statistics for the electrons since in most cases amplification is observed in materials where the electron density is low enough for the electrons to obey classical statistics. In the limits of long wavelength $ql\ll 1$ and short wavelength $ql\gg 1$, we obtain the same results as previously,[8] taking into account the modifications imposed because of the use of degenerate statistics in the older work. However, it becomes clear from our calculation that the amplification of the acoustic wave in the region $ql\ll 1$ arises because of a field-dependent imaginary electron temperature while in the region $ql\gg 1$ it arises because of a drifted electron distribution. In addition, in the limit where the drift velocity is greater than the average electron velocity we compare our results to those of Solymar.[7]

II. CALCULATION

The Boltzmann equation for electrons interacting with an acoustic wave of frequency ω and wave vector $\mathbf{q}$ in the presence of a dc electric field $\boldsymbol{\mathcal{E}}_0$ is

$$\frac{df}{dt}+\mathbf{v}\cdot\frac{df}{d\mathbf{r}}-\frac{e}{m}\left(\boldsymbol{\mathcal{E}}_0+\boldsymbol{\mathcal{E}}_1+\frac{\mathbf{v}}{c}\times\mathbf{B}_1\right)\cdot\frac{df}{d\mathbf{v}}=-\frac{(f-f_s)}{\tau}, \quad (2.1)$$

where $\boldsymbol{\mathcal{E}}_1$ and $\mathbf{B}_1$ are the electric and magnetic fields induced by the acoustic wave, τ is the electron relaxation time, and f_s is the distribution to which the electrons relax in the presence of the wave. This distribution is[9]

$$f_s(\mathbf{v})=f_0(\mathbf{v}-d\xi/dt,\, n_0+n_1)\approx f_0(\mathbf{v})-(d\xi/dt)\cdot(df_0/d\mathbf{v})+n_1(df_0/dn_0). \quad (2.2)$$

Here $f_0(\mathbf{v})$ is the equilibrium distribution of the electrons, ξ is the amplitude of the acoustic wave, and the second and third terms on the right side of (2.2) arise from the collision-drag effect and from the fact that the scattering is local[9,10] and therefore does not change the electron density. In semiconductors where the interaction of the wave with the electrons is either via the deformation potential or piezoelectric coupling, the second term can be neglected.[2] We also take the elec-

[1] D. L. White, J. Appl. Phys. **33**, 2547 (1962).
[2] For other references in this area see H. N. Spector, Solid State Phys. **19**, 291 (1966).
[3] A. R. Hutson, J. H. McFee, and D. L. White, Phys. Rev. Letters **7**, 237 (1961).
[4] M. Pomerantz, Phys. Rev. Letters **13**, 308 (1964).
[5] K. W. Nill and A. L. McWhorter, J. Phys. Soc. Japan Suppl. **21**, 755 (1966); K. W. Nill, thesis, M.I.T., 1966 (unpublished).
[6] K. W. Nill, Bull. Am. Phys. Soc. **12**, 423 (1967).
[7] L. Solymar, Solid State Electron. **9**, 879 (1966).
[8] H. N. Spector, Phys. Rev. **127**, 1084 (1962).
[9] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960).
[10] T. Holstein, Phys. Rev. **113**, 479 (1959).

tric field $\mathcal{E}_1$ induced by the wave to be a longitudinal field so that $\mathbf{B}_1 = c\mathbf{q}/\omega \times \mathcal{E}_1 = 0$. This is valid since in semiconductors where deformation potential coupling dominates, the induced electric field is longitudinal regardless of whether the acoustic wave itself is longitudinal or transverse. Also, when piezoelectric coupling dominates, the interaction is strongest for those waves which induce longitudinal electric fields. The solution of (2.1) can be written as

$$f = f_{\mathrm{dc}}(\mathbf{v}) + g(\mathbf{v}) \exp i(\mathbf{q}\cdot\mathbf{r} - \omega t), \qquad (2.3)$$

where the first term represents the electron distribution function in the presence of the dc electric field, but in the absence of the acoustic wave, while the second term represents the part of the distribution function which is induced by the wave. The dc distribution can be found to first order in $\mathcal{E}_0$ by solving the dc part of (2.1).[8] However, it can be seen that such a solution is given by the first two terms in the expansion of $f_0(\mathbf{v} - \mathbf{v}_d)$ for small $\mathbf{v}_d$ where $\mathbf{v}_d = -(e\tau/m)\mathcal{E}_0$ is the drift velocity of the electrons in the dc field. We therefore take $f_{\mathrm{dc}}(\mathbf{v}) = f_0(\mathbf{v} - \mathbf{v}_d)$. The part of the distribution function which is directly proportional to the amplitude of the acoustic wave obeys the equation

$$[\tau^{-1} + i(\mathbf{q}\cdot\mathbf{v} - \omega) + (\mathbf{v}_d/\tau)\cdot(d/d\mathbf{v})]g(\mathbf{v}) = (e\mathcal{E}_1/m)\cdot(df_{\mathrm{dc}}/d\mathbf{v}) + (n_1/\tau)(df_0/dn_0). \qquad (2.4)$$

If we take the direction of the dc electric field, and therefore also $\mathbf{v}_d$ to be the z axis of our coordinate system, then (2.4) has the solution

$$g(\mathbf{v}) = \frac{\tau}{v_d}\int_{-\infty}^{v_z} ds \exp\left\{-\int_s^{v_z} \frac{dt\tau}{v_d}[i(\mathbf{q}\cdot\mathbf{v} - \omega) + \tau^{-1}]\right\} \times \left(\frac{e\mathcal{E}_1}{m}\cdot\frac{df_{\mathrm{dc}}}{d\mathbf{v}} + \frac{n_1}{\tau}\frac{df_0}{dn_0}\right). \qquad (2.5)$$

We take the general case where $\mathbf{q}$ lies in the xz plane and treat the electrons as obeying Boltzmann statistics, i.e.,

$$f_0(\mathbf{v}) = n_0(m/2\pi k_B T)^{3/2} \exp(-mv^2/2k_B T). \qquad (2.6)$$

Then the ac distribution $g(\mathbf{v})$ has the form

$$g(\mathbf{v}) = -\frac{\sigma_0}{e\pi v_d}\frac{c^{5/2}}{(c-b)^{1/2}} \exp -c[v_\perp^2 + (v_z - v_d)^2] \times\left\{\mathcal{E}_{1x} v_\perp \cos\phi F(u) - \frac{\mathcal{E}_{1z}}{\pi^{1/2}(c-b)^{1/2}} \times\left[1 - \pi^{1/2}\frac{(\alpha + i\beta v_\perp \cos\phi)}{2(c-b)^{1/2}} F(u)\right]\right\} + \frac{n_1}{2\pi v_d}\frac{c^{3/2}}{(c-b)^{1/2}} \exp -c(v_\perp^2 + v_z^2) F(\gamma), \qquad (2.7)$$

where we have expressed $\mathbf{v}$ in cylindrical coordinates

$$\sigma_0 = n_0 e^2\tau/m, \quad \alpha = a + 2bv_d, \quad a = (1 - i\omega\tau)/v_d,$$
$$b = iq_z\tau/2v_d, \quad \beta = (q_x\tau)/v_d, \quad c = m/2k_B T,$$
$$u = \{(\alpha + i\beta v_\perp \cos\phi)/[2(c-b)^{1/2}]\} - (c-b)^{1/2}(v_z - v_d),$$
$$\gamma = \{(a + i\beta v_\perp \cos\phi)/[2(c-b)^{1/2}]\} - (c-b)^{1/2} v_z,$$

and

$$F(u) = \exp u^2 \operatorname{erfc}(u) = (2/\pi^{1/2}) \exp u^2 \int_u^\infty dt \exp -t^2.$$

To first order in $\mathbf{v}_d$, (2.7) reduces to the result derived in our previous work.[8] The ac current induced by the wave is

$$\mathbf{j} = -e\int d\mathbf{v}\,\mathbf{v}g(\mathbf{v}) = \boldsymbol{\sigma}\cdot\mathcal{E}_1 - \mathbf{R}n_1 e v_s, \qquad (2.8)$$

where $\boldsymbol{\sigma}$ and $\mathbf{R}$ can be evaluated by using (2.7) together with (2.8). The integrals necessary to obtain $\boldsymbol{\sigma}$ and $\mathbf{R}$ are evaluated in the Appendix. The expressions obtained are

$$\sigma_{xx} = (2\boldsymbol{\sigma}_0/v_d)(x/\alpha)^3\{(\beta^2 x/c)[1 - \pi^{1/2}xF(x)] + 2\pi^{1/2}(b/c)(c-b)F(x)\}, \qquad (2.9a)$$

$$\sigma_{zz} = \frac{\boldsymbol{\sigma}_0}{2bv_d}\frac{c}{(c-b)}\left\{\left[a - \frac{2}{c}\left(\frac{\beta x}{\alpha}\right)^2(a + bv_d) + \frac{(\beta x)^4}{c^2\alpha^3}\right] \times[1 - \pi^{1/2}xF(x)] + \frac{2\pi^{1/2}b(c-b)}{c^2}\beta^2\left(\frac{x}{\alpha}\right)^3 F(x)\right\}, \qquad (2.9b)$$

$$\sigma_{zx} = \frac{i\boldsymbol{\sigma}_0\beta}{bv_d}\left(\frac{x}{\alpha}\right)^2\left\{\left[a + \frac{(\beta x)^2}{c\alpha}\right][1\cdot\pi^{1/2}xF(x)] - 2\pi^{1/2}\frac{b}{c}(c-b)\frac{x}{\alpha}F(x)\right\}, \qquad (2.9c)$$

$$\boldsymbol{\sigma}_{xz} = \frac{-i\boldsymbol{\sigma}_0\beta}{v_d(c-b)}\frac{x^2}{\alpha}\left\{\left[1 - c^{-1}\left(\frac{\beta x}{\alpha}\right)^2\right][1 - \pi^{1/2}xF(x)] - 2\pi^{1/2}\frac{b}{c}\frac{(c-b)}{\alpha^2}xF(x)\right\}, \qquad (2.9d)$$

$$R_x = (-i\beta/cv_s v_d)(y/a)^2[1 - \pi^{1/2}yF(y)], \qquad (2.9e)$$

$$R_z = (2v_s b v_d)^{-1}[1 - c^{-1}(\beta y/a)^2][1 - \pi^{1/2}yF(y)], \qquad (2.9f)$$

where

$$x = \frac{1 - i(\omega - \mathbf{q}\cdot\mathbf{v}_d)\tau}{ql(1 - ie\mathcal{E}_0\cdot\mathbf{q}/q^2 k_B T)^{1/2}},$$

$$y = \frac{1 - i\omega\tau}{ql(1 - ie\mathcal{E}_0\cdot\mathbf{q}/q^2 k_B T)^{1/2}},$$

and $l = (2k_B T/m)^{1/2}\tau$ is the mean free path. The components of $\boldsymbol{\sigma}$ and $\mathbf{R}$ which are important in determining the interaction of acoustic waves with conduction electrons in semiconductors, where deformation potential coupling or piezoelectric coupling of acoustic waves to a longitudinal electric field are the dominant mechanisms, are the components along the direction of propagation of the wave, which we denote σ_{11} and R_1. The relation between these components and those derived in (2.9) can be obtained by rotating our coordinate system;

$$\sigma_{11}=\sigma_{xx}(q_x/q)^2+\sigma_{zz}(q_z/q)^2+(q_xq_z/q^2)(\sigma_{xz}+\sigma_{zx}),$$

$$R_1=R_x(q_x/q)+R_z(q_z/q). \tag{2.10}$$

After a lengthy but straightforward calculation we find that

$$\sigma_{11}=\frac{2(1-i\omega\tau)\sigma_0}{(ql)^2(1-ie\mathcal{E}_0\cdot\mathbf{q}/q^2k_BT)^{1/2}}\{1-\pi^{1/2}xF(x)\}, \tag{2.11a}$$

$$R_1=(-i/\omega\tau)[1-\pi^{1/2}yF(y)]. \tag{2.11b}$$

The effective conductivity tensor which comes into the calculation of the acoustic absorption coefficient and the change in the velocity of sound is

$$\sigma_{11}'=\frac{\sigma_{11}}{1-R_1}$$

$$=\frac{-2i\omega\tau\sigma_0[1-\pi^{1/2}xF(x)]}{(ql)^2[1-ie\mathcal{E}_0\cdot\mathbf{q}/q^2k_BT][1-\pi^{1/2}yF(y)/(1-i\omega\tau)]}. \tag{2.12}$$

The effect of the dc electric field comes into the expression for the effective conductivity tensor (2.12) in two ways, first in that the use of a drifted dc electron distribution introduces a Doppler-shifted frequency $\omega_{\text{eff}}=\omega-\mathbf{q}\cdot\mathbf{v}_d$, and second in that a field-dependent effective temperature $T_{\text{eff}}=T-ie\mathcal{E}_0\cdot\mathbf{q}/q^2k_B$ is introduced. When $ql\ll1$, x, $y\gg1$ and $F(x)\approx(1/\pi^{1/2}x)\times(1-1/2x^2)$, so that (2.12) reduces to the form

$$\sigma_{11}'=\frac{\sigma_0}{\{1-\mathbf{q}\cdot\mathbf{v}_d/\omega+\frac{1}{2}i[(ql)^2/\omega\tau]\}}. \tag{2.13}$$

This is just the result obtained by using the phenomenological approach. In this case, the terms leading to a negative absorption coefficient (amplification) arise from the imaginary part of the effective temperature. When $ql\gg1$ and $v_d\ll v_T+v_s$, where $v_T=(2k_BT/m)^{1/2}$ is the mean thermal velocity of the electrons, then x, $y\ll1$ and $F(x)\approx1$, so that (2.12) reduces to the form

$$\sigma_{11}'=-i\omega(q_d/q)^2\{1+i\pi^{1/2}[(\omega-\mathbf{q}\cdot\mathbf{v}_d)/qv_T]\}, \tag{2.14}$$

where $q_d=(n_0e^2/k_BT)^{1/2}$ is the Debye wave number of the electrons and we have taken $e\mathcal{E}_0\cdot\mathbf{q}/q^2k_BT\ll1$. Here the terms leading to acoustic amplification arise from the use of a drifted dc distribution function. The third case of interest, which was treated by Solymar,[7] occurs when $v_d\gg v_T$ and $\omega\tau\gg1$. Under these conditions $x\gg1$ and $y\ll1$, so that (2.12) reduces to the form

$$\sigma_{11}'=i\omega n_0e^2/m(\omega-\mathbf{q}\cdot\mathbf{v}_d)^2. \tag{2.15}$$

This is exactly the same result derived by Solymar using degenerate statistics and a drifted dc distribution function for the electrons. In this limit, the statistics which the electrons obey are irrelevant to the calculation of the conductivity since the phase velocity of the wave in the rest frame of the electrons is $\hat{\mathbf{q}}\cdot\mathbf{v}_d$ and thus exceeds the average electron velocity. The conductivity can then be calculated just by taking moments of the Boltzmann equation since the individual electrons cannot reach resonance with the wave and there is no Landau damping. In fact, this third case is the one which is least likely to be observed in practice since the electrons must be given a drift velocity which exceeds the average electron velocity in the semiconductor.

III. DISCUSSION

In this paper we have solved the Boltzmann equation for electrons interacting with an acoustic wave in the presence of strong dc electric fields. We have calculated the components of the conductivity tensor which come in to the calculation of the absorption coefficient and the change in the sound velocity. The advantage of our present calculation over our previous solution of the Boltzmann equation[8] is twofold. First, we have made our calculations using classical statistics for the electrons, which is more realistic than using degenerate statistics in most high-resistivity semiconductors, where experiments have been done. Second, we have not limited ourselves to a solution which is only correct to first order in the dc electric fields. The presence of the dc electric fields leads to the introduction of a drifted dc distribution for the electrons and a complex, field-dependent effective temperature. This agrees with the predictions of Pines and Schrieffer[11] in treating a similar problem concerning the two-stream instability in solid-state plasmas. In the limit $ql\ll1$, our result agrees with that derived previously.[8] However, from our treatment it becomes clear that the terms leading to acoustic amplification arise from the field-dependent effective temperature. When $ql\gg1$, our results reduce to that derived previously as long as $e\mathcal{E}_0\cdot\mathbf{q}/q^2\ll k_BT$ and $v_d<v_T$. Here the terms leading to acoustic amplification arise from the use of a drifted distribution function. The condition $e\mathcal{E}_0\cdot\mathbf{q}/q^2k_BT\ll1$ can be rewritten in terms of v_d and v_T, i.e., $2\hat{\mathbf{q}}\cdot\mathbf{v}_d/qlv_T\ll1$, where $\hat{\mathbf{q}}$ is a unit vector in the direction of $\mathbf{q}$. It can therefore be seen that for $ql\gg1$ and $v_d<v_T$, the field-dependent part of the effective temperature can be

[11] D. Pines and J. R. Schrieffer, Phys. Rev 124, 1387 (1961).

neglected because it plays no important role in determining the amplification coefficient. The only effect it has is to reduce slightly the value of the conductivity. The limit $\omega\tau\gg1$ and $v_d>v_T$ was discussed above and is a limit which would be very difficult to obtain in practice. It therefore seems that the present theory which takes account of arbitrary electric field is in good agreement with our earlier linear theory. Thus any anomalous phenomena arising in the acoustic amplification in strong electric fields[6] must result from effects other than the direct effect of the dc field on the ac conductivity. The most serious approximations we have made is to take the dc electron distribution to be a drifted Boltzmann distribution for all dc electric fields and to use a relaxation time in treating the collision term in the Boltzmann equation. The first approximation is commonly made. It is certainly a valid approximation at low electric fields, while for high fields Frohlich and Paranjape[12] have shown that it is valid when the electron-electron collisions are more effective than the electron-lattice interactions in establishing the energy losses of the carriers. Some of the criteria for using a drifted Maxwellian distribution in semiconductors like InSb are discussed in Ref. 11. The second approximation should become unimportant when dealing with amplification of high-frequency phonons where $ql\gg1$. However, a field-dependent relaxation time could explain anomalous behavior in the region $ql\ll1$ and the division between the regions $ql\ll1$ and $ql\gg1$ might be different in the presence of strong fields than in their absence.

The reason why the linear theory agrees with the present theory is that the electric field only introduces a Doppler-shifted frequency ω_{eff}, and an effective temperature T_{eff}. In the limits $ql\ll1$ and $ql\gg1$, with $v_d<v_T$, the terms containing the electric field come in only to first order, yielding an exact agreement with the linear theory.

Note also that the expressions for the components of $\boldsymbol{\sigma}$ in strong dc fields derived in this paper are valid for any excitation which generates longitudinal electric fields in nondegenerate semiconductors. Therefore, these expressions could also be used in describing the interaction of conduction electrons with optical phonons or plasma waves in semiconductors. The only difference would be in the initial dispersion of the wave, i.e., the dependence of ω on q.

APPENDIX

To obtain the expressions for $\boldsymbol{\sigma}$ and $\mathbf{R}$ in (2.9) from (2.7) and (2.8) we must evaluate integrals of the following form:

$$I_1=\int_{-\infty}^{+\infty}dx\,\exp-cx^2F(\lambda-\delta x),\tag{A1}$$

$$I_2=\int_{-\infty}^{+\infty}dx\,x\,\exp-cx^2F(\lambda-\delta x),\tag{A2}$$

$$I_3=\int_0^{+\infty}dv_\perp\,v_\perp\exp-cv_\perp^2\int_0^{2\pi}d\phi\,F(\lambda+i\delta v_\perp\cos\phi),\tag{A3}$$

$$I_4=\int_0^{+\infty}dv_\perp\,v_\perp^2\exp-cv_\perp^2\int_0^{2\pi}d\phi\,\cos\phi F(\lambda+i\delta v_\perp\cos\phi),\tag{A4}$$

$$I_5=\int_0^{+\infty}dv_\perp\,v_\perp^3\exp-cv_\perp^2\int_0^{2\pi}d\phi\,\cos^2\phi F(\lambda+i\delta v_\perp\cos\phi).\tag{A5}$$

To evaluate these integrals we use the following integral form for $F(z)$[13]:

$$F(z)=\pi^{-1}\int_0^{\infty}d\psi\int_{-\infty}^{+\infty}dt\,\exp-t^2\exp-(z+it)\psi.\tag{A6}$$

Using (A6) in (A1) and (A2), and performing the integrations over x, t, and ψ in that order, we obtain

$$I_1=[\pi/(c-\delta^2)]^{1/2}F[\lambda/(1-\delta^2/c)^{1/2}],\tag{A7}$$

$$I_2=\frac{\delta}{c^{1/2}(c-\delta^2)}\left[1-\frac{\pi^{1/2}\lambda}{(1-\delta^2/c)^{1/2}}F\left(\frac{\lambda}{(1-\delta^2/c)^{1/2}}\right)\right].\tag{A8}$$

To obtain I_3, I_4, and I_5 we use (A6) in (A3)–(A5) and perform the integrations over ψ, ϕ, $v_\perp$, and t in that order. We then get the result

$$I_3=\frac{\pi}{c^{1/2}(c+\delta^2)^{1/2}}F\left(\frac{\lambda}{(1+\delta^2/c)^{1/2}}\right),\tag{A9}$$

$$I_4=\frac{-i\pi^{1/2}\delta}{c(c+\delta^2)}\left[1-\frac{\pi^{1/2}\lambda}{(1+\delta^2/c)^{1/2}}F\left(\frac{\lambda}{(1+\delta^2/c)^{1/2}}\right)\right],\tag{A10}$$

$$I_5=\frac{\pi^{1/2}\lambda\delta^2}{c(c+\delta^2)^2}\left[1-\frac{\pi^{1/2}\lambda}{(1+\delta^2/c)^{1/2}}F\left(\frac{\lambda}{(1+\delta^2/c)^{1/2}}\right)\right]+\frac{\pi}{2c^{1/2}(c+\delta^2)^{3/2}}F\left(\frac{\lambda}{(1+\delta^2/c)^{1/2}}\right).\tag{A11}$$

The evaluation of the integrals I_1–I_5 allows us to write down the results (2.9) for $\boldsymbol{\sigma}$ and $\mathbf{R}$.

[12] H. Frohlich and B. Paranjape, Proc. Phys. Soc. (London) **B69**, 21 (1956).

[13] *Handbook of Mathematical Functions*, edited by M. Abramovitz and I. A. Stegun (U.S. Department of Commerce, National Bureau of Standards, Washington, D.C., 1964), Appl. Math. Ser. 55, p. 297.