Alternating-Current-Induced Voltages in Superconducting Wires

HENRY F. TAYLOR*
Rice University, Houston, Texas
(Received 19 June 1967)

A model introduced by Bean and extended by Kim is applied to predict voltages induced in a superconducting wire carrying low-frequency alternating current. A basic premise of Bean's model is that the magnitude of the local current density, J, is determined by the magnitude of the local magnetic induction, B. The function $J(B)$ is assumed to have the form deduced by Kim from magnetization measurements, $J=\alpha/(B+B_0)$, with α and B_0 treated as adjustable constants. Relations for voltage as a function of current have been derived for ac and for ac plus superimposed dc. Predicted V-versus-I curves are found to agree fairly well with oscilloscope tracings obtained for defect-saturated wires of Nb and various Nb alloys.

INTRODUCTION

A LARGE number of experimental studies of the ac losses in hard superconducting wires have been reported.[1] Grasmehr and Finzi,[2] and Heinzel and Voigt,[3] have presented evidence which suggests that a model proposed by Bean[4] to explain the magnetization behavior of hard superconductors could also serve as a basis for explaining the ac loss phenomenon. Grasmehr and Finzi have applied Bean's model to predict the voltage across a wire carrying ac as a function of time for the case that a large magnetic field is directed transverse to the axis of the wire. Predicted curves are in good agreement with measurements they made on a Nb-25% Zr sample. Heinzel and Voigt have applied the model to explain the results of a magnetic hysteresis experiment which is closely analogous to ac loss experiments. They measured the losses in a Nb-25% Zr wire sample placed in an alternating magnetic field.

The case of a wire carrying varying current and subject only to its self-field is considered in this paper. In particular, an extended version, proposed by Kim *et al.*,[5] of the model introduced by Bean, is applied to predict the voltage induced in a wire carrying ac. Explicit relations giving the voltage across the wire as a function of the current are derived. Predicted voltage-current curves are compared with experimental curves obtained for Nb and some Nb alloys. Such comparisons provide a significant test of the applicability of the model to the case of a hard superconducting wire carrying ac.

PREDICTION OF THE DEPENDENCE OF VOLTAGE ON CURRENT

A basic premiss of the model introduced by Bean is that the magnitude of the local current density, J, is a function only of the magnitude of the local magnetic induction, B. The derivation presented in this section is carried out for the case that the dependence of $|J|$ on $|B|$ is given by

$$|J|=\alpha/(|B|+B_0). \tag{1}$$

This form for $J(B)$ was initially deduced by Kim[5] from magnetization measurements on hard superconductors. The quantities α and B_0 are constants which would presumably be determined by the composition of a particular sample and by how it was prepared.

A second premise of Bean's model is that the direction of J is such as to shield the interior of the sample from changes in the value of B at the surface of the sample. The surface magnetic induction B_s is given by

$$B_s=\mu_0 I/2\pi R, \tag{2}$$

for a wire of radius R carrying a current I.

The only components of J and B in cylindrical coordinates are assumed to be J_z and B_θ. So Ampere's law

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}$$

can be written

$$(\partial/\partial r)(rB)=\mu_0 rJ. \tag{3}$$

Substituting (1) into (3) leads to

$$(\partial/\partial r)(rB)=\pm\mu_0 r\alpha/(|B|+B_0). \tag{4}$$

The sign in (4) is determined by the shielding criterion. In order to simplify (4) to a form which can be solved easily, it will be assumed that the maximum depth of current penetration is small. That is, $R-r_0\ll R$, where r_0 is defined by $J\equiv0$ for $r<r_0$, $J\neq0$ for $r_0<r<R$. This assumption reduces (3) and (4) to

$$\partial B/\partial r=\mu_0 J, \tag{5}$$

$$\partial B/\partial r=\pm\mu_0\alpha/(|B|+B_0). \tag{6}$$

Figure 1 shows how the spatial distribution of J and B is predicted to change as the total current I is carried through half of an ac cycle, from I_m to $-I_m$. The plots were obtained by solving (6) for $B(r)$ with suitable attention to the shielding criterion and the boundary condition and then obtaining $J(r)$ from (5). Solutions depicted in Fig. 1 represent the case that $B_m/B_0=3$, where $B_m=\mu_0 I_m/2\pi R=$magnetic induction at the surface of the wire for $I=I_m$.

* This work has been supported in part by the National Aeronautics and Space Administration.

[1] See, for example, W. R. Wisseman, L. A. Boatner, and F. J. Low, J. Appl. Phys. **35**, 2649 (1964); R. G. Rhodes, E. C. Rogers, and R. J. A. Seebold, Cryogenics **4**, 206 (1964); T. Pech, J. P. Duflot, and G. Fournet, Phys. Letters **16**, 201 (1965).
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[5] Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **129**, 528 (1963).

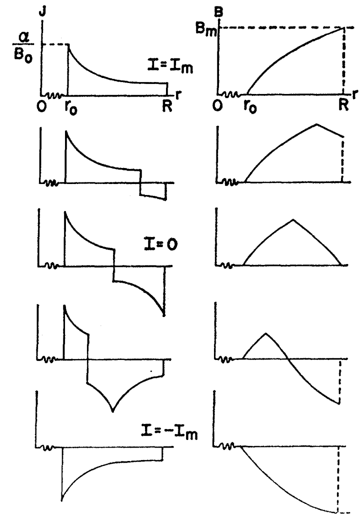

FIG. 1. Variation of $J(r)$ and of $B(r)$ as I is decreased from I_m to $-I_m$. The ratio $B_m/B_0=3$.

The function $B(r)$ can be integrated to obtain ϕ_{wire}, the total flux contained within the material of the wire:

$$\phi_{\text{wire}}=l\int_0^R B dr, \tag{7}$$

with l=length of the sample. The quantity ϕ_{wire} is related to the electric field E at the surface of the sample by the equation

$$\int_a^b \mathbf{E}\cdot d\mathbf{l}=\frac{d\phi_{\text{wire}}}{dt}, \tag{8}$$

with the integral taken along the surface between points a and b at either end of the sample. This can be shown by applying Faraday's law to the path of integration p shown in Fig. 2:

$$\oint_p \mathbf{E}\cdot d\mathbf{l}=-\frac{d\phi_{\text{wire}}}{dt}. \tag{9}$$

But

$$-\oint_p \mathbf{E}\cdot d\mathbf{l}=\int_a^b \mathbf{E}\cdot d\mathbf{l}+\int_b^c \mathbf{E}\cdot d\mathbf{l}+\int_c^d \mathbf{E}\cdot d\mathbf{l}+\int_d^a \mathbf{E}\cdot d\mathbf{l}. \tag{10}$$

It has been assumed that $J\equiv 0$ for $r<r_0$. This cannot be true unless $E\equiv 0$ in this cylindrical region of "virgin superconductor" which contains the axis of the wire. So

$$\int_c^d \mathbf{E}\cdot d\mathbf{l}=0.$$

But

$$\int_b^c \mathbf{E}\cdot d\mathbf{l}+\int_d^a \mathbf{E}\cdot d\mathbf{l}=0$$

by symmetry, and (10) can be written

$$-\oint_p \mathbf{E}\cdot d\mathbf{l}=\int_a^b \mathbf{E}\cdot d\mathbf{l}. \tag{11}$$

Combining (11) and (9) leads to (8), the desired result.

It is convenient for comparing experimental with predicted curves to have the observed voltage V equal to $d\phi_{\text{wire}}/dt$, a quantity which can be predicted on the basis of the Bean–Kim model. In order to accomplish this, a coil is used to exactly cancel a large inductive component of voltage which is picked up in the voltage-sensing circuit. The signal across the coil, which is proportional to dI/dt, is divided using a potentiometer. The voltage across the oscilloscope terminals is given by

$$V=\int_a^b \mathbf{E}\cdot d\mathbf{l}+K dI/dt, \tag{12}$$

with the value of the constant K determined by the potentiometer setting. Substituting (8) into (12) yields

$$V=d\phi_{\text{wire}}/dt+K dI/dt. \tag{13}$$

The potentiometer is set to give $V\equiv 0$ for low ac amplitudes. It is expected that $\phi_{\text{wire}}\equiv 0$ for low enough ac amplitudes, since all the current is presumed to flow very near the surface as surface shielding currents. But $\phi_{\text{wire}}\equiv 0$ and $V\equiv 0$ imply that $K=0$ in (13), and so

$$V=d\phi_{\text{wire}}/dt \tag{14}$$

for all ac amplitudes. It is worth repeating that $K=0$ in (13) as a consequence of setting the canceling coil potentiometer to give $V\equiv 0$ for low ac amplitudes.

Explicit relations have been derived for the dependence of V on I. The quantity $\dot{I}$ in these equations is related to I by $|\dot{I}|=\omega(I_m^2-I^2)^{1/2}$ for the case that $I=I_m \sin\omega t$. The constant $C=l\mu_0^2/(2\pi R)^3\alpha$.

A. Pure ac:

$$\begin{aligned}
V(I)&=C\{I-I_0+2^{-1/2}[(I_m+I_0)^2+(I-I_0)^2]^{1/2}\}\\
&\qquad\times[I_0-I]\dot{I}\qquad \dot{I}>0,\qquad I<0\\
&=C\{I-I_0+2^{-1/2}[2I_0^2+(I_m+I_0)^2-(I+I_0)^2]^{1/2}\}\\
&\qquad\times[I+I_0]\dot{I}\qquad \dot{I}>0,\qquad I>0\\
&=C\{-I-I_0+2^{-1/2}[2I_0^2+(I_m+I_0)^2-(I_0-I)^2]^{1/2}\}\\
&\qquad\times[-I+I_0]\dot{I}\qquad \dot{I}<0,\qquad I<0\\
&=C\{-I-I_0+2^{-1/2}[(I_m+I_0)^2+(I+I_0)^2]^{1/2}\}\\
&\qquad\times[I+I_0]\dot{I}\qquad \dot{I}<0,\qquad I>0.
\end{aligned} \tag{15}$$

B. Superimposed dc $I_d>I_m$:

$$\begin{aligned}
V(I)&=C\{I+I_0-2^{-1/2}[(I_d-I_m+I_0)^2+(I+I_0)^2]^{1/2}\}\\
&\qquad\times[I+I_0]\dot{I},\qquad \dot{I}>0\\
&=-C\{I+I_0-2^{-1/2}[(I+I_0)^2+(I_m+I_d+I_0)^2]^{1/2}\}\\
&\qquad\times[I+I_0]\dot{I},\qquad \dot{I}<0.
\end{aligned} \tag{16}$$

It is of some interest to compare the shape of the $V(I)$ curve for pure ac for two limiting cases: $I_0/I_m\to 0$

[corresponding to $J(B)=\alpha/B$] and $I_0/I_m \to \infty$ [corresponding to $J(B)=\alpha/B_0=\text{const}$]. This is done in Fig. 3.

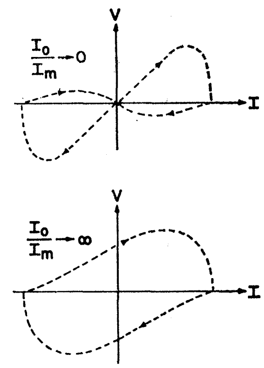

FIG. 3. Shapes of predicted $V(I)$ curves for two limiting cases. These curves are for pure ac, Eq. (15).

EXPERIMENTAL

All samples had a high defect concentration due to the cold working process used in their production. Supplier was the National Research Corporation.

The experimental setup is not complicated. Large copper input leads carry current to the superconducting wire samples, which are maintained at 4.2°K. The samples are arranged in a U configuration. Calculations have indicated that the magnetic field at the surface of the wire should be uniform to within 3% for the configuration used. This implies that, for purposes of comparing experimental results with predictions of the model, the samples can be considered as long, straight wires.

The voltage across the sample is observed on the vertical scale of an oscilloscope (sensitivity$=2\times10^{-4}$ V/cm). The current in the sample, proportional to the voltage across a standard resistor, is plotted on the horizontal oscilloscope scale.

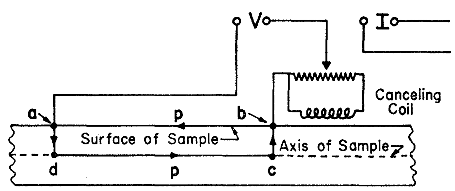

FIG. 2. Schematic diagram of wire sample and canceling coil The path of integration p (see text) extends along the surface of the sample between points b and a and along the axis between d and c.

The canceling coil is positioned to pick up a large signal proportional to dI/dt. A potentiometer is used to divide the voltage across the canceling coil. The potentiometer is adjusted until a horizontal line is observed on the oscilloscope (i.e., $V\equiv0$) for low ac amplitudes. This adjustment is not changed during the balance of the experiment. The inductive signal which is cancelled is from 5 to 25 times as large as the signal of interest.

RESULTS

Voltage-versus-current curves predicted by the Bean–Kim model are compared with oscilloscope tracings for Nb-25% Zr in Fig. 4 and for Nb in Fig. 5. Note the qualitative features of agreement between predicted and observed curves. For pure ac, there is a small maximum in the voltage as I is increased from $-I_m$, a minimum near $I=0$, and a large maximum for $I>0$. Note also that $V=0$ at $I=I_m$ and $I=-I_m$. For dc superimposed on the ac, the observed curves are again predicted fairly well for both materials.

Agreement between predicted V-versus-I curves and experimental curves has also been good for other samples studied. These include Nb-33% Zr, Nb-50% Ti, and a smaller diameter Nb wire. Figure 6 compares

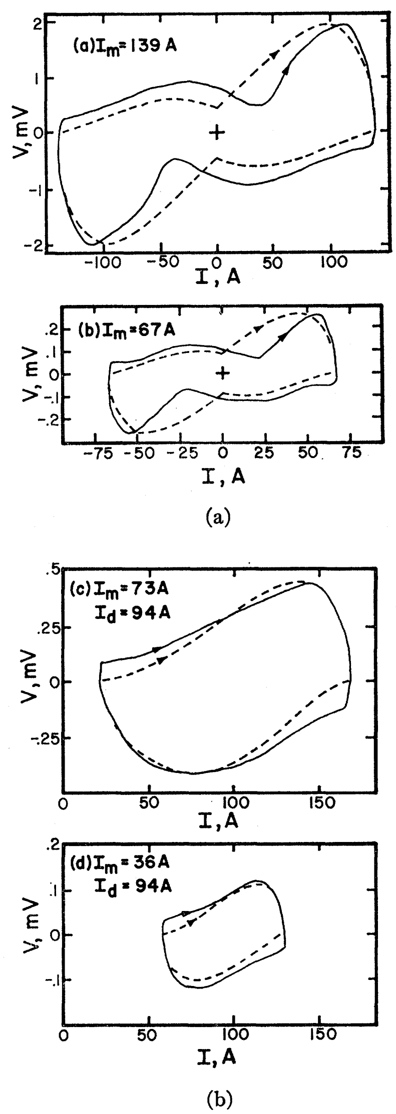

FIG. 4. Oscilloscope tracings (solid lines) for a 95-cm length of 10-mil Nb-25%Zr wire carrying 60~ ac compared with predicted curves (dashed lines). The adjustable parameters were chosen to have the values $\alpha=1.10\times10^6$ kG A/cm^2, $B_0=0.55$ kG for plotting predicted curves.

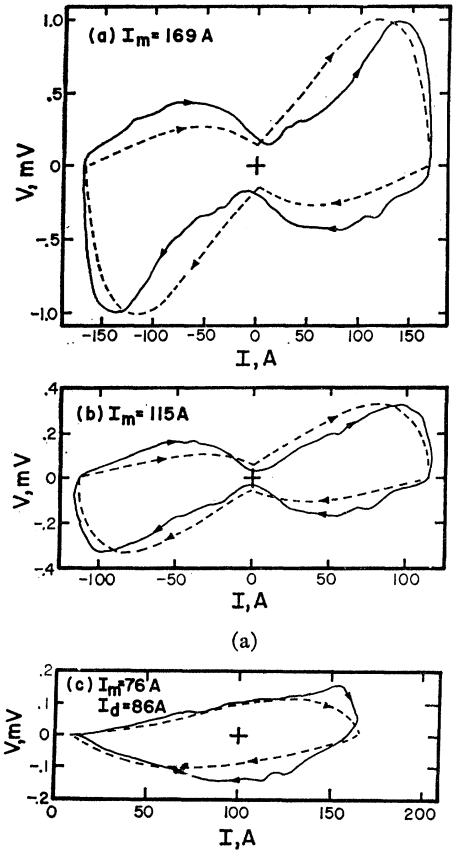

FIG. 5. Oscilloscope tracings (solid lines) for a 95-cm length of 10-mil Nb wire carrying 60~ ac compared with predicted curves (dashed lines). The adjustable parameters were chosen to have the values $\alpha = 3.2\times10^6$ kG A/cm², $B_0 = 0.35$ kG for plotting the predicted curves.

energy-loss-per-cycle data for all five samples studied with curves predicted by the model. Energy loss per cycle $W_f = \oint VIdt$ is computed by numerical integration of V-versus-I curves. Values of α and B_0 have been chosen for each material to give a good fit of the predicted energy loss to the data. Values of α chosen to fit energy-loss data for Nb-25% Zr and Nb differ slightly from values used to fit the V-versus-I curves of Figs. 4 and 5.

In Fig. 7 are compared oscilloscope tracings for Nb-25% Zr measured at two different frequencies. The 412~tracings have been "reduced" by a factor of 60/412 for comparison with 60~data. The Bean–Kim model predicts

$$V(I, I_m, I_d, s) = g(I, I_m, I_d, s)\dot{I},$$

where the function g, which depends on s, the sign of $\dot{I}$, as well as on I, I_m, and I_d, can be determined from (15) and (16). If $I = I_m \sin\omega t$,

$$V(I, I_m, I_d, s) = \omega g(I, I_m, I_d, s) I_m \cos\omega t.$$

This implies that

$$V_{60\sim}(I, I_m, I_d, s) = (60/412)\, V_{412\sim}(I, I_m, I_d, s),$$

which explains the factor 60/412 by which 412~data have been reduced for comparison with 60~data in Fig. 7.

Figure 8 shows energy-loss-per-cycle data for the two different frequencies compared with the predicted curve for Nb-25% Zr. The model predicts that the energy loss per cycle W_f should be independent of frequency:

$$W_f = \oint g(I, I_m, I_d, s) I\dot{I}dt,$$

$$W_f = \oint g(I, I_m, I_d, s) IdI,$$

$$W_f = W_f(I_m, I_d).$$

The Bean–Kim model predicts that the depth of current penetration $R - r_0$ is given by

$$R - r_0 = (\mu_0/8\pi^2R^2\alpha)[(I_m + I_0)^2 - I_0^2].$$

Substituting values of the adjustable constants α and I_0 used to fit the data leads to estimates that $R - r_0 = 0.18R$ for Nb-25% Zr at $I_m = 139$ amps [Fig. 4(a)]

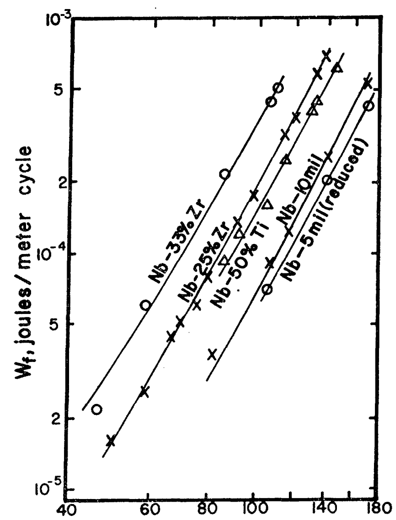

FIG. 6. Energy loss per cycle as a function of ac amplitude for 10-m wires of various materials and a 5-m Nb wire. Data points were taken at 60~; the solid lines represent predicted dependence of W_f on I_m. Values of α and B_0 used for plotting the predicted curves are

	α(kGA/cm²)	B_0 (kG)
Nb-33%Zr	1.00×10^6	1.4
Nb-25%Zr	1.34×10^6	0.55
Nb-50%Ti	2.0×10^6	0.95
Nb10-mil	3.9×10^6	0.47
Nb 5-mil	4.8×10^6	0.63

Data for the 5-ml wire are reduced for comparison with the 10-ml data; multiply W_f and I_m scales by $\frac{1}{2}$ to obtain values actually measured for these quantities.

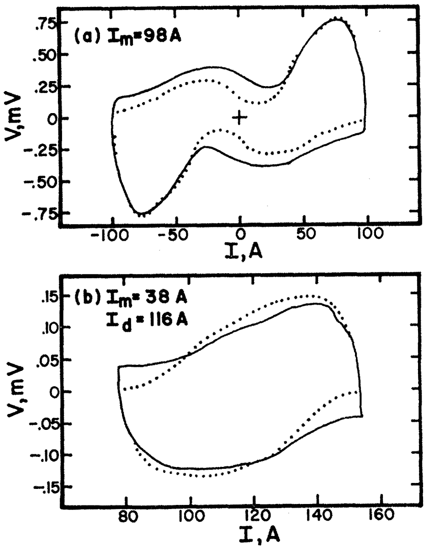

FIG. 7. 60~oscilloscope tracings (solid curves) compared with "reduced" 412~tracings (dotted curves) for a 95-cm length of 10-m Nb-25%Zr wire. The measured 412~voltage has been reduced by a factor of 60/412.

and $R-r_0=0.09R$ for Nb at $I_m=168$ A [Fig. 5(a)]. Penetration depths would be less than the above values for the other curves depicted in Figs. 4 and 5. A check is thus provided for an assumption used to derive equations for $V(I)$—that the maximum depth of current penetration is small.

Each V-versus-I curve can be used to construct an "equivalent" B-versus-H loop. Such loops are constructed in Fig. 9 from the curves in Fig. 4(a). An "average magnetic induction" $\langle B\rangle$ is defined as

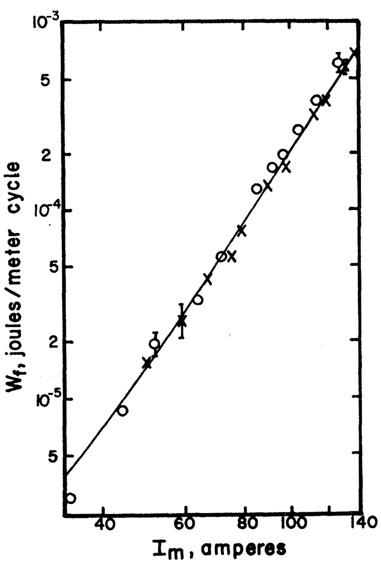

FIG. 8. Energy loss per cycle data for a 10-mil Nb-25%Zr wire taken at 60 cps (×) and 412 cps (○). The solid line is a curve predicted for the case that $\alpha=1.34\times10^6$ kG A/cm², $B_0=0.55$ kG.

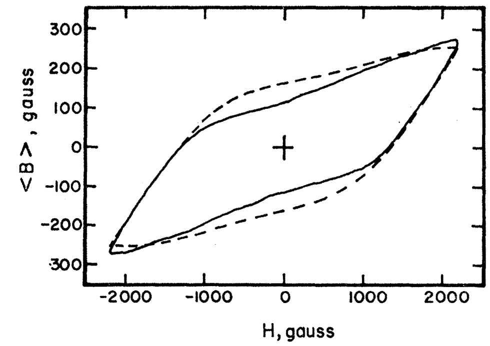

FIG. 9. "Equivalent" B versus H loops plotted from the V-versus-I curves of Fig. 4(a). The solid curve is derived from data; the dashed curve is predicted.

$$\langle B\rangle=(1/R)\int_0^R Bdr=\phi_{\text{wire}}/lR,$$

and H is taken equal to the field at the surface of the wire: $H=\mu_0 I/2\pi R$. The value of ϕ_{wire} at any point in the ac cycle is given by

$$\phi_{\text{wire}}=\phi_{\max}=+\int_{I_m}^{I} Vdt,$$

with

$$\phi_{\max}=\tfrac{1}{2}\int_{-I_m}^{I_m} Vdt.$$

Note that the procedure of setting the canceling coil potentiometer to give $V\equiv0$ for low ac amplitudes is analogous to a procedure commonly used in the calibration of magnetization data for superconducting samples—that of assuming the sample to be perfectly diamagnetic for low values of the applied field.

CONCLUSIONS

The comparison of predicted $V(I)$ curves with experimental curves provides a test of the applicability of the Bean–Kim model to the case of a wire carrying ac. Comparisons have been made over a wide range of ac amplitudes for ac and ac plus superimposed dc and for two different frequencies. Agreement has been generally good, and it is concluded that the model does lead to reasonably accurate predictions of the spatial distribution of J and B in the samples studied.

ACKNOWLEDGMENT

I would like to thank my thesis director, Dr. W. V. Houston, for the valuable suggestions, clarifying discussions, and kindly encouragement which have proven essential to this work.