Quantum Theory of a Gas Laser. II*

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We generalize the method of deriving a kinetic equation for a gas laser developed in the first paper of this series to include radiation-matter correlations. As a consequence, we are able to show that the first Born approximation with asymptotic conditions which contain radiation-matter correlations is sufficient to explain the nonthermal photon distribution observed in photon-counting experiments. Our derivation includes the spatial and velocity effects of the motion of the gas atoms. After adiabatically eliminating the matter variables, we obtain a master equation for the radiation density matrix R with nonlinear coefficients. Our radiation master equation for stationary atoms without spatial effects is the same as the generalized Fokker-Planck equation of the Langevin-noise-equation treatment of lasers. We show that the generalized Fokker-Planck equation is the result of the first-Born-approximation treatment of the $\mathbf{j} \cdot \mathbf{A}$ interaction and is not dependent upon the microscopic structure of the reservoirs. In the first Born approximation, the diagonal and off-diagonal matrix elements of R do not interact with each other. Consequently, an initially diagonal radiation density matrix remains diagonal. However, we show that even though the stationary state is diagonal it is necessary to know the propagator of the off-diagonal part of R to answer time-dependent questions about quantities, such as the line shape, that depend on phase.

I. INTRODUCTION

THE experiments with gas lasers fall into one of two rough categories. The first category consists of those experiments which are mainly explainable by knowing the dependence of the electromagnetic intensity of the laser light on such parameters as the frequencies of the cavity modes, the pressure of the gas atoms, the number of modes, the magnetic field, and the inversion density of the atoms. The second category of experiments consists of the photon-counting experiments which measure the nonthermal photon-distribution function itself and not just the intensity which is the second moment of the photon-distribution function. A complete theory of the full distribution function contains an explanation of the intensity experiments. However, it is often useful with the many variables that appear in intensity experiments to ignore the complications involved in the full photon-distribution function and retain a description on the intensity level.

We previously showed[1] that the first Born approximation for the radiation-matter interaction in a gas laser was sufficient to explain the first category of experiments for all realizable pump power. In order to explain hole-burning phenomena (phenomena which depend on the correlation between the atom's motion and its internal excitation), we found that it was necessary to require that the density matrix of the matter was not factorizable into a center-of-mass (c.m.) matrix and an internal variable matrix at any time even in the asymptotic past. The main result of the present paper is that the first-Born-approximation treatment of the radiation-matter interaction is sufficient to describe the nonthermal laser distribution if we require that the density matrix of matter and radiation is not factorizable even in the infinite past. In other words, by simply changing the asymptotic conditions in the derivation of the kinetic equations for a laser, we find that the kinetic equation for a small parameter is sufficient to explain the nonthermal laser distribution.

We make the discussion in the above paragraph precise by comparing the starting equations of I with the starting equations of this paper. Equations (I2.3a) and (I2.3b) for the radiation density matrix R and the matter density matrix ρ are

$$\partial R/\partial t = - (i/\hbar)[H_f, R] + \mathcal{K}_r R - (\gamma \omega_0)^2 N \, \mathrm{tr}_\sigma \mathrm{tr}_x \mathrm{tr}_{\mathrm{res}} \int_0^\infty d\tau \, [H_1, [H_1(\tau), R\rho(x, t)\mathcal{P}]],$$

$$[\partial \rho(x, t)/\partial t] = - (i/\hbar)[h(1), \rho(x, t)] - (i/\hbar)[(P^2/2m), \rho(x, t)] + \mathcal{K}_{\mathrm{int}}\rho(x, t) + \mathcal{L}_c(x, t)\rho(x, t)$$

$$- (\gamma \omega_0)^2 \, \mathrm{tr}_\sigma \mathrm{tr}_{\mathrm{res}} \int_0^\infty d\tau \, [H_1, [H_1(\tau), R\rho(x, t)\mathcal{P}]],$$

where H_f is the electromagnetic field Hamiltonian, $\mathcal{K}_r$ is the radiation reservoir operator, γ is the dimensionless dipole moment, N is the number of two-level systems, $\hbar\omega_0$ is the two-level energy difference, H_1 is the $\mathbf{j} \cdot \mathbf{A}$ interaction between the two-level systems and the radiation in the rotating wave approximation, ρ is the one-particle density matrix, $\mathcal{P}$ is the density matrix of all the reservoirs, $h(1)$ is the two-level system Hamiltonian, $\mathcal{K}_{\mathrm{int}}$ is the matter reservoir operator including pump, and $\mathcal{L}_c$ is the c.m. collision operator.

[1] C. R. Willis, Phys. Rev. **156**, 320 (1967).

The fundamental equations of the more general theory of the present paper are

$$\partial R/\partial t = -(i/\hbar)[H_f, R] + \mathcal{K}_r R - (\gamma\omega_0)^2 N \, \text{tr}_\sigma \text{tr}_x \text{tr}_{\text{res}} \int_0^\infty d\tau \, [H_1, [H_1(\tau), F_1\mathcal{P}]], \tag{3.6}$$

$$\partial F_1/\partial t = -(i/\hbar)[h(1) + H_f, F_1] - (i/\hbar)[P^2/2m, F_1] + \mathcal{K}_m F_1 + \mathcal{L}_c F_1 - (\gamma\omega_0)^2 \, \text{tr}_a \text{tr}_{\text{res}} \int_0^\infty d\tau \, [H_1, [H_1(\tau), F_1\mathcal{P}]], \tag{3.8}$$

where F_1 is the one-particle radiation density matrix and depends on the radiation variables a, the internal atomic variables σ, and the c.m. variables X.

Equation (3.6) differs from Eq. (I2.3a) only in that the product density matrix $R\rho$ is replaced by the density matrix F_1 which contains correlations. The trace of Eq. (3.8) over the radiation variables reduces the equation for F_1 to the equation for ρ. The new equation for ρ, Eq. (3.7), differs from Eq. (I2.3b) only in that the product density matrix $R\rho$ is replaced by F_1. The replacement of $R\rho$ by F_1 results from a change of the asymptotic condition or, alternatively, from a different truncation of the hierarchy.

Recently, several authors[2-5] have presented theories for models of lasers that lead to nonthermal distributions. Except for Ref. 3, these models refer to stationary atoms and omit spatial effects. Haken and co-workers[3] consider a traveling wave-gas laser and give results to third order in the electric field.

Although Refs. 2 and 3 are not identical, they have in common the same fundamental approach through a Langevin formulation. The authors of Refs. 2 and 3 take the exact equations of motion for the operators with the $j \cdot A$ interaction in the rotating wave approximation, and add a systematic dissipative term and a fluctuating noise term defined through its stochastic properties. These nonlinear Langevin equations then represent the foundation of their theories. These are called noise theories. They then obtain Fokker-Planck (FP) formulations, which are equivalent to the Langevin equation, using a generalization of the method used to treat the motion of a Brownian particle. In this paper, we show that their resultant FP equation is equivalent to the first-Born-approximation master equation of a system of N atoms interacting with each other and with reservoirs. Consequently, to order γ^2, the Born-approximation master equation and the Langevin noise approach give the same results if multiparticle effects are omitted from the master equation. In addition, our derivation holds for moving atoms and spatial effects due to standing waves.

The starting points and initial assumptions of the Langevin-equation and Born-approximation-master-equation approaches are different. It is difficult to formulate asymptotic conditions concerning correlations in the Langevin-equation approach. Stochastic noise variables do not appear in the master equation. The microscopic structure of the reservoirs in the sense of whether or not they are Gaussian plays no role in the master equation. The FP structure of the master equation is due mainly to the Born-approximation treatment of the $j \cdot A$ interaction, not to detailed reservoir properties.

Recently, Fleck[4] has completed a series of papers based on a master-equation approach that has many similarities with the present work. His work includes a discussion of many-body effects and super-radiance effects within the first Born approximation. In addition, he has found numerical solutions for many of his equations. We have generalized the master-equation approach to include off-diagonal matrix elements, correlated c.m. motion, spatial effects, and an analysis of the structure of the time-dependent equations. We find the method Fleck[4] developed for finding stationary states remains valid in the presence of these additional complications.

It is easy to generalize the Bogoliubov method of derivation of kinetic equations to include asymptotic conditions that contain correlations. Bogoliubov[6] found it necessary to include asymptotic correlations to treat Debye shielding in plasmas. However, during the derivation, much time is spent on formal questions. Prigogine[7] has given a careful analysis of the derivation of the Born-approximation master equation for a wide range of asymptotic conditions. The derivations of the Born-approximation master equation are just Fermi's golden rule applied to the density matrix. The crucial requirement is the existence of a time long compared with an "interaction time" and short compared with a "relaxation time," or equivalently, a suitably defined interaction time must be small compared with the relaxation time. The gas lasers and some solid lasers would seem to be simple examples of the Born-approximation master equation because there exists a small parameter which is the dimensionless dipole moment. However, the definition of the interaction time is

[2] M. Lax, Phys. Rev. **157**, 213 (1967). This reference contains complete references to the author's earlier work on noise and lasers.

[3] V. Artz, H. Haken, H. Risken, H. Sauermann, Ch. Schmid, and W. Weidlich, Z. Physik **197**, 207 (1966). This reference contains complete references to earlier work of Haken and co-workers.

[4] J. A. Fleck Jr., Phys. Rev. **149**, 309 (1966); **149**, 322 (1966); **152**, 278 (1966).

[5] M. Scully and W. E. Lamb, Phys. Rev. Letters **16**, 833 (1966).

[6] N. N. Bogoliubov, in *Studies in Statistical Mechanics*, edited by J. de Boer and G. E. Uhlenbeck (North-Holland Publishing Co., Amsterdam, 1962), pp. 5–118.

[7] I. Prigogine, *Non-Equilibrium Statistical Mechanics* (Interscience Publishers Inc., New York, 1962).

complicated by the fact that the lifetime of the atom plays a dual role as an interaction time in one part of the problem and as a relaxation time in another part of the problem.

We show in Sec. II that the parameters of a gas laser are such that the inequality for the validity of the Born-approximation master equation is satisfied even in the presence of the atomic-lifetime complication. In Sec. III we apply the well-known form of the Born-approximation master equation to the gas laser. Section IV contains a solution for the stationary state of the radiation density matrix for three physical situations: (a) moving atoms with traveling waves, (b) stationary atoms with traveling waves, and (c) stationary atoms with standing waves. The case of moving atoms with standing waves is considerably more difficult and an approximate solution is found in Sec. V. We adiabatically eliminate the matter variables and study the time dependence of the diagonal radiation density matrix in Sec. VI. Section VII contains a derivation of the equations of motion for the off-diagonal matrix elements of R. In Sec. VIII, we compare our theory with the Langevin-equation theories. Appendix A contains a derivation of the truncation of the hierarchy. In Appendix B, we analyze the frequency-shift operator.

II. VALIDITY OF THE MASTER EQUATION

We begin our analysis of the first-Born-approximation master equation by considering an arbitrary system with the Hamiltonian $H_0 + \lambda V$, where the dimensionless parameter is small compared with unity. For convenience, we will refer to the first-Born-approximation master equation throughout the rest of this paper as simply the master equation, although the term "master equation" is often used for a more general equation. If a master equation exists, it has the following form:

$$[\partial \rho(t)/\partial t] - i[\rho(t), H_0] = \lambda^2 \int_0^\infty [V, [V(\tau), \rho(t)]] \, d\tau,$$

$$(2.1)$$

where
$$V(\tau) = \exp(iH_0\tau) V \exp(-iH_0\tau)$$

and $\rho(t)$ is the density matrix of the system. There is sometimes a term linear in λ which either vanishes or can be transformed away. The right side of Eq. (2.1) has a real and imaginary part. The imaginary part can be written as a commutator, and represents the shift of H_0 due to the perturbation. The real part causes transitions and represents the lifetime of the unperturbed system. The derivation of Eq. (2.1) consists of applying Fermi's golden rule of perturbation theory to the quantum-mechanical density-matrix equation for ρ. The crucial condition for the validity of Eq. (2.1) is $\tau_{int} \ll \lambda l \ll \tau_{rel}$ or, more simply, $\tau_{int} \ll \tau_{rel}$. The relaxation time τ_{rel} is proportional to λ^{-2}, and so is large as λ goes to zero. The interaction time τ_{int} is a rather subtle but very important quantity. Applying

Fermi's golden rule for the transition probability per unit time W, we obtain

$$W = |t|^{-1} \int_{-\infty}^\infty f(\omega) \frac{\sin^2\omega |t|}{\omega^2} d\omega \approx \pi f(0). \quad (2.2)$$

The condition for validity of the approximate equality in Eq. (2.2) is that $|t| \gg (\Delta\omega)^{-1}$, where $\Delta\omega$ is the width of $f(\omega)$ which is mainly a product of density over states times the square of the interaction Hamiltonian. The definition of τ_{int} is $\tau_{int} \equiv (\Delta\omega)^{-1}$. In single-body problems such as an atom in an external field, it is rather easy to find τ_{int} and τ_{rel} and to see if the inequality $\tau_{int} \ll \tau_{rel}$ is satisfied. In a problem with many degrees of freedom, the difficulty usually is that the inequality is satisfied for most degrees of freedom, but violated by a special set of degrees of freedom. In a plasma, such a special set are those particles whose velocity is the same as the wave velocity. In a gas laser, the special set are those atoms whose Doppler-shifted frequency is equal to the cavity frequency; i.e., $\omega_0 - \mathbf{k} \cdot \mathbf{V}_\alpha = \Omega$.

A frequently occurring physical situation in which the master equation is valid is a system weakly interacting with a reservoir. The resultant equation is

$$(\partial\rho/\partial t) - i[\rho, H_0{}^s] = \lambda^2 \mathrm{tr}_{res} \int_0^\infty [V, [V(\tau), \rho\mathcal{P}]] \, d\tau \equiv \mathcal{K}\rho,$$

$$(2.3)$$

where

$$V(\tau) = \exp[i(H_0{}^s + H_0{}^r)\tau] V \exp[-i(H_0{}^s + H_0{}^r)\tau].$$

The terms $H_0{}^s$ and $H_0{}^r$ are the system and reservoir Hamiltonians, respectively, V is the system-reservoir interaction Hamiltonian, and $\mathcal{P}$ is the reservoir density matrix. The two new features that have appeared in the reservoir master equation are a trace over the reservoir variables and the assumption that in the infinite past the system and reservoir were expressible as a product $\rho\mathcal{P}$. The reservoir master equation is usually easier to justify than a master equation for internal interactions because the reservoir often provides a physical mechanism for a small τ_{int}.

The Hamiltonian for the combined system and reservoirs of a schematic model for a laser is

$$H = H_0{}^s + H_0{}^r + \gamma H_1 + \epsilon_m V_m + \epsilon_r V_r, \quad (2.4)$$

where $H_0{}^s$ and $H_0{}^r$ are the unperturbed system and reservoir Hamiltonians, and γ, ϵ_m, and ϵ_r are dimensionless constants measuring the strength of the $\mathbf{j} \cdot \mathbf{A}$ interaction of the laser atoms, the strength of the internal atomic variable-matter reservoir interaction, and the strength of the radiation–radiation reservoir interaction, respectively. If we attempt to derive a master equation in the simultaneous limits $\gamma \to 0$, $\epsilon_m \to 0$, and $\epsilon_r \to 0$, we find that the reservoir terms $\epsilon_m{}^2$ and $\epsilon_r{}^2$ are well behaved, but the particles whose Doppler frequencies equal the cavity frequency, $\omega_0 - \mathbf{k} \cdot \mathbf{V}_\alpha = \Omega$, cause the γ^2 term to diverge. This divergence occurs because τ_{int} is infinite

for a permanent two-level system exactly on resonance with the radiation field, and thus the conditions for the first Born approximation are not satisfied. One physical resolution of the problem is clear. In a gas laser, the atom's lifetime in the laser states is finite because the atom decays to the atomic ground state with lifetime ν_2^{-1}. If the effect of the radiation-matter interaction is small during the lifetime of the atom, then we can still have a master equation. As we show in Sec. III, the small dimensionless parameter $N\gamma^2\omega_0^2/\omega_D\nu_2$ in the gas laser is precisely the ratio of the two-level system lifetime to the effective radiation-matter relaxation time between the two laser states.

The formal difficulty is that ν_2^{-1}, which is proportional to ϵ_m^2, plays two roles. The first is as the relaxation time of the atom to its ground state. The second role is as τ_{int} for laser absorptions and emissions between the two levels of our two-level system. Consequently, if we naively let $\epsilon_m\to0$, then τ_{int} for the two-level system becomes larger than the laser interaction relaxation time, and we cannot describe H_1 by the Born approximation. The solution is to let $\gamma^2\to0$ holding ϵ_m^2 fixed, then let $\epsilon_m^2\to0$. Actually, we never have to let $\epsilon_m^2\to0$ unless the physics indicates this limit is reasonable. The crucial requirement is that $\gamma^2\to0$ relative to ϵ_m^2. The requirement for the validity of a reservoir Born-approximation master equation for V_m is the same as the requirement for the validity of the Wigner–Weisskopf approximation for the decay of an atom to its ground state in a nonresonant cavity.

As we see in Sec. III, we need terms such as $[H_1H_1(\tau)]_{\text{av}}\equiv f(\tau)$, where the average is over the matter reservoir. However, because we do not let ϵ_m go to zero until after γ^2 goes to zero, the τ dependence of $f(\tau)$ is a function of ϵ_m, i.e.,

$$H_1(\tau) = \exp[i(H_0^s+H_0^r+\epsilon_mV_m)\tau]H_1$$
$$\times\exp[-i(H_0^s+H_0^r+\epsilon_mV_m)\tau].$$

If we had let ϵ_m go to zero at the same time as γ went to zero, the ϵ_mV_m would not have appeared in the τ dependence of $H_1(\tau)$, and $f(\tau)$ would not have gone to zero as τ went to infinity. The inclusion of the ϵ_mV_m term in the τ dependence of $H_1(\tau)$ corresponds to the decay of the excited atom to the atomic ground state.

The particular terms we need for the explicit evaluation of the master equation are averages such as

$[\sigma^\dagger(\tau)\sigma]_{\text{av}}$. Because of the term ϵ_mV_m in the τ dependence, the correlation $[\sigma^\dagger(\tau)\sigma]_{\text{av}}$ decays with τ. We can define the two-level lifetime ν_2^{-1} as the width of $[\sigma^\dagger(\tau)\sigma]_{\text{av}}$. Lax[2] has shown that if there is a master equation for the matter reservoir, then $[\sigma^\dagger(\tau)\sigma]_{\text{av}}$ equals $\sigma^\dagger\sigma\exp(-\nu_2\tau)$. This result corresponds to letting ϵ_m^2 go to zero after letting γ^2 go to zero.

If the system Hamiltonian H_0^s contains N degrees of freedom, then ρ will depend on N variables, and the interaction potential will consist of a sum of N terms; otherwise, there is no change in the formal development analyzed in this section.

III. MASTER EQUATION FOR A LASER

We use the same Hamiltonian as we used in I for N two-level systems interacting with a single mode of the electromagnetic field. This Hamiltonian is

$$H(N) = h(N)+H_f+H_{\text{c.m.}}+H_i,$$

where

$$h(N) = \tfrac{1}{2}\hbar\omega_0\sum_\alpha^N\sigma_\alpha, \qquad H_f = \hbar\Omega(a^\dagger a+\tfrac{1}{2})$$

$$H_i = \hbar\omega_0\gamma\sum_\alpha\Gamma(X_\alpha)(a^\dagger\sigma_\alpha+a\sigma_\alpha^\dagger)\equiv\sum_\alpha H_1(\alpha),$$

$$H_{\text{c.m.}} = \sum_\alpha(P_\alpha^2/2m)+\tfrac{1}{2}\sum_{\alpha,\beta}V(X_\alpha-X_\beta)$$

$$+\sum_\alpha^N\sum_0^N U(X_\alpha-\eta_i),$$

$$\gamma = (\hbar\omega_0)^{-1}(\hbar\Omega)\tfrac{1}{2}e\langle a\mid\boldsymbol{\varepsilon}\cdot\mathbf{r}\mid b\rangle(4\pi/V)^{1/2}$$

$$\Gamma(X_\alpha) = E(X_\alpha)V^{1/2}. \quad (3.1)$$

The normalized eigenfunction of the cavity corresponding to the frequency Ω evaluated at the position of the αth particle is $E(X_\alpha)$. A full discussion of the terms in the Hamiltonian is given in I and in Ref. 8.

The equation of motion for the density matrix of the combined system and reservoir density $\mathfrak{F}$ is

$$i\hbar(\partial\mathfrak{F}/\partial t)+[\mathfrak{F}, H(N)+H_{\text{res}}+\epsilon V_{\text{res}}]=0, \quad (3.2)$$

where H_{res} represents the Hamiltonian of the reservoirs and V_{res} is the system-reservoir interaction.

The master equation in the limit that γ and ϵ_r (the system-radiation-reservoir interaction parameter) simultaneously go to zero is

$$\partial F_N/\partial t = -(i/\hbar)[H_f+\sum_\alpha h(\alpha)+\sum_\alpha(P_\alpha^2/2m), F_N]+\mathfrak{K}_rF_N+\mathfrak{L}_cF_N$$

$$-\gamma^2\omega_0^2\sum_\alpha\sum_\beta\text{tr}_{\text{res}}\int_0^\infty[H_1(\beta),[H_1(\alpha,\tau), F_N\mathcal{O}]]\,d\tau+\int_0^t\mathfrak{L}_m(t-t')F_N(t')\,dt', \quad (3.3)$$

where

$$H_1(\alpha,\tau)\equiv\exp\{i[H_{\text{res}}+H_f+h(\alpha)+(P_\alpha^2/2m)+\epsilon_mV_m]\tau\}H_1(\alpha)\exp\{-i[H_{\text{res}}+H_f+h(\alpha)+(P_\alpha^2/2m)+\epsilon_mV_m]\tau\},$$

and $F_N(\sigma_1X_1, \sigma_2X_2, \cdots, \sigma_N, X_N, a, t)$ is the density matrix of the N two-level systems with internal degrees of freedom σ_j and c.m. degrees of freedom X_j and of the radiation oscillator a. We assumed that the reser-

[8] C. R. Willis, Phys. Rev. **147**, 406 (1966).

voirs and the system were uncoupled in the infinite past. Consequently, the radiation reservoir master equation $\mathcal{K}_r$ is a linear time-independent operator on F_N. We did not let the system-matter-reservoir interaction parameter go to zero; thus $H_1(\alpha, \tau)$ depends on the system-matter reservoir interaction potential V_m. Also, we have written a general time-dependent kernel for the system-matter-reservoir kernel that depends only on the fact that the system and reservoir were uncoupled in the infinite past. We now let ϵ_m go to zero, and the last term in Eq. (3.3) becomes $\mathcal{K}_m F_N$, where $\mathcal{K}_m$ is a linear time-independent kernel.

The operator $\mathcal{L}_c F_N$ represents the effect of the collisions of the Ne atoms with the He atoms and other Ne atoms and is a result of letting the density go to zero. For gas lasers, $\mathcal{L}_c$ is basically a linear Boltzmann operator. Collisions also influence $H_1(\alpha, \tau)$ through $\Gamma[X_\alpha(\tau)]$, which is the mode function evaluated at the c.m. of the αth particle at the time τ. In the remainder of this paper, we consider collisionless atoms, but the inclusion of collisions does not fundamentally alter the qualitative results of this paper.

As a result of the above discussion, Eq. (3.3) becomes

$$\partial F_N/\partial t = -(i/\hbar)[H_f + \sum_\alpha h(\alpha) + \sum_\alpha (P_\alpha{}^2/2m), F_N]$$

$$+ \mathcal{K}_r F_N + \mathcal{K}_m F_N - \gamma^2 \omega_0{}^2 \mathrm{tr}_{\mathrm{res}}$$

$$\times \sum_\alpha \sum_\beta \int_0^\infty [H_1(\beta), [H_1(\alpha, \tau), F_N \mathcal{P}]] d\tau. \quad (3.4)$$

In Eq. (3.4), the terms with $\alpha \neq \beta$ in the interaction term represent two particle-correlation effects. Dynamically, the first Born approximation correlates a single particle with the radiation field and we must go to a second-Born-approximation master equation, i.e., to γ^4, in order to obtain dynamically induced two-body correlations. However, there are two ways that two-body correlations could appear kinematically. One is by superradiant effects due to the degeneracy of two or more particles within a wavelength of the radiation of each other. The relevant dimensionless parameter for superradiant effects is $(N/V)\lambda^3$, which is about 10^{-5} for gas lasers. The second way is that the pump might create two-body correlations. However, in a gas laser, where the pump produces inversion by collisions, the pump does not create two-body correlations among the pumped atoms. Consequently, the terms with $\alpha \neq \beta$ are unimportant in a gas laser. However, Eq. (3.4) describes kinematically produced two-body correlations in those solids for which a first Born approximation is valid. A good example where two-body effects are important in Eq. (3.4) is the Q-spoiled laser solved numerically by Fleck.[4]

For a gas laser, the double sum in Eq. (3.4) reduces to a single sum. When we sum Eq. (3.4) over all particles but one, we obtain

$$\partial F_1(1)/\partial t = -(i/\hbar)[H_f + h(1) + (P_1{}^2/2m), F_1(1)]$$

$$+ \mathcal{K}_r F_1(1) + \mathcal{K}_m F_1(1) - \gamma^2 \omega_0{}^2 \mathrm{tr}_{\mathrm{res}}$$

$$\times \int_0^\infty [H_1(1), [H_1(1, \tau), F_1(1) \mathcal{P}]] d\tau$$

$$- \gamma^2 \omega_0{}^2 (N-1) \mathrm{tr}_{\mathrm{res}} \mathrm{tr}_2$$

$$\times \int_0^\infty [H_1(2), [H_1(2, \tau), F_2(1, 2) \mathcal{P}]] d\tau, \quad (3.5)$$

where tr_2 means the trace over all variables of the second particle including c.m. variables. The subscript i on the F_i means that the distribution function depends on i particles in addition to the radiation oscillator.

The summation of Eq. (3.4) over all particle variables yields

$$\partial R/\partial t = -(i/\hbar)[H_f, R] + \mathcal{K}_r R - \gamma^2 \omega_0{}^2 N$$

$$\times \mathrm{tr}_1 \int_0^\infty [H_1(1), [H_1(1, \tau), F_1 \mathcal{P}]] d\tau, \quad (3.6)$$

where we use $R \equiv \mathrm{tr}_1 F_1$ and $\mathrm{tr}_{\mathrm{mat}} \mathcal{K}_m F_N = 0$.

Equations (3.5) and (3.6) do not constitute a closed theory because Eq. (3.5) depends on F_2, which depends on F_3, and so on until we reach F_N again. The term proportional to $N-1$ is very large compared with the fourth term on the right side of Eq. (3.5). However, as we show in Appendix A, the $\gamma^2(N-1)$ term is almost exactly cancelled by the $\mathcal{K}_r F_1$ term. Consequently, the terms remaining after the cancellation of the $\mathcal{K}_r$ and $\gamma^2(N-1)$ terms depend only on F_1, and Eqs. (3.5) and (3.6) constitute a complete closed theory.

The trace of Eq. (3.5) over the radiation variable after cancelling the $\mathcal{K}_r$ term and the $\gamma^2(N-1)$ term gives the following equation for the one-particle density matrix:

$$\partial \rho/\partial t = -(i/\hbar)[h(1) + (P_1{}^2/2m), \rho] + \mathcal{K}_m \rho$$

$$- \gamma^2 \omega_0{}^2 \mathrm{tr}_{\mathrm{res}} \mathrm{tr}_a \int_0^\infty [H_1(1), [H_1(1, \tau), F_1 \mathcal{P}]] d\tau. \quad (3.7)$$

We do not need the equation for ρ in this paper; however, if we replace F_1 by $R\rho$ on the right side of Eqs. (3.6) and (3.7), we obtain the initial Eqs. (I2.3a) and (I2.3b).

We have treated the c.m. variables as quantum-mechanical; however, since they are essentially classical, we replace their commutators with Poisson brackets. We showed in I that the neglect of the Poisson brackets in the interaction term is equivalent to neglecting the recoil of the atom on emission and absorption which constitutes an error of less than one part in 10^{-3}.

The trace of Eq. (3.5) over the reservoir variables of Eq. (3.5), omitting the $\mathcal{K}_r$ and $\gamma^2(N-1)$ terms, is

$$\partial F_1/\partial t = -(i/\hbar)[h(1)+\hbar\Omega a^\dagger a, F_1] - V(\partial F_1/\partial X) + \mathcal{K}_m F_1$$

$$-\gamma^2\omega_0^2 \int_0^\infty [a^\dagger\sigma+a\sigma^\dagger, [a^\dagger\sigma \exp(i\Delta\tau)+a\sigma^\dagger \exp(-i\Delta\tau), F_1]]\Gamma\Gamma(\tau) \exp(-\nu_2\tau)d\tau, \quad (3.8)$$

where $\Delta\equiv\omega_0-\Omega$ and $\mathrm{tr_{res}}\sigma^\dagger(\tau)\sigma\mathcal{P}=\sigma^\dagger\sigma \exp(-i\Delta\tau) \exp(-\nu_2\tau)$ for a Born-approximation reservoir. The separation of the right side of Eq. (3.8) gives

$$\partial F_1/\partial t+(i/\hbar)[h(1)+\hbar\Omega a^\dagger a+\Delta H_1, F_1] - V(\partial F_1/\partial X) = \mathcal{K}_m F_1 + K(X, V)[a^\dagger\sigma+a\sigma^\dagger, [a^\dagger\sigma+a\sigma^\dagger, F_1]], \quad (3.9)$$

where

$$K(X, V) \equiv \int_0^\infty \exp(-\nu_2\tau) \cos\Delta\tau\Gamma\Gamma(\tau)d\tau$$

and

$$\Delta H_1 \equiv \gamma^2\omega_0^2\hbar \int_0^\infty \exp(-\nu_2\tau) \sin\Delta\tau\Gamma\Gamma(\tau)d\tau(aa^\dagger\sigma^\dagger\sigma - a^\dagger a\sigma\sigma^\dagger).$$

In I, we treated the equations for the density matrix in their operator form. We find it somewhat more illuminating to solve the equations in this paper in the number representation. The simplification arises because in the first Born approximation the diagonal matrix elements interact only with diagonal matrix elements. The off-diagonal matrix elements divide into mutually exclusive sets such that matrix elements in a set interact only with members of the same set and do not interact with diagonal matrix elements.

The diagonal matrix elements of Eqs. (3.6) and (3.9) are

$$(\partial F_+/\partial\tau)(n, y, \xi, \tau)+\xi(\partial F_+/\partial y)(n, y, \xi, \tau) = -\bar\nu_1[F_+(n, y, \xi, \tau)-R(n)\rho^0(+)]$$

$$-2(n+1)\bar K(y, \xi)[F_+(n, y, \xi, \tau)-F_-(n+1, y, \xi, \tau)], \quad (3.10a)$$

$$(\partial F_-/\partial\tau)(n, y, \xi, \tau)+\xi(\partial F_-/\partial y)(n, y, \xi, \tau) = -\bar\nu_1[F_-(n, y, \xi, \tau)-R(n)\rho^0(-)]$$

$$-2n\bar K(y, \xi)[F_-(n, y, \xi, \tau)-F_+(n-1, y, \xi, \tau)], \quad (3.10b)$$

and

$$\partial R_n/\partial\tau = -\bar\nu_r[nR_n-(n+1)R_{n+1}]+2N \int dy\, d\xi\bar K(y, \xi)$$

$$\times\{(n+1)[F_-(n+1, y, \xi, \tau)-F_+(n, y, \xi, \tau)]+n[F_+(n-1, y, \xi, \tau)-F_-(n, y, \xi, \tau)]\}, \quad (3.10c)$$

where we introduced the dimensionless variables $y\equiv kx$, $\xi\equiv V/V_T$, and $\tau\equiv\omega_D t$, and a frequency with a bar over it has been made dimensionless with the Doppler frequency ω_D. We represent the diagonal matrix elements of F_1, $\langle\pm, n | F_1(y, \xi, \tau)|\pm, n\rangle$ by the symbols $F_\pm(n, y, \xi, \tau)$, and the diagonal matrix elements of R, R_{nn} by a single subscript R_n. The solution for R does not depend very sensitively on the precise form of $\mathcal{K}_r$ because its magnitude $\bar\nu_r$ is the smallest reservoir parameter in the problem. For $\mathcal{K}_r$, we take a zero-temperature harmonic-oscillator reservoir. A finite-temperature reservoir can be included with ease. We take a simple form for $\mathcal{K}_m$ that preserves the commutation relations. The pump creates a population inversion $[\rho^0(+)-\rho^0(-)]$ without creating particle-field correlations. For convenience, we take $\rho^0(-)$ equal to zero and for moving atoms $\rho^0(+)=(2\pi)^{-1/2} \exp(-\frac{1}{2}\xi^2)$.

We conclude this section with a discussion of the dimensionless parameters in the theory. In I, we showed that in the absence of collisions and with rectangular

geometry, $\bar K$ is

$$\bar K\equiv\gamma^2\omega_0^2 \int_0^\infty \exp(-\bar\nu_2\tau) \cos\bar\Delta\tau\Gamma\Gamma(\tau)d\tau$$

$$=\sin^2 y(\gamma^2\omega_0^2/\omega_D^2)\tfrac{1}{2}\bar\nu_2$$

$$\times\{[(\xi+\bar\Delta)^2+\bar\nu_2^2]^{-1}+[(\xi-\bar\Delta)^2+\bar\nu_2^2]^{-1}\}$$

$$\approx\tfrac{1}{2}(\gamma^2\omega_0^2/\omega_D^2)\bar\nu_2[(\xi^2+\bar\nu_2^2)]^{-1}. \quad (3.11)$$

When Eq. (3.11) is substituted in Eq. (3.10c) and after an integration over a Maxwellian population inversion, we obtain the parameter $\gamma^2\omega_0^2 N/\omega_D\nu_2$. Consequently, our relaxation time $\gamma^2\omega_0^2 N/\omega_D$ consists of two parts. The first part $\gamma^2\omega_0 N$ is the probability that a two-level system spontaneously emits a photon and goes to the ground state of the two-level system, *not* to the ground state of the atom. The spontaneous emission time is divided by ω_D/ω_0, which is the motional broadening of the spontaneous two-level system emission line shape due to Doppler motion. The interaction

time τ_{int} is the lifetime for the decay of the atom to the ground state of the atom, i.e., $\tau_{int}=\nu_2^{-1}$. Consequently, the requirement for the validity of the Born approximation $(\tau_{int}/\tau_{rel}\ll 1)$ is satisfied if $\gamma^2\omega_0^2 N/\omega_D\omega_2$ is much less than unity. For a He–Ne laser, the ratio is approximately 2×10^{-2}. For a He–Ne laser, ν_2^{-1} is able to fulfill its dual role as τ_{int} for the laser transition and τ_{rel} for the decay of the atom to its ground state. The main reason that ν_2^{-1} can be a "short time" for the laser transition and a "long time" for the decay of the atom to its ground state is that the density of the final states and the matrix elements of the decay of the atom to the ground state are much larger than the corresponding transition from the laser excited state to the laser two-level system ground state.

IV. STATIONARY STATE

We now find the stationary-state solutions of Eq. (3.10) for a number of physically different cases. We consider the following four physical cases of lasers: (a) moving atoms with traveling waves, (b) stationary atoms with traveling waves, (c) stationary atoms with standing waves, and (d) moving atoms with standing waves. The standing-wave cases are more difficult to solve than the traveling-wave ones because the former are spatially-dependent. The first three cases are algebraic equations in y because either $\xi=0$ [cases (b) and (c)], or the interaction term is spatially independent [cases (a) and (b)]. We discuss case (d) in Sec. V because in this case we have a differential equation in the spatial variable y.

In the absence of collisions, the expressions for $\bar{K}$ in the four cases are

(a) $\quad \bar{K}=\gamma^2\bar{\omega}_0^2\frac{1}{4}\bar{\nu}_2\{[(\xi+\bar{\Delta})^2+\bar{\nu}_2^2]^{-1}+[(\xi-\bar{\Delta})^2+\bar{\nu}_2^2]^{-1}\},$ (4.1a)

(b) $\quad \bar{K}=\gamma^2\omega_0^2/[2(\Delta^2+\nu_2^2)],$ (4.1b)

(c) $\quad \bar{K}=\sin^2 y\gamma^2\omega_0^2/(\Delta^2+\nu_2^2),$ (4.1c)

(d) $\quad \bar{K}=\gamma^2\bar{\omega}_0^2\bar{\nu}_2\frac{1}{2}(\sin^2 y)$
$\{[(\xi+\bar{\Delta})^2+\bar{\nu}_2^2]^{-1}+[(\xi-\bar{\Delta})^2+\bar{\nu}_2^2]^{-1}\}.$ (4.1d)

For stationary atoms ξ equals zero, ω_D is replaced by ν_2, and $\rho^0(+)+\rho^0(-)=(2\pi)^{-1/2}\exp(-\frac{1}{2}\xi^2)$ is replaced by $\rho^0(+)+\rho^0(-)=1$.

The simplest case and the one most authors use is case (b). We use the method developed for case (b) by Fleck[4] to solve Eqs. (3.10) in the three cases (a), (b), and (c). When we set the time and spatial derivatives of Eqs. (3.10a) and (3.10b) equal zero, we obtain

$[\bar{K}(n+1)+\bar{\nu}_1]\eta_+(n)=\bar{K}(n+1)\eta_-(n+1)+\bar{\nu}_1\rho^0(+)2^{-1},$ (4.2a)

$[\bar{K}(n+1)+\bar{\nu}_1]\eta_-(n+1)=\bar{K}(n+1)\eta_+(n)+\bar{\nu}_1\rho^0(-)2^{-1},$ (4.2b)

where $\eta_\pm(n)\equiv F_\pm(n)/R_n$ and where we have set $R_{(n+1)}/R_n$ equal to one in Eq. (4.2b).

The solution for $\eta_+(n)$ is

$\eta_+(n)=\dfrac{[1+\alpha(n+1)]\rho^0(+)}{1+2\alpha(n+1)}+\dfrac{\alpha(n+1)\rho^0(-)}{1+2\alpha(n+1)},$ (4.3)

where $\alpha\equiv 2K/\nu_1$. For convenience, we set $\rho^0(-)$ equal zero, i.e., the pump excites atoms only in the upper state. When we substitute Eq. (4.3) in Eq. (3.10c) and perform the indicated spatial and velocity integrations, we obtain the following recursion relations for cases (a), (b), and (c).

Case (a):

$$R_{n+1}=\left(2N\int d\xi \bar{K}(\xi)\eta_+(\xi,n)\Big/\bar{\nu}_r+2N\int d\xi \bar{K}(\xi)\eta_-(\xi,n+1)\right)R_n.$$ (4.4a)

The formal solution is

$$\frac{R_n}{R_0}=\prod_{n'=0}^{n'=n}\left[\frac{\gamma^2\bar{\omega}_0^2 N\bar{\nu}_2}{(2\pi)^{1/2}}\int\frac{d\xi e^{-\frac{1}{2}\xi^2}}{(\xi^2+\bar{\nu}_2^2)}\left(\frac{\xi^2+\bar{\nu}_2^2[1+(\gamma^2\omega_0^2/\nu_1\nu_2)(n'+1)]}{\xi^2+\nu_2'^2}\right)\Big/\bar{\nu}_r+\frac{\gamma^2\bar{\omega}_0^2 N\bar{\nu}_2}{(2\pi)^{1/2}}\int\frac{d\xi e^{-\frac{1}{2}\xi^2}}{(\xi^2+\bar{\nu}_2^2)}\left(\frac{\bar{\nu}_2^2(\gamma^2\omega_0^2/\nu_1\nu_2)(n'+1)}{\xi^2+\nu_2'^2}\right)\right],$$

where (4.4b)

$$\bar{\nu}_2'^2\equiv\bar{\nu}_2^2[1+(2\gamma^2\omega_0^2/\nu_1\nu_2)(n'+1)].$$

We have set $\bar{\Delta}=0$ for convenience in Eqs. (4.4). The $\bar{\Delta}\neq 0$ case can be obtained simply by replacing $\xi^2+\bar{\nu}_2^2$ everywhere by $\frac{1}{2}\{[(\xi+\bar{\Delta})^2+\bar{\nu}_2^2]+[(\xi-\bar{\Delta})^2+\bar{\nu}_2^2]\}$.

Case (b):

$$R_{n+1}=\frac{N(\gamma^2\omega_0^2/\nu_2^2)\eta_+(n)}{\bar{\nu}_r+N(\gamma^2\omega_0^2/\nu_2^2)\eta_-(n+1)}R_n.$$ (4.5a)

The formal solution is

$$R_n=\prod_{n'=n}^{n'=n}\frac{N(\gamma^2\omega_0^2/\nu_2^2)\eta_+(n')}{\bar{\nu}_r+N(\gamma^2\omega_0^2/\nu_2^2)\eta_-(n'+1)}R_0.$$ (4.5b)

Case (c):

$$R_{n+1}=\left(\frac{\gamma^2\omega_0^2}{\nu_2^2}N\frac{2}{\pi}\int_0^\pi dy\sin^2 y\eta_+(y,n)\Big/\bar{\nu}_r+\frac{\gamma^2\omega_0^2}{\nu_2^2}N\frac{2}{\pi}\int_0^\pi dy\sin^2 y\eta_-(y,n+1)\right)R_n,$$ (4.6a)

where

$$\eta_+(y, n) \equiv \frac{1+2\,\sin^2 y(\gamma^2\omega_0^2/\nu_1\nu_2)}{1+4\,\sin^2 y(\gamma^2\omega_0^2/\nu_1\nu_2)}.$$

The formal solution is

$$R_n = \prod_{n'=0}^{n'=n} \left(\frac{\gamma^2\omega_0^2}{\nu_2^2} N \frac{2}{\pi} \int_0^\pi dy\,\sin^2 y\,\eta_+(y, n') \,\Big/\, \bar{\nu}_r + \frac{\gamma^2\omega_0^2}{\nu_2^2} N \frac{2}{\pi} \int_0^\pi dy\,\sin^2 y\,\eta_-(y, n'+1) \right) R_0. \qquad (4.6b)$$

If the numerator is less than the denominator for $n=0$ in the three cases, R_n is a monotonic decreasing function of n. If the denominator is less than or equal to the numerator at $n=0$, R_n first increases, then decreases with n. The dividing line between the two types of behavior occurs when the numerator equals the denominator for $n=0$. The value of N for which the denominator equals the numerator is the threshold number of systems required for a peaked (nonthermal) distribution. The value of N turns out to be the same as N_T, the threshold number of systems in I. The results for the three cases are as follows.

Case (a):

$$\bar{\nu}_r = \frac{\gamma^2\omega_0^2}{\omega_D^2} N\bar{\nu}_2 \int \frac{d\xi}{(\xi^2+\bar{\nu}_2^2)} \left[\eta_+(\xi, 0) - \eta_-(\xi, 0) \right]$$

$$= \frac{\gamma^2\omega_0^2}{\omega_D^2} N\bar{\nu}_2 \int \frac{d\xi\,\exp(-\tfrac{1}{2}\xi^2)}{(2\pi)^{1/2}(\xi^2+\bar{\nu}_2^2)}$$

$$= (\gamma^2\omega_0^2/\omega_D^2) I(\bar{\nu}_2) N, \qquad (4.7)$$

where

$$I(\bar{\nu}_2) \equiv \int_0^\infty \exp(-\bar{\nu}_2 x)\,\exp(-\tfrac{1}{2}x^2)\,dx$$

and the threshold N_T is $(\nu_r \omega_D/\gamma^2\omega_0^2) I^{-1}(\bar{\nu}_2)$.

Case (b):

$$\bar{\nu}_r = (\gamma^2\omega_0^2 N/\nu_2^2)[\eta_+(0) - \eta_-(0)]$$

$$= (\gamma^2\omega_0^2/\nu_2^2) N,$$

$$N_T = \nu_r \nu_2/\gamma^2\omega_0^2. \qquad (4.8)$$

Case (c):

$$\bar{\nu}_r = \frac{\gamma^2\omega_0^2}{\nu_2^2} N \frac{2}{\pi} \int_0^\pi \sin^2 y [\eta_+(y, 0) - \eta_-(y, 0)]dy$$

$$= (\gamma^2\omega_0^2/\nu_2^2) N,$$

$$N_T = \nu_r \nu_2/\gamma^2\omega_0^2. \qquad (4.9)$$

The value of n at which $R(n)$ has its maximum value, n_0, is the solution of the equation $R(n+1) = R(n)$. In other words, n_0 is the value of n where the numerator equals the denominator in Eqs. (4.4a), (4.5a), and (4.6a).

The results for the three cases are as follows.

Case (a):

$$n_0 = \frac{\nu_1\nu_2}{2\gamma^2\omega_0^2} \left[\left(\frac{NI(\bar{\nu}_2')}{N_T I(\bar{\nu}_2)} \right)^2 - 1 \right] \xrightarrow{\bar{\nu}_2 \to 0} \frac{\nu_1\nu_2}{2\gamma^2\omega_0^2} \left[\left(\frac{N}{N_T} \right)^2 - 1 \right], \qquad (4.10a)$$

where

$$\bar{\nu}_2' = \bar{\nu}_2 [1 + (2\gamma^2\omega_0^2/\nu_1\nu_2)(n_0+1)]^{1/2}.$$

Case (b):

$$n_0 = (\nu_1\nu_2/2\gamma^2\omega_0^2)[(N/N_T) - 1]. \qquad (4.10b)$$

Case (c):

$$n_0 = \frac{\nu_1\nu_2}{8\gamma^2\omega_0^2} \left[\left(4\frac{N}{N_T} - 1 \right) - \left(1 + 8\frac{N}{N_T} \right)^{1/2} \right]. \qquad (4.10c)$$

We use Fleck's[4] technique of finding the best Gaussian fit centered around n_0 by first rewriting Eq. (4.4b) as

$$R_n = R_{n_0} \prod_{n'=n_0}^{n} \left[2N \int d\xi\,\bar{K}(\xi)\eta_+(\xi, n') \,\Big/\, \bar{\nu}_r + 2N \int d\xi\,\bar{K}(\xi)\eta_-(\xi, n'+1) \right]^{\pm 1}, \qquad (4.11)$$

where the positive exponent holds for $n > n_0$ and the negative exponent holds for $n < n_0$. Next we expand the logarithm of R_n in a Taylor series about n_0. The result is

$$R_n = (\tfrac{1}{2}\pi)^{-1/2}\sigma^{-1} \exp[-(n-n_0)^2/2\sigma^2],$$

where $\sigma^2 \equiv \{(n-n_0)^2\}_{\text{av}}$. The expressions for σ^2 in the three cases are as follows. We are also giving the results in the limit as $N \to N_T$ because this is the limit in which the photon-counting experiments are done.

Case (a):

$$\sigma^2 = \frac{\nu_1\nu_2}{2\gamma^2\omega_0^2} \frac{I(\bar{\nu}_2')}{|I'(\bar{\nu}_2')|} \left(\frac{N}{N_T} + 1 \right) \xrightarrow{N \to N_T} \frac{\nu_1\nu_2}{2\gamma^2\omega_0^2} \left(\frac{2}{\pi} \right)^{1/2} \left(\frac{N}{N_T} + 1 \right),$$

$$\frac{n_0}{\sigma^2} = \frac{\{[NI(\bar{\nu}_2')/N_T I(\bar{\nu}_2)]^2 - 1\}}{[I(\bar{\nu}_2')/|I'(\bar{\nu}_2')|](N/N_T + 1)}$$

$$\xrightarrow{N \to N_T} \left(\frac{2}{\pi} \right)^{1/2} \left(\frac{N}{N_T} - 1 \right) \equiv \left(\frac{2}{\pi} \right)^{1/2} \epsilon, \qquad (4.12)$$

where $\epsilon \equiv (N - N_T)/N_T$.

Case (b):

$$\sigma^2 = \frac{\nu_1\nu_2}{4\gamma^2\omega_0^2}\left(\frac{N}{N_T}\right)\left(\frac{N}{N_T}+1\right),$$

$$\frac{n_0}{\sigma^2} = \frac{2[1-(N_T/N)]}{(N/N_T)+1} \xrightarrow{N\to N_T} \epsilon. \qquad (4.13)$$

Case (c):

$$\sigma^2 = \frac{n_0[1+(N_T/N)]}{8\{4(N_T/N)-\frac{1}{2}[1-(2N_T n_0\gamma^2\omega_0^2/N\nu_1\nu_2)]^3\}},$$

$$n_0/\sigma^2 \xrightarrow{N\to N_T} \frac{4}{3}\epsilon. \qquad (4.14)$$

In all three cases, σ^2 is greater than n_0 and approaches n_0 as N increases. However, as N becomes of the order of $1.5N_T$, the distribution becomes more sharply peaked than a Gaussian distribution. Case (a) has the broadest distribution, case (c) has the most peaked distribution, and case (b) lies roughly halfway in between, at least near threshold. In Sec. V, we show case (d) of moving atoms with standing waves has the same value of n_0/σ^2 near threshold as case (b).

In case (b), when we take the limit as N approaches N_T, the expression $[1+(\gamma^2\omega_0^2/\nu_1\nu_2)(n+1)]^{-1}$ is replaced by $1-(\gamma^2\omega_0^2/\nu_1\nu_2)(n+1)$ in Eq. (4.5b). The resultant equation for R_n is the same as Risken's[9] and Lax's[10] master equation integrated over the phase variable for the rotating-wave Van der Pol equation. Risken's[9] stationary solution is

$$W(r) = (N/2\pi)\,\exp[-\tfrac{1}{4}r^4+\tfrac{1}{2}(ar^2)].$$

In our notation, $\frac{1}{2}(ar^2)=1-N_T/N=n_0/\sigma^2$, and his parameter β/q equals $\gamma^2\omega_0^2/\nu_1\nu_2$. Our result for n_0/σ^2 is $2(1-N_T/N)(N/N_T+1)^{-1}$ which reduces to $1-N_T/N$ near threshold. However, the threshold limit is the limit in which we are permitted to replace $\eta_+(n)$ and $\eta_-(n)$ by the first two terms of their Taylor series. Thus near threshold the best Gaussian fit is exact. For a typical He–Ne laser, a value of 10 for Risken's a corresponds to an ϵ between 10^{-3} and 10^{-2}. Conse-

quently, the photon-counting experiments are done in a range where our ϵ is less than one percent. Although cases (a) and (c) do not correspond to Risken's model, they have the same functional form with different values of the ratio n_0/σ^2.

V. STATIONARY STATE: MOVING ATOMS WITH STANDING WAVES

It is more complicated to obtain the stationary state for case (d), moving atoms with standing waves, because we must solve a differential equation for $\eta_\pm$, instead of just an algebraic equation as we did in cases (a), (b), and (c). We illustrate the difficulties by first solving for $[\eta_+(y,\xi)-\eta_-(y,\xi)]$. The time-independent solution of Eqs. (3.10a) and (3.10b) is

$$\xi(\partial/\partial y)[\eta_+(y,\xi,n)-\eta_-(y,\xi,n)]$$
$$= -\bar{\nu}_1[\eta_+(y,\xi,n)-\eta_-(y,\xi,n)-(2\pi)^{-1/2}\exp(-\tfrac{1}{2}\xi^2)]$$
$$-[4\sin^2 y)/(\xi^2+\bar{\nu}_2^2)]\bar{\nu}_2\gamma^2\omega_0^2[\eta_+(y,\xi,n)-\eta_-(y,\xi,n)].$$
$$(5.1)$$

The solution of Eq. (5.1) is

$$\eta_+(y,\xi,n)-\eta_-(y,\xi,n)$$
$$= (2\pi)^{-1/2}\exp(-\tfrac{1}{2}\xi^2)\,\exp\left[-\frac{\bar{\nu}_1 y}{\xi}-\frac{A\bar{\nu}_1}{\xi}\int_{,1}^{y}\sin^2 z\,dz\right]$$
$$\times\left[1+\frac{\bar{\nu}_1}{\xi}\int_0^y\exp\left(\frac{\bar{\nu}_1 x}{\xi}\right)\exp\left(\bar{\nu}_1 xA\int_0^x\frac{\sin^2 z\,dz}{\xi}\right)dx\right], \quad (5.2)$$

where $A\equiv 4\gamma^2\bar{\omega}_0^2\nu_2 n/(\xi^2+\bar{\nu}_2^2)\nu_1$ and where we have assumed in the solution of the homogeneous equation that

$$\eta_+(y=0)-\eta_-(y=0)=(2\pi)^{-1/2}\exp(-\tfrac{1}{2}\xi^2).$$

We can combine Eq. (5.2) with

$$\eta_+(y,\xi,n)+\eta_-(y,\xi,n)=(2\pi)^{-1/2}(-\tfrac{1}{2}\xi^2),$$

solve for η_+ and η_-, and substitute the results in the time-independent form of Eq. (3.10c) for R. However, we are unable to perform the resultant integrals over y and ξ in closed form. Consequently, we expand the solution of Eq. (5.1) to order γ^2 and obtain

$$\eta_+(y,\xi,n)-\eta_-(y,\xi,n)\approx\frac{\exp(-\tfrac{1}{2}\xi^2)}{(2\pi)^{1/2}}\left\{1-\frac{\xi^2 A}{4\xi^2+\bar{\nu}_1^2}\left[\left(\left(\frac{\bar{\nu}_1}{\xi}\right)^2\sin^2 y-\frac{2\bar{\nu}_1}{\xi}\sin y\cos y+2\right)-2\exp\frac{\bar{\nu}_1 y}{\xi}\right]\right\}. \qquad (5.3)$$

In order to obtain the equation for R, we need integrals of Eq. (5.3) multiplied by functions of y and ξ. One integral we need is

$$g=\frac{\bar{\nu}_2}{(2\pi)^{1/2}}\int\frac{\exp(-\tfrac{1}{2}\xi^2)}{(\xi^2+\bar{\nu}_2^2)}2\sin^2 y$$
$$\times[\eta_+(y,\xi,n)-\eta_-(y,\xi,n)]dy\,d\xi. \qquad (5.4)$$

The result of the substitution of Eq. (5.3) into

Eq. (5.5) is

$$g=\frac{\bar{\nu}_2}{(2\pi)^{1/2}}\int\frac{\exp(-\tfrac{1}{2}\xi^2)}{(\xi^2+\bar{\nu}_2^2)}\left\{1-\tfrac{1}{2}A\left[1+\frac{\bar{\nu}_1^2}{2(4\xi^2+\bar{\nu}_1^2)}\right]\right\}$$
$$= I(\bar{\nu}_2)+\gamma^2\bar{\omega}_0^2\left(\frac{\nu_2}{\nu_1}\right)\frac{\partial}{\partial\bar{\nu}_2}\left(\frac{I(\bar{\nu}_2)}{\bar{\nu}_2}\right)$$
$$-\frac{\gamma^2\bar{\omega}_0^2 n\bar{\nu}_1^2}{2(2\pi)^{1/2}}\frac{\nu_2}{\nu_1}\frac{\partial}{\partial\bar{\nu}_2}\left(\int\frac{\exp(-\tfrac{1}{2}\xi^2)d\xi}{(\xi^2+\bar{\nu}_2^2)(4\xi^2+\bar{\nu}_2^2)}\right). \qquad (5.5)$$

[9] H. Risken and H. D. Volmer, Z. Physik **201**, 323 (1967).
[10] R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967).

The integral in the last term is not expressible in closed form. However, we can estimate the ratio of the third term of Eq. (5.5) to the second term quite accurately because the Lorentzian linewidths are much narrower than the Gaussian linewidth. For the parameters of a He-Ne laser the ratio of the third term of Eq. (5.5) to the second term is approximately

13/98. With this result, Eq. (5.5) becomes

$$\mathcal{I} = I(\bar{\nu}_2)\left\{1 + \frac{\gamma^2\bar{\omega}_0^2}{I(\bar{\nu}_2)}\left(\frac{\nu_2}{\nu_1}\right)\frac{111}{98}\frac{\partial}{\partial\bar{\nu}_2}\left[\frac{I(\bar{\nu}_2)}{\bar{\nu}_2}\right]\right\}. \quad (5.6)$$

We use the method of Sec. IV to obtain the steady-state solution for R_n for moving atoms with standing waves:

$$\frac{R_n}{R_0} = \prod_{n'=0}^{n}\left(\frac{2\gamma^2\bar{\omega}_0^2 N\bar{\nu}_2}{(2\pi)^{1/2}}\int\frac{\sin^2 y\eta_+(y,\xi,n')\,dy\,d\xi}{(\xi^2+\bar{\nu}_2^2)}\middle/\bar{\nu}_r + \frac{2\gamma^2\bar{\omega}_0^2 N\bar{\nu}_2}{(2\pi)^{1/2}}\int\frac{\sin^2 y\eta_-(y,\xi,n'+1)\,dy\,d\xi}{(\xi^2+\bar{\nu}_2^2)}\right). \quad (5.7)$$

We obtain the integrals of η_+ and η_- from $\mathcal{I}$ by using the following relationships:

$$\eta_+(y,\xi,n) + \eta_-(y,\xi,n) = (2\pi)^{-1/2}\exp(-\tfrac{1}{2}\xi^2),$$

$$\eta_+(y,\xi,n) = \tfrac{1}{2}\{(\eta_+(y,\xi,n) - \eta_-(y,\xi,n)) + [\exp(-\tfrac{1}{2}\xi^2)/(2\pi)^{1/2}]\}. \quad (5.8)$$

The solution of Eq. (5.7) to order γ^4 is then

$$R_n = \prod_{n'=0}^{n}\frac{1 + (\gamma^2\bar{\omega}_0^2\nu_2 n/2\nu_1)(111/98)(\partial/\partial\bar{\nu}_2)[I(\bar{\nu}_2)/\bar{\nu}_2]}{N_T/N + (\gamma^2\bar{\omega}_0^2\nu_2 n/2\nu_1)(111/98)(\partial/\partial\bar{\nu}_2)[I(\bar{\nu}_2)/\bar{\nu}_2]}R_0, \quad (5.9)$$

where $N_T \equiv (\nu_r\omega_D)[\gamma^2\omega_0^2 I(\bar{\nu}_2)]^{-1}$.

When we use the methods of Sec. IV to find n_0 and the best Gaussian fit, we obtain

$$n_0 = (a\nu_1\nu_2/\gamma^2\omega_0^2)(1 - N_T/N),$$

$$a \equiv (98/111)(1 + 2^{1/2}\pi^{-1/2}\bar{\nu}_2)^{-1} \approx \tfrac{4}{5}, \quad (5.10)$$

$$\sigma^2 = \tfrac{1}{2}(N_T/N + 1)(\nu_1\nu_2 a/\gamma^2\omega_0^2),$$

$$\frac{n_0}{\sigma^2} = \frac{1 - N_T/N}{\tfrac{1}{2}[(N_T/N) + 1]} \approx \epsilon. \quad (5.11)$$

The above result for n_0/σ^2 is important and rather surprising. It states that near threshold, where the photon-counting experiments are done, the ratio n_0/σ^2 for moving atoms with standing waves is the same as for stationary atoms with traveling waves. Consequently, the inclusion of both velocity and spatial effects cancel each other near threshold. The inclusion of the motion of the atoms broadens the photon-distribution function. However, the separate values of n_0 and σ^2 are different for the two cases. In conclusion, we see that the case of moving atoms with standing waves is much more complicated than the other cases because of the operator $\xi\partial/\partial y$.

VI. TIME DEPENDENCE OF THE MASTER EQUATION

The master equation for the diagonal matrix elements of R, Eq. (3.10c), has a complicated time dependence. In some lasers the rate of change of F_1 is fast compared with the time dependence of R, and so the matter variables can be eliminated adiabatically. In the He-Ne laser, a rough measure of the ratio of the time dependence of R to the time dependence of F_1 is given by $\nu_r[(N/N_T) - 1]/\nu_1$. For the photon-counting experiments, this ratio is between 10^{-4} and 10^{-3}. Consequently, the adiabatic approximation is valid. However, well above threshold the ratio may approach 10^{-1}, and a more careful investigation of the adiabatic approximation is needed.

The adiabatic approximation consists simply of putting $F_1(\tau)$ equal to $\eta R(\tau)$, where η is the stationary solution of Eqs. (4.2a) and (4.2b), and inserting the result in the time-dependent equation for R. The resultant equation for R is

$$\partial R_n/\partial\tau = (n+1)R_{n+1}[\bar{\nu}_r + C_-(n+1)] + nR_{n-1}[C_+(n)]$$
$$- R_n\{n[\bar{\nu}_r + C_-(n)] + (n+1)C_+(n+1)\}, \quad (6.1)$$

where

Case (a)

$$C_+(n) \equiv \frac{\gamma^2\omega_0^2}{\nu_2\omega_D}2N\int d\xi\,K(\xi)\eta_+(\xi,n)$$

$$= \frac{\gamma^2\omega_0^2 N\bar{\nu}_2}{\nu_2\omega_D(2\pi)^{1/2}}\int\frac{d\xi}{(\xi^2+\bar{\nu}_2^2)}\left[\frac{\xi^2 + \bar{\nu}_2^2(1 + \gamma^2\omega_0^2 n/\nu_1\nu_2)}{\xi^2 + \bar{\nu}_2'^2}\right]\exp(-\tfrac{1}{2}\xi^2),$$

$$C_-(n) \equiv \frac{\gamma^2\omega_0^2}{\nu_2\omega_D}2N\int d\xi\,K(\xi)\eta_-(\xi,n)$$

$$= \frac{\gamma^2\omega_0^2 N\bar{\nu}_2}{\nu_2\omega_D(2\pi)^{1/2}}\int\frac{\exp(-\tfrac{1}{2}\xi^2)[\bar{\nu}_2^2(\gamma^2\omega_0^2 n/\nu_1\nu_2)]}{(\xi^2+\bar{\nu}_2^2)(\xi^2+\bar{\nu}_2'^2)}d\xi. \quad (6.2a)$$

Case (b)

$$C_+(n) \equiv \left(\frac{\gamma^2\omega_0^2 N}{\nu_2^2}\right) \frac{1+\gamma^2\omega_0^2 n/\nu_1\nu_2}{1+2\gamma^2\omega_0^2 n/\nu_1\nu_2},$$

$$C_-(n) \equiv \left(\frac{\gamma^2\omega_0^2 N}{\nu_2^2}\right) \frac{\gamma^2\omega_0^2 n/\nu_1\nu_2}{1+2\gamma^2\omega_0^2 n/\nu_1\nu_2}. \tag{6.2b}$$

Case (c)

$$C_+(n) \equiv \frac{2\gamma^2\omega_0^2 N}{\pi\nu_2^2} \int_0^\pi dy \sin^2 y \frac{1+2\sin^2 y(\gamma^2\omega_0^2 n/\nu_1\nu_2)}{1+4\sin^2 y(\gamma^2\omega_0^2 n/\nu_1\nu_2)},$$

$$C_-(n) \equiv \frac{2\gamma^2\omega_0^2 N}{\pi\nu_2^2} \int_0^\pi dy \sin^2 y \frac{2\sin^2 y(\gamma^2\omega_0^2 n/\nu_1\nu_2)}{1+4\sin^2 y(\gamma^2\omega_0^2 n/\nu_1\nu_2)}. \tag{6.2c}$$

We demonstrate that the adiabatic approximation leads to a proper kinetic equation for R by showing the kinetic equation for R is negative semidefinite. Furthermore, it is Hermitian as a consequence of a generalized detail balance condition.

We write Eq. (6.1) in the following form:

$$\partial R_n/\partial\tau = \sum_{n'}[L(n\mid n')R_{n'} - L(n'\mid n)R_n], \tag{6.3}$$

where

$$L(n\mid n') \equiv \delta_{n',n+1}(n+1)[\bar{\nu}_r + C_-(n+1)$$
$$+\delta_{n',n-1}nC_+(n),$$

$$\sum_{n'}L(n'\mid n) = n[\bar{\nu}_r + C_-(n)] + (n+1)C_+(n+1).$$

The "amount entering minus the amount leaving" form of Eq. (6.3) with the $L(n\mid n')$ positive guarantees that Eq. (6.1) for R is negative semidefinite and preserves the normalization.[11]

The $L(n\mid n')$ are not symmetric, but they are symmetrizable because they satisfy the following condition:

$$L(n\mid n')R_{n'}^s = L(n'\mid n)R_n^s, \tag{6.4}$$

where R^s is the stationary solution of Eq. (6.1). The proof of Eq. (6.4) follows from the definition of $L(n\mid n')$ and the stationary state, Eq. (4.4b).

When we substitute the definition $r_n(\tau) \equiv R_n(\tau)/R_n^s$ into Eq. (6.3), we obtain

$$R_n^s[dr_n(\tau)/d\tau] = \sum l(n\mid n')[r_{n'}(\tau) - r_n(\tau)], \tag{6.5}$$

where $l(n\mid n') \equiv L(n\mid n')R_{n'}^s$ is symmetric because of Eq. (6.4). The eigenvalues of Eq. (6.5) are real, and the eigenfunctions are orthogonal. The eigenfunctions satisfy the equation

$$\sum_{n'}l(n\mid n')[u_\beta(n') - u_\beta(n)] = -\lambda_\beta R_n^s u_\beta(n),$$

where

$$\lambda_\beta > 0, \qquad \sum_n u_\alpha^*(n)u_\beta(n)R_n^s = \delta_{\alpha\beta},$$

$$\sum_\beta u_\beta^*(n)u_\beta(n') = \delta_{nn'}.$$

[11] C. R. Willis, Phys. Rev. **127**, 1405 (1962).

The Green's function for Eq. (6.1) is

$$\mathcal{G}_d[n\mid\bar{n};\tau] = \sum_\beta \exp(-\lambda_\beta\tau)u_\beta(n)u_\beta(\bar{n}) \qquad \text{for } \tau>0$$
$$=0 \qquad\qquad\qquad\qquad \text{for } \tau<0, \tag{6.6}$$

or symbolically,

$$\mathcal{G}_d(\tau) = \sum_\beta \mid\beta\rangle \exp(-\lambda_\beta\tau)\langle\beta\mid \qquad \text{for } \tau>0$$
$$=0 \qquad\qquad\qquad\qquad \text{for } \tau<0,$$

where d represents the fact that Eq. (6.6) is the Green's function for the diagonal matrix elements of R. Once we know $\mathcal{G}(\tau)$, we can answer all those questions that depend on the propagation of diagonal matrix elements of R. As an example, we consider the intensity correlation which is

$$\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle$$
$$= \text{tr}\, a^\dagger(\tau)a(\tau)aR^s a^\dagger$$
$$= \sum_n n\mathcal{G}_d(n\mid n;\tau)(n+1)R_{n+1}^s$$
$$= \sum_\alpha \exp(-\lambda_\alpha\tau)\sum_n [nu_\alpha(n)][(n+1)u_\alpha(n)]R_{n+1}^s. \tag{6.7}$$

However, if we try to treat phase fluctuations, we find that our diagonal equation for R is not sufficient. Consider the following expression for the line shape:

$$\int_0^\infty \exp(i\omega t)\langle a^\dagger(\tau)a\rangle d\tau$$
$$= \text{tr}\left(a^\dagger \int_0^\infty \exp(i\omega t)\mathcal{G}(\tau)d\tau aR^s\right)$$
$$= \sum_n \langle n\mid a^\dagger\mid n-1\rangle\mathcal{G}_{nd}[n, n-1\mid n-1, n;\omega]$$
$$\times\langle n-1\mid a\mid n\rangle R_n^s$$
$$= \sum_n n\mathcal{G}_{nd}[n, n-1\mid n-1, n;\omega]R_n^s, \tag{6.8}$$

where nd indicates that $\mathcal{G}_{nd}$ is the Green's function for the nondiagonal matrix elements of R. Even though R^s is diagonal in n, we still need to know how the off-

diagonal matrix elements of the density matrix propagate. In Sec. VII, we obtain the equations of motion for the off-diagonal matrix elements of R_{mn} of F.

VII. OFF-DIAGONAL MATRIX ELEMENTS OF R

The equations of motion for the diagonal matrix elements of R, Eq. (6.1), do not depend on the off-diagonal matrix elements of R, and, as we show in this section, the equations of motion for the off-diagonal matrix elements of R do not depend on the diagonal matrix elements of R. If R is diagonal at $t=0$, it remains diagonal at least in the Born approximation. Even when R is diagonal, we need the propagator for the off-diagonal matrix elements to compute multitime correlation functions.

The diagonal and off-diagonal matrix elements do not interact with each other in the first Born approximation because the general condition on the kernel of the equation for the density matrix requires that a relationship exists between the matrix elements that influence each other.[12] The condition on the kernel K

in the Born-approximation master equation is represented by the restrictions on the sum in the following equation:

$$i(\partial R_{mn}/\partial \tau) + (R, H_0)_{mn} = \sum_{pq}{}' K_{mn}{}^{pq} R_{pq},$$

where the prime means that p and q must satisfy $E_m{}^0 - E_p{}^0 = E_n{}^0 - E_q{}^0$. This condition on the sum divides the states into mutually exclusive sets which do not influence each other. In the Born approximation, the sets are the diagonal matrix elements and elements that lie on the same line parallel to the main diagonal. The equations of motion for the off-diagonal matrix elements of R depend on matrix elements of F_1, which are off-diagonal in the radiation variables but diagonal in the matter variables because of the trace over the matter variables in Eq. (3.6). We apply the same method to solve for the off-diagonal matrix elements $\langle m \mid F_{\pm}(y, \xi, \tau) \mid n \rangle$ as we used to solve for the diagonal matrix elements $F_{+}(y, \xi, n)$ in Sec. IV. The solutions are

$$\langle m \mid F_{\pm}(y, \xi, \tau) \mid n \rangle = C_{\pm}(m, n) R_{mn}(\tau),$$

$$C_{+}(m, n) + C_{-}(m, n) = (2\pi)^{-1/2} \exp(-\tfrac{1}{2}\xi^2),$$

$$C_{+}(m, n) = \frac{\bar{\nu}_1 (2\pi)^{-1/2} \exp(-\tfrac{1}{2}\xi^2)[i\Omega(m-n) + \bar{\nu}_1 + \bar{K}(m+1+n+1)]}{[i\Omega(m-n) + \bar{\nu}_1] + \bar{K}(m+1+n+1)^2 - 4\bar{K}^2(m+1)(n+1)} \rightarrow \frac{\bar{\nu}_1 (2\pi)^{-1/2} \exp(-\tfrac{1}{2}\xi^2)}{i\Omega(m-n) + \bar{\nu}_1 + \bar{K}(m+1+n+1)},$$

where the arrow indicates the result when we neglect depletion;—i.e., we set F_{-} equal to zero—and where

$$\bar{K} \equiv \gamma^2 \omega_0^2 \int_0^{\infty} d\tau \exp(-\bar{\nu}_2 \tau) \cos\bar{\Delta}\Gamma\Gamma(\tau). \quad (7.1)$$

When we substitute Eq. (7.1) in Eq. (3.6) for R, we obtain

$$\partial R_{mn}/\partial \tau + i(m-n)(\bar{\Omega} + \delta\bar{\Omega}) R_{mn}$$

$$= -N R_{mn} \left((m+n) \int K(\xi) C_{-}(m, n, \xi) d\xi \right.$$

$$\left. + (m+1+n+1) \int K(\xi) C_{+}(m, n, \xi) d\xi \right)$$

$$+ 2N R_{m-1,n-1}(mn)^{1/2} \int K(\xi) C_{+}(m-1, n-1, \xi) d\xi$$

$$+ 2N R_{m+1,n+1}[(m+1)(n+1)]^{1/2}$$

$$\times \int K(\xi) C_{-}(m+1, n+1, \xi) d\xi + \bar{\nu}_r[(m+1)(n+1)]^{1/2}$$

$$\times R_{m+1,n+1} - \tfrac{1}{2}\bar{\nu}_r(m+n) R_{mn}. \quad (7.2)$$

We derive the expression for $\delta\bar{\Omega}$ in Appendix B. We need the Green's function $\mathcal{G}[n, m \mid \bar{n}, \bar{m}; \tau]$ of Eq. (7.2) to answer questions about the propagation of off-diagonal phase information.

VIII. DISCUSSION

The double commutator structure of the first Born approximation for a harmonic oscillator leads to a FP structure in the variables $a^{\dagger}$ and a. However, in the laser problem we also have a double commutator structure in the matter variables. The adiabatic elimination of the matter variables changes the FP equation with linear coefficients to the generalized FP equation with nonlinear coefficients which have the form

$$[1 + (\gamma^2 \omega_0^2 / \nu_1 \nu_2) a^{\dagger} a]^{-1}.$$

The generalized FP structure which results from the Born approximation of the $\mathbf{j} \cdot \mathbf{A}$ interaction is completely independent of the radiation reservoir. The form of the radiation-reservoir term, $\mathcal{K}_r R$, depends only on the existence of the radiation reservoir and weak coupling of our system to the reservoir. Nowhere do we require the existence of any microscopic property of a reservoir such as requiring the reservoir to be Gaussian. Our

[12] C. R. Willis and P. G. Bergmann, Phys. Rev. **128**, 391 (1962).

generalized FP equation is not determined by the type of reservoirs but comes only from the first Born approximation, from the existence of the matter reservoir which provides the lifetime for our two-level systems, and from the adiabatic elimination of the matter variables.

If we consider the case of stationary atoms with traveling waves and let the lower state population F_- go to zero, then our equation for R is equivalent to Lax's equation[2] for ρ in the n representation. Lax[2] showed the equivalence of the quantum-mechanical nonlinear FP equation with the equation for ρ in the n representation. Consequently, our equations including the case of moving atoms with spatial dependence are equivalent to the quantum-mechanical nonlinear FP equation. All four cases have the same structure in the n, m dependence but the coefficients depend in different ways on such parameters as N/N_T. For example, the average radiation energy depends quadratically on N/N_T in the cases of moving atoms and linearly on N/N_T in the cases of stationary atoms.

An interesting alternative way to derive the nonlinear FP equation directly would be to eliminate the matter variables adiabatically from the operator Eqs. (3.6) and (3.9), and express the resultant equation of motion for R in the representation where the creation operator $a^{\dagger}$ is diagonal[13] instead of in the number representation.

Lax[14] showed that near threshold the nonlinear FP equation reduces to the rotating-wave Van der Pol Fokker–Planck equation. Thus, our master equation reduces to the FP equation for the rotating-wave Van der Pol equation, and our time-dependent Green's functions are related to the eigenvalues and the eigenfunctions found by Lax and Hempstead[10] and Risken and Volmer.[9] Our Green's function for the diagonal matrix elements of R corresponds to the classical Green's function integrated over the phase variable which is the case of $\lambda=0$ of Ref. 10. The Green's function for the off-diagonal matrix elements corresponds to the Fourier transforms of the classical Green's function with respect to the phase variable. In particular, the line shape defined in Eq. (6.8) requires $\lambda=1$. The easiest way to obtain the Green's function near threshold is to express Lax's and Risken's results in the $P(\alpha)$ representation and transform to the n representation with the known $\langle n \mid \alpha \rangle$.

The first Born approximation depends crucially on the assumption that atoms change little during their lifetimes. It is possible for some atoms exactly on resonance to change a great deal during their lifetimes. For these atoms, the Born approximation is no longer valid, and it is necessary[15] to go to a Boltzmann equation that treats the individual particle-radiation interaction to all orders in γ, not just to order γ^2. For the range of

13 R. J. Glauber, Phys. Rev. 131, 2766 (1963).
14 M. Lax, Phys. Rev. 145, 110 (1966).
15 C. R. Willis (to be published).

parameters where the photon-counting experiments are performed, the results of the Boltzmann-equation approach are numerically very close to the Born-approximation results.

The steady-state deviation of the correlated distribution function $F_+(n)$ from the product $R_{(n)}\rho$ for the case of stationary atoms with traveling waves is

$$
\chi(+) = \frac{F_+(n) - R(n)\rho(+)}{R(n)\rho(+)}
$$
$$
= \frac{(1+x)/(1+2x) - \langle (1+x)/(1+2x) \rangle}{\langle (1+x)/(1+2x) \rangle}, \quad (8.1)
$$

where $x = (\gamma^2 \omega_0^2 / \nu_1 \nu_2)\, n$.

From Eq. (8.1), we see that the correlation between radiation and matter is large for photon numbers n which deviate a great deal from $\langle n \rangle$ or n_0. In the photon-counting experiments, x is small through most of the range of n, and we have $\chi(+) \approx \langle x \rangle - x$, which is positive for $n < \langle n \rangle$ and negative for $n > \langle n \rangle$. It is this relatively large correlation for large and small n that causes the photon distribution function to be sharply peaked. As the distribution function becomes more sharply peaked, the correlation between radiation and matter decreases and the coherence of the radiation increases.

APPENDIX A. TRUNCATION OF THE HIERARCHY

We show the ansatz

$$
F_N = R\rho(1) \cdots \rho(N) + \sum_{j=1}^{N} [F_1(j) - R\rho(j)] \prod_{i \neq j}^{N} \rho(i)
$$
$$
\equiv R\rho(1) \cdots \rho(N)[1 + \sum_{j=1}^{N} \chi(j)], \quad (A1)
$$

where

$$
\chi(j) \equiv [\eta(j) - \rho(j)] \rho^{-1}(j)
$$

truncates the hierarchy so that we obtain the closed Eqs. (3.6) and (3.9) for R and F_1. In particular, Eq. (A1) for F_2 causes the following inequality to be satisfied:

$$
\mathfrak{K}_r F_1 - \gamma^2 (N-1) \bar{\omega}_0^2 \mathrm{tr}_2 \mathrm{tr}_{\mathrm{res}} \int_0^{\infty} d\tau
$$
$$
\times [H_1(2), [H_1(2, \tau), F_2\mathcal{P}]] \ll \mathfrak{K}_m F_1. \quad (A2)
$$

We can replace Eq. (3.5) by Eq. (3.9) only if Eq. (A2) is valid. We show the inequality is true for the stationary state, and thus it is true for the time-dependent equation because in a gas laser the pump term increases and the left side of Eq. (A2) decreases with time.

When we substitute Eq. (A1) into the left side of

Eq. (A2), we obtain

$$\mathfrak{K}_r F_1 - \gamma^2 N \bar{\omega}_0{}^2 \, \text{tr}_2 \text{tr}_{\text{res}} \int_0^\infty [H_1(2), [H_1(2,\tau), (F_1(2)\rho(1)+F_1(1)\rho(2)-R\rho(1)\rho(2))\mathcal{P}]]d\tau$$

$$+\gamma^2 \bar{\omega}_0{}^2 \, \text{tr}_2 \text{tr}_{\text{res}} \int_0^\infty [H_1(2), [H_1(2,\tau), F_2\mathcal{P}] \, d\tau = \mathfrak{K}_r[F_1(1)-R\rho(1)]$$

$$-\gamma^2 N \bar{\omega}_0{}^2 \, \text{tr}_2 \text{tr}_{\text{res}} \int_0^\infty [H_1(2), [H_1(2,\tau), (F_1(1)-R\rho(1))\rho(2)\mathcal{P}]]d\tau$$

$$+\gamma^2 \bar{\omega}_0{}^2 \, \text{tr}_2 \text{tr}_{\text{res}} \int_0^\infty [H_1(2), [H_1(2,\tau), F_2\mathcal{P}]]d\tau \approx \mathfrak{K}_r[F_1(1)-R\rho(1)]$$

$$-\gamma^2 N \bar{\omega}_0{}^2 \, \text{tr}_2 \text{tr}_{\text{res}} \int_0^\infty [H_1(2), [H_1(2,\tau), (F_1(1)-R\rho(1))\rho(2)\mathcal{P}]]d\tau, \quad (A3)$$

where we used the exact time-independent Eq. (3.6) for R multiplied by $\rho(1)$, which is

$$\rho(1)\mathfrak{K}_r R = \rho(1)\gamma^2 \bar{\omega}_0{}^2 N \, \text{tr}_2 \text{tr}_{\text{res}} \int_0^\infty [H_1(2), [H_1(2,\tau), \mathcal{P}F_1(2)]]d\tau.$$

The approximate equality in Eq. (A3) follows from neglect of the F_2 term, which we later show is negligible.
The (n, n) diagonal matrix element of Eq. (A3) is

$$\bar{\nu}_r \left[1-\frac{N}{N_T}(\rho(+)-\rho(-))\right]\frac{\partial}{\partial n}\{n(F_1(1,n)-R_n\rho(1))\}$$

$$=\frac{\bar{\nu}_r}{\langle n\rangle}\left[n-\frac{N}{N_T}\langle n\rangle(\rho(+)-\rho(-))\right]\frac{\partial}{\partial n}\{n(F_1(1,n)-R_n\rho(1))\}$$

$$=\frac{\nu_r}{\langle n\rangle}\frac{N}{N_T}[\langle n/(1+2x)\rangle-\langle n\rangle\langle(1+2x)^{-1}\rangle]\frac{\partial}{\partial n}\{n(F_1(1,n)-R_n\rho(1))\}, \quad (A4)$$

where $x \equiv \gamma^2\omega_0{}^2 n/\nu_1\nu_2$, and where we use the exact equation for $\langle n\rangle$. The derivatives with respect to n appear in Eq. (A4) for the following reason. If we replace R_{n+1} and $F_1(1, n+1)$ by R_n and $F_1(1, n)$ in Eq. (A3), the inequality Eq. (A2) is satisfied because the left side vanishes. However, the inequality is being used to justify the discarding of terms that are proportional to N, which is very large, and the retaining of the $\gamma^2\bar{\omega}_0{}^2$ term in Eq. (3.5) for F_1. Therefore, to make sure that the inequality is strongly satisfied, we replace $R_{n+1}=\exp(\partial/\partial n)R_n$ not by just R_n but by $R_n+\partial R_n/\partial n$.

Below threshold and way above threshold the square bracketed term vanishes. In the region near threshold where the photon-counting experiments are done, the ratio of Eq. (A4) to $\mathfrak{K}_m F_1$ is

$$\mathfrak{K}_r[f_1(1)]-\gamma^2\bar{\omega}_0{}^2 N \, \text{tr}_2 \text{tr}_{\text{res}} \int_0^\infty [H_1(2), [H_1(2,\tau), f_1(1)\rho(2)\mathcal{P}]]d\tau \Big/ \mathfrak{K}_m F_1$$

$$\approx \frac{\nu_r}{\nu_1}\frac{N}{N_T}\frac{\gamma^2\omega_0{}^2}{\nu_1\nu_2}\langle n\rangle\left\{\frac{\langle n^2\rangle-\langle n\rangle^2}{\langle n\rangle^2}\right\}\approx 3\times10^{-6}\frac{\langle n\rangle}{V}\left\{\frac{\langle n^2\rangle-\langle n\rangle^2}{\langle n\rangle^2}\right\}\ll 3\times10^{-3}, \quad (A5)$$

where $f(1)\equiv F_1(1)-R\rho(1)$. The inequality occurs because for a thermal distribution the curly brackets equal 1, and they become much less than 1 when the distribution becomes sharply peaked as it does in the laser.
The n, n diagonal matrix element of the term dropped in Eq. (A3) is

$$\text{tr}_2\text{tr}_{\text{res}}\gamma^2\bar{\omega}_0{}^2\left\langle n\left|\int_0^\infty [H_1(2), [H_1(2,\tau), \mathcal{P}F_2(1,2,\tau)]]d\tau\right|n\right\rangle$$

$$=2\bar{K}\{n[F_2(1,+,n)+F_2(1,-,n)-F_2(1,+,n-1)-F_2(1,-,n+1)]+F_2(1,+,n)-F_2(1,-,n+1)\}$$

$$\approx 0+\bar{K}(\partial/\partial n)\{n[F_2(1,+,n)-F_2(1,-,n)]\}. \quad (A6)$$

The leading terms cancel exactly, and the ratio of the remainder to $\mathcal{K}_m F_1$ is of the order $\gamma^2\omega_0^2/\nu_1\nu_2\approx10^{-6}$, and so is negligible.

Thus the ansatz Eq. (A1) satisfies Eq. (A2), and so the term proportional to $\gamma^2(N-1)$ in Eq. (3.5) is effectively cancelled by the term $\mathcal{K}_r F_1$. Since $\gamma^2(N-1)$ is the only term that depends on F_2, Eq. (3.5) is a linear inhomogeneous equation for F_1.

We conclude with some remarks about our ansatz Eq. (A1). Since the trace of $F_1(j)$ over the matter variables is R and the trace of $F_1(j)$ over the radiation variables is $\rho(j)$, the trace of F_N over all the variables is unity. If there are no two-particle correlations, the most general ansatz of the master equation is

$$F_N = R \prod_{j=1}^{N} \frac{F_1(j)}{R} = R \prod_{j=1}^{N} \eta(j)$$

$$= R\rho(1)\cdots\rho(N)\prod_{j=1}^{N}[1+\chi(j)]. \quad (A7)$$

Consequently, our ansatz is just the linearization of the most general ansatz without two-particle correlations. When the ansatz Eq. (A7) satisfies the inequality Eq. (A2), the resultant equations for F_1 and R are the same as Eqs. (3.6) and (3.9), and the differences in Eqs. (A1) and (A7) are inconsequential. For n near the center of the photon distribution, the ansatz Eq. (A7) is essentially as good as Eq. (A1). In the wings of the photon distribution, the ansatz Eq. (A7) is less exact than Eq. (A1).

APPENDIX B. FREQUENCY-SHIFT OPERATOR

The imaginary part of the last term on the right side of Eq. (3.6) in dimensionless variables is

$$-i\gamma^2\bar{\omega}_0^2N\,\mathrm{tr}_1\int_0^\infty \exp(-\bar{\nu}_2\tau)\,\sin\bar{\Delta}\tau\Gamma\Gamma(\tau)$$

$$\times[(a^\dagger a\sigma^\dagger\sigma,\,F_1)+(F_1,\,a^\dagger a\sigma\sigma^\dagger)+(\sigma^\dagger\sigma,\,F_1)]$$

$$= -i\gamma^2\bar{\omega}_0^2N\int_{-\infty}^{\infty} d\xi \int_0^\infty \exp(-\bar{\nu}_2\tau)\,\sin\bar{\Delta}\tau\Gamma\Gamma(\tau)\,d\tau$$

$$\times[a^\dagger a,\,(\eta_+(\xi)-\eta_-(\xi))R]$$

$$= -i\gamma^2\bar{\omega}_0^2N\int_{-\infty}^{\infty} d\xi \int_0^\infty \exp(-\bar{\nu}_2\tau)\,\sin\bar{\Delta}\,\cos\xi$$

$$\times\left[a^\dagger a,\,\frac{\xi^2+\bar{\nu}_2^2}{\xi^2+\bar{\nu}_2^2[1+(\gamma^2\omega_0^2/\nu_1\nu_2)a^\dagger a]}R\right]$$

$$\equiv -i[a^\dagger a,\,\delta\bar{\Omega}_{\mathrm{op}}R]. \quad (B1)$$

As a consequence of the correlation between the radiation and matter variables, the frequency-shift operator $\delta\bar{\Omega}_{\mathrm{op}}$ is an operator in the radiation variables instead of just a frequency shift. When F_1 is a product $R\rho$, the frequency-shift operator in Eq. (B1) becomes a simple number

$$\delta\bar{\Omega}(t) = N\gamma^2\bar{\omega}_0^2\int_{-\infty}^{\infty} d\xi \int_0^\infty d\tau\,\exp(-\bar{\nu}_2\tau)$$

$$\times\sin\bar{\Delta}\tau\Gamma\Gamma(\tau)\,d\tau[\rho_+(\xi,\,t)-\rho_-(\xi,\,t)], \quad (B2)$$

which is the same as Eq. (I3.8).

In a strict sense, there is no simple frequency shift when there is a correlation between radiation and matter variables, and one has to find the complex eigenvalues of the Green's function for the off-diagonal matrix elements of R. However, we obtain a very good approximation of the frequency-shift operator by replacing

$$\rho_+(\xi,\,t)-\rho_-(\xi,\,t) = \{\xi^2+\bar{\nu}_2^2[1+(\gamma^2\omega_0^2/\nu_1\nu_2)\langle n\rangle]\}^{-1}$$

by

$$\langle\eta_+(\xi,\,t)-\eta_-(\xi,\,t)\rangle = \langle\{\xi^2+\bar{\nu}_2^2[1+(\gamma^2\omega_0^2/\nu_1\nu_2)n]\}^{-1}\rangle$$

$$(B3)$$

in Eq. (B2). We obtain Eq. (B3) when we approximate $\langle a^\dagger(\eta_+-\eta_-)\rangle$ by $\langle a^\dagger\rangle\langle\eta_+-\eta_-\rangle$ in the equation of motion for $a^\dagger$.