Size-Dependent Transport-Coefficient Effects in Fermi Liquids

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Landau's theory of a Fermi liquid is applied to investigate the size-limited transport coefficients of a Fermi liquid contained in a long narrow channel of diameter d small compared with the interquasiparticle mean free path λ. By characterizing the scattering of quasiparticles (QP) from the walls of the channel in a phenomenological way, expressions are obtained for the coefficients of spin diffusion, D, and of thermal conductivity, K. These expressions show, not unexpectedly, that D is a constant independent of T, giving a direct measure of the QP group velocity v_0, while K is proportional to T and coincides with the result that would be obtained if we had considered a noninteracting Fermi gas. Mass flow through the channel, which for $d \ll \lambda$ is not characteristic of viscous flow, is also considered. Under the action of an externally applied pressure gradient, it is found that mass is discharged at a temperature-independent rate G proportional to d^3: in the viscous regime ($d \gg \lambda$) G varies as d^4. The expression obtained for G is actually the Fermi-gas analog of the classical result originally obtained by Knudsen. The extension of the results to the regime $d \sim \lambda$ if is briefly considered. As a consequence of competition between Knudsen flow and Poiseuille flow, a "Knudsen minimum" will appear in the temperature dependence of G. Some numerical estimates are made for liquid He^3.

1. SYNOPSIS

SIZE-VARIATION effects in transport coefficients occur whenever the mean free path λ of the elementary carriers becomes comparable in magnitude to the dimensions of the system under study. The theory of these effects for a classical gas,[1] and for the electron and phonon gases of a crystal,[2] has been extensively developed on the basis of the kinetic theory of gases, or equivalently on the basis of Boltzmann's transport equation. Following closely the methods used in these theories we present in this paper a theory of the size-variation effects which can be anticipated to occur in the transport coefficients of the quasiparticle gas of a Fermi liquid[3] confined in a long narrow channel of constant cross section. In practice such effects might be detected in He^3 by measuring the transport coefficients of the liquid in the narrow pores of vycor glass[4] ($d \sim 100$ Å), where at sufficiently low temperatures ($T \sim 100$ m°K) the quasiparticle mean free path will be comparable to the mean pore diameter d. Experiments of this nature are currently in progress.[5]

The theory to be presented here will deal mainly with the situation for which $\lambda \gg d$ (collisionless regime) so that the quasiparticle (QP) transport processes will be determined solely by scattering at the walls of the channel. In this case the collisionless transport equation of Landau[3] may be used to solve for the QP distribution function $n_{\mathbf{p}\sigma}$. The boundary conditions required here are treated phenomenologically by assuming that a proportion ν of quasiparticles incident on the walls are scattered specularly while the remainder $(1-\nu)$ are scattered diffusely. Within this highly simplified scheme the coefficients of thermal conductivity K and spin-diffusion D for transport along the channel axis are found to be $K=\frac{1}{3}C_v v_0 \Lambda_B$ and $D=\frac{1}{3}(1+Z_0/4)v_0\Lambda_B$ where v_0 denotes the QP group velocity on the Fermi surface, C_v the specific heat of the QP gas, Z_0 the Landau interaction parameter which enters the Pauli susceptibility, and Λ_B a geometrical mean free path, equal to $[(1+\nu)/(1-\nu)]d$ for the case of a perfectly circular channel cross section. Thus, in the collisionless regime, D is a constant and gives a direct measure of the QP group velocity v_0: The thermal conductivity, which may be rewritten as $K=\frac{1}{3}\Lambda_B\pi^2k_B^2nT/p_0$, where p_0 denotes the Fermi momentum, n the number of fermions per unit volume and k_B Boltzmann's constant, is proportional to T and coincides with the result that would be obtained if we had considered a noninteracting Fermi gas.

The theory of mass flow through the channel, which, in the collisionless regime, is not characteristic of viscous flow,[6] is also considered. Under a pressure gradient $-\partial p/\partial \mathbf{r}$ applied along the axis of the channel, it is found that mass is discharged at a temperature-independent rate $G=|(\partial p/\partial \mathbf{r})|(\pi m d^2\Lambda_B/4p_0)$, where m is the bare fermion mass. Like the thermal conductivity, this result is coincident with that of a noninteracting Fermi gas and is the Fermi gas analog of the result originally obtained by Knudsen[6] for a highly rarefied classical gas. It predicts G to vary as the cube of the channel diameter d: In the viscous regime ($d \gg \lambda$) G varies as the fourth power of d.

The extension of the above results to the intermediate regime ($\lambda \sim d$) is briefly considered. As a consequence of competition between Knudsen flow and classical Poiseuille flow a "Knudsen minimum" will appear in the

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[1] E. Bloch, *The Kinetic Theory of Gases* translated by P. A. Smith (Methuen and Co. Ltd., London, 1924).
[2] See J. M. Ziman, *Electrons and Phonons* (Clarendon Press, Oxford, England, 1960), Chap. 11.
[3] L. Landau, Zh. Eksperim i Teor. Fiz. 30, 1058 (1956) [English transl.: Soviet Phys.—JETP 3, 920 (1957)]; Zh. Eksperim. i Teor. Fiz. 32, 59 (1957) [English transl.: Soviet Phys.—JETP 5, 101 (1957)].
[4] D. F. Brewer, Proceedings of the Tenth International Conference on Low Temperature (Moscow, 1966) (to be published).
[5] D. F. Brewer and D. S. Betts (private communication).
[6] M. Knudsen, Ann. Phys. 28, 75 (1909). See also, Ref. 1.

temperature dependence of G. A numerical calculation of the temperature dependences expected for the coefficients of thermal conductivity and spin diffusion of liquid He^3, contained in a cylinder of diameter 100 Å, is given.

2. THERMAL CONDUCTION AND SPIN DIFFUSION IN THE COLLISIONLESS REGIME

In the collisionless regime the steady state QP distribution function $n_{\mathbf{p}\sigma}(\mathbf{r})$ satisfies the Landau-Boltzmann transport equation[3,7]

$$\nabla_p \epsilon_{\mathbf{p}\sigma} \cdot \nabla_r n_{\mathbf{p}\sigma} - \nabla_r \epsilon_{\mathbf{p}\sigma} \cdot \nabla_p n_{\mathbf{p}\sigma} = 0\,, \tag{2.1}$$

where $\epsilon_{\mathbf{p}\sigma}$ denotes the energy of a quasiparticle of momentum $\mathbf{p}$ and spin σ. For small deviations from equilibrium, close to absolute zero, the QP energy depends upon space $\mathbf{r}$ through its dependence on the QP distribution function[3]:

$$\epsilon_{\mathbf{p}\sigma} = \epsilon_{\mathbf{p}\sigma}{}^0 + \frac{V}{(2\pi)^3} \sum_{\sigma'} \int d^3p' f(\mathbf{p}\sigma; \mathbf{p}'\sigma') \delta n_{\mathbf{p}'\sigma'} \qquad (V = \text{volume of system})\,, \tag{2.2}$$

where $\epsilon_{\mathbf{p}\sigma}{}^0$ is the equilibrium QP energy at $T=0$, $\delta n_{\mathbf{p}\sigma} = n_{\mathbf{p}\sigma} - n_F(\epsilon_{\mathbf{p}\sigma}{}^0)$ and $f(\mathbf{p}\sigma; \mathbf{p}'\sigma')$ is Landau's characteristic function for the Fermi liquid. $n_F(\epsilon)$ is used to denote the equilibrium Fermi function

$$n_F(\epsilon) = 1/e^{(\epsilon-\mu)/k_BT} + 1\,, \tag{2.3}$$

where μ denotes the chemical potential. In the following discussion both T and μ will be regarded as being slowly varying functions of space so that the quantity $\nabla_r n_F(\epsilon_{\mathbf{p}\sigma}{}^0)$ will involve a measure of small spacial inhomogeneities $\nabla_r T$ and $\nabla_r \mu$. The actual local equilibrium distribution function for the quasiparticles is $n_F(\epsilon_{\mathbf{p}\sigma})$. It is accordingly the deviation, $g_{\mathbf{p}\sigma}$, of $n_{\mathbf{p}\sigma}$ from this distribution that has to enter the appropriate expressions for the mean QP fluxes in the nonequilibrium situation. Because of the dependence of $\epsilon_{\mathbf{p}\sigma}$ on $n_{\mathbf{p}\sigma}$ as specified by (2.2), $g_{\mathbf{p}\sigma}$ and $\delta n_{\mathbf{p}\sigma}$ are in general not the same but are related by

$$\delta n_{\mathbf{p}\sigma} = g_{\mathbf{p}\sigma} + \frac{\partial n_F(\epsilon_{\mathbf{p}\sigma}{}^0)}{\partial \epsilon_{\mathbf{p}\sigma}{}^0} \frac{V}{(2\pi)^3} \times \sum_{\sigma'} \int d^3p' f(\mathbf{p}\sigma; \mathbf{p}'\sigma') \delta n_{\mathbf{p}'\sigma'}\,. \tag{2.4}$$

Following Abrikosov and Khalatnikov[7] the transport equation for $g_{\mathbf{p}\sigma}$ may be obtained from (2.1) by entering (2.1) with $n_{\mathbf{p}\sigma} = n_F(\epsilon_{\mathbf{p}\sigma}{}^0) + \delta n_{\mathbf{p}\sigma}$ and linearizing with respect to $\delta n_{\mathbf{p}\sigma}$. This procedure gives

$$\mathbf{v}_{\mathbf{p}\sigma} \cdot \nabla_r \left\{ \delta n_{\mathbf{p}\sigma} - \frac{\partial n_F(\epsilon_{\mathbf{p}\sigma}{}^0)}{\partial \epsilon_{\mathbf{p}\sigma}{}^0} \frac{V}{(2\pi)^3} \times \sum_{\sigma'} \int d^3p' f(\mathbf{p}\sigma; \mathbf{p}'\sigma') \delta n_{\mathbf{p}'\sigma'} \right\} = -\mathbf{v}_{\mathbf{p}\sigma} \cdot \nabla_r n_F(\epsilon_{\mathbf{p}\sigma}{}^0)$$

or, in view of (2.4),

$$\mathbf{v}_{\mathbf{p}\sigma} \cdot \nabla_r g_{\mathbf{p}\sigma} = -\mathbf{v}_{\mathbf{p}\sigma} \cdot \nabla_r n_F(\epsilon_{\mathbf{p}\sigma}{}^0)\,, \tag{2.5}$$

where $\mathbf{v}_{\mathbf{p}\sigma} = \nabla_p \epsilon_{\mathbf{p}\sigma}{}^0$, which, for $|\mathbf{p}|$ close to the Fermi momentum p_0, may be written in terms of the QP effective mass m^* viz. $\mathbf{v}_{\mathbf{p}\sigma} = \mathbf{p}/m^* \equiv \mathbf{v}_0$. The right hand side of (2.5) is to be understood to be linear in the spacial inhomogeneities. Then by (2.3) Eq. (2.5) becomes

$$\mathbf{v}_{\mathbf{p}\sigma} \cdot \nabla_r g_{\mathbf{p}\sigma} = \frac{\partial n_F(\epsilon_{\mathbf{p}\sigma}{}^0)}{\partial \epsilon_{\mathbf{p}\sigma}{}^0} \left\{ \nabla_r \mu + \frac{(\epsilon_{\mathbf{p}\sigma}{}^0 - \mu)}{T} \nabla_r T \right\} \cdot \mathbf{v}_{\mathbf{p}\sigma} \tag{2.6}$$

$$\equiv \mathbf{A}_{\mathbf{p}\sigma} \cdot \mathbf{v}_{\mathbf{p}\sigma}\,, \tag{2.7}$$

where the right-hand side may now be regarded as being independent of space. Equation (2.6) is of the same form of the transport equation that has to be solved in the analogous boundary-scattering problem for electrons in a metal.[2]

In order to solve Eq. (2.6) it will be necessary to specify the boundary conditions which describe the effects of the walls of the channel in which the Fermi liquid is assumed to be enclosed. Following the methods developed in Ref. 2 we employ a simple phenomenological description in which a proportion ν of quasiparticles incident on the walls are assumed to be scattered specularly according to the laws of elastic reflection whilst the remainder $(1-\nu)$ are assumed to be scattered diffusely, that is, the quasiparticles are considered to be absorbed by the walls and re-emitted at an equilibrium rate appropriate to the temperature of the walls. It will be essential for the subsequent discussion of spin diffusion to make the additional assumption that the scattering at the walls occurs without change of spin. Measuring the normal to the surface at a point $\mathbf{r}_B$ on the walls into the channel, the appropriate boundary condition may be written as

$$g_{\mathbf{p}\sigma}(\mathbf{r}_B; v_i > 0) = \nu g_{\mathbf{p}'\sigma}(\mathbf{r}_B; v_i < 0)\,, \tag{2.8}$$

where v_i denotes the normal component of the QP group velocity and $\mathbf{p}'$ is related to $\mathbf{p}$ by simple elastic reflection of the QP group velocity in the surface. For a critical derivation of Eq. (2.8) the reader is referred to the treatise of Ziman.[2]

We first consider the case $\nu=0$ in which the scattering at the walls is completely diffuse. In this case Eqs. (2.7) and (2.8) have the simple solution.

$$g_{\mathbf{p}\sigma}(\mathbf{r}) = \mathbf{A}_{\mathbf{p}\sigma} \cdot |\mathbf{r} - \mathbf{r}_B| \mathbf{v}_{\mathbf{p}\sigma} / |\mathbf{v}_{\mathbf{p}\sigma}| \tag{2.9}$$

[7] A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. 22, 329 (1959).

which, physically, describes the path of a quasiparticle subsequent to leaving the surface at the point $\mathbf{r}_B$ in the direction $\mathbf{r}-\mathbf{r}_B$. All such paths will figure in the calculation of the mean QP fluxes. Thus, for example, the total mean QP flux, of specified spin orientation σ, along the axis of the channel, is given by

$$J_\sigma=\frac{1}{S}\int\frac{d^3p}{(2\pi)^3}\int\frac{d\Omega_B}{4\pi}\int d\mathbf{S}\cdot\mathbf{v}_{\mathbf{p}\sigma}g_{\mathbf{p}\sigma}(\mathbf{r}-\mathbf{r}_B), \quad (2.10)$$

where $\int d\mathbf{S}$ denotes an integration over the cross-section S of the channel and $d\Omega_B$ an element of solid angle giving the direction of $(\mathbf{r}-\mathbf{r}_B)$. Similarly, the linear heat flux along the channel is

$$J_H=\sum_\sigma\frac{1}{S}\int\frac{d^3p}{(2\pi)^3}\int\frac{d\Omega_B}{4\pi}\int d\mathbf{S}\cdot v_{\mathbf{p}\sigma}\times(\epsilon_{\mathbf{p}\sigma}{}^0-\mu)g_{\mathbf{p}\sigma}(\mathbf{r}-\mathbf{r}_B). \quad (2.11)$$

For a magnetization gradient, maintained at constant temperature along the axis of the channel, we have[8]

$$\mathbf{A}_{\mathbf{p}\sigma}=\frac{\partial n_F(\epsilon_{\mathbf{p}\sigma}{}^0)}{\partial\epsilon_{\mathbf{p}\sigma}{}^0}\nabla_r\mu=\frac{\partial n_F(\epsilon_{\mathbf{p}\sigma}{}^0)}{\partial\epsilon_{\mathbf{p}\sigma}{}^0}\frac{\partial\mu}{\partial n_\sigma}\nabla_r n_\sigma, \quad (2.12)$$

where n_σ denotes the concentration of fermions of specified spin σ. If the spin dependence of the Fermi liquid function f is assumed to be of exchange origin we may use the relation

$$\partial\mu/\partial n_\sigma=(2\pi^2/m^*p_0)\tfrac{1}{4}(1+Z_0) \quad (2.13)$$

due to Hone,[8] where Z_0 is the zeroth coefficient in the Legendre expansion of the exchange part of f and enters the Fermi liquid theory expression for the Pauli spin susceptibility.[7] Using (2.12) and (2.13) in (2.9) the integration over momenta in (2.10) may be performed to yield the diffusion law

$$J_\sigma=-D|\nabla_r n_\sigma|$$

with the spin-diffusion coefficient D given by

$$D=\tfrac{1}{3}\Lambda_B(1+Z_0/4)v_0, \quad (2.14)$$

where Λ_B is an effective QP mean free path given by

$$\Lambda_B=\frac{3}{S}\int\int|d\mathbf{S}|\frac{d\Omega_B}{4\pi}\cos^2\theta\cdot|\mathbf{r}-\mathbf{r}_B|, \quad (2.15)$$

where θ is the angle between the direction of $\mathbf{r}-\mathbf{r}_B$ and the axis of the channel. For a perfectly cylindrical channel of diameter d Eq. (2.15) may be evaluated to give $\Lambda_B=d$.[9] The thermal flux J_H which results from a temperature gradient, maintained at constant pressure along the axis of the channel, may be calculated by setting $\mathbf{A}_{\mathbf{p}\sigma}$ equal to the term linear in $\nabla_r T$ on the right-hand side of Eq. (2.6).[10] An integration over $\mathbf{p}$ in (2.11) then gives

$$J_H=-K|\nabla_r T|$$

with the coefficient of thermal conductivity K equal to

$$K=\tfrac{1}{3}\pi^2\Lambda_B nTk_B{}^2/p_0, \quad (2.16)$$

where n denotes the number of fermions per unit volume and k_B Boltzmann's constant. Since the specific heat C_v of the QP gas is $C_v=\pi^2nk_B{}^2T/m^*v_0{}^2$, Eq. (2.16) may be written as

$$K=\tfrac{1}{3}\Lambda_B v_0 C_v. \quad (2.17)$$

The generalization of the above results to the case for which $\nu\neq0$ involves a consideration of the various orders of multiple reflections from the walls made by the proportion ν of quasiparticles that are specularly scattered. The net result is simply that in the expressions (2.14) and (2.16) previously obtained for D and K respectively, the mean free path Λ_B is to be replaced by $[(1+\nu)/(1-\nu)]\Lambda_B$. This result may be derived by following precisely the same treatment of the general boundary condition [Eq. (2.8)] as given in Ref. 2 and need not be reproduced here.

The coefficients of spin-diffusion D_0 and thermal conductivity K_0 of the *infinite* Fermi liquid ($d\gg\lambda$) may be written in the form[11]

$$D_0=\tfrac{1}{3}\lambda_D v_0(1+Z_0/4);\quad K_0=\tfrac{1}{3}C_v\lambda_K v_0, \quad (2.18)$$

where λ_D and λ_K denote inter-QP mean free paths for spin diffusion and thermal conduction, respectively. Then, according to Eqs. (2.14) and (2.17), and the remarks made in the preceding paragraph, the spin diffusion and thermal conductivity in the collisionless regime ($d\ll\lambda$) are obtainable from Eq. (2.18) by substituting the geometrical mean free path $[(1+\nu)/(1-\nu)]\Lambda_B$ in place of λ_D and λ_K. Thus, in the collisionless regime D is a constant and gives a direct measure of the QP group velocity v_0: The thermal conductivity K is proportional to T and, as may be seen from Eq. (2.16), coincides with the result that would be obtained if we had considered a noninteracting Fermi gas with the same value of n. Since the same mean free path appears in D and K the ratio K/DT is a constant, independent of ν and the geometrical cross section of the channel, given by

$$K/DT=\left(\frac{\pi^2n}{9}\right)^{1/3}\frac{k_B{}^2m^*}{\tfrac{1}{4}(1+Z_0)},$$

where we have used the relation $p_0=(3\pi^2n)^{1/3}$.

[8] D. Hone, Phys. Rev. **121**, 669 (1961).

[9] H. B. G. Casimir, Physica, **5**, 495 (1938).

[10] In the presence of a temperature gradient we should also include in $\mathbf{A}_{\mathbf{p}\sigma}$ the term linear in $\nabla_r\mu$. The neglect of this term, however, introduces a relative error of order $(k_BT/\mu)^2$, as can be checked by invoking the constraint on the thermal conductivity that there should be no net QP current.

[11] J. C. Wheatley in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Co., Amsterdam, 1966).

3. KNUDSEN FLOW

In the collisionless regime mass flow through the channel will not be characteristic of ordinary viscous flow as the rarity of inter-QP collisions prevents the establishment of an organized hydrodynamic flow of the Poiseuille type. Resistance to flow is, instead, completely determined by the scattering of quasiparticles by the walls of the channel. Thus, in order to maintain a constant net drift velocity along the axis of the channel it will be necessary to apply an appropriate external pressure gradient to compensate the rate of QP momentum loss to the walls that may be caused by any finite degree of diffuse scattering. This is the phenomenon of Knudsen flow, originally investigated by Knudsen[6] for a highly rarefied classical gas.

A theory of this effect in a Fermi liquid can be given on the same lines as the theory of the previous section in which the nonequilibrium QP distribution function $g_{p\sigma}(\mathbf{r})$ is determined by Eq. (2.6) and (2.8). For an externally applied pressure gradient, $\partial p/\partial \mathbf{r}$, maintained along the axis of the channel at constant temperature, (2.6) takes the form

$$\mathbf{v}_{p\sigma}\cdot\nabla_{\mathbf{r}}g_{p\sigma}=\frac{\partial n_F(\epsilon_{p\sigma}^0)}{\partial\epsilon_{p\sigma}^0}\left(\frac{\partial\mu}{\partial p}\right)_T\mathbf{v}_{p\sigma}\cdot\frac{\partial p}{\partial\mathbf{r}}$$

$$=\frac{1}{n}\frac{\partial n_F(\epsilon_{p\sigma}^0)}{\partial\epsilon_{p\sigma}^0}\mathbf{v}_{p\sigma}\cdot\frac{\partial p}{\partial\mathbf{r}} \tag{3.1}$$

$$\equiv\mathbf{A}_{p\sigma}\cdot\mathbf{v}_{p\sigma}, \tag{3.2}$$

where we have made use of the thermodynamic identity $(\partial p/\partial\mu)_T=n$. This has a solution of the form (2.9) for the case of completely diffuse scattering, $\nu=0$. In this case the rate at which mass is transferred through the channel is given by

$$G=m\sum_\sigma\int\frac{d^3p}{(2\pi)^3}\int\frac{d\Omega_B}{4\pi}\int d\mathbf{S}\cdot\mathbf{v}_{p\sigma}g_{p\sigma}(\mathbf{r}-\mathbf{r}_B), \tag{3.3}$$

where m denotes the mass of a bare fermion. This expression has been written down on the basis that the mass flux in the Fermi liquid is given by m times the QP number flux, i.e. we are invoking here the one to one correspondence between quasiparticles and bare fermions.[3] Substituting in (3.3) the expression for $g_{p\sigma}$ which results from (2.9), (3.1) and (3.2) and performing the integration over $\mathbf{p}$ we obtain

$$G=-\frac{S\Lambda_B m}{p_0}\left|\frac{\partial p}{\partial\mathbf{r}}\right|, \tag{3.4}$$

where Λ_B is defined by Eq. (2.15). As with D and K in Sec. 2, the result [Eq. (3.4)] for G is generalized to the case $\nu\neq 0$ by simply substituting $[(1+\nu)/(1-\nu)]\Lambda_B$ in place of Λ_B.

For a perfectly cylindrical channel of diameter d we have $\Lambda_B=d^9$ so that

$$G=-\frac{\pi d^3m}{4p_0}\left(\frac{1+\nu}{1-\nu}\right)\left|\frac{\partial p}{\partial\mathbf{r}}\right|. \tag{3.5}$$

Thus in the collisionless, or Knudsen regime, mass is discharged from a cylindrical channel at a temperature-independent rate proportional to the *cube* of the channel diameter; in the viscous, or collision-dominated regime, where Poiseuille's formula will hold for G,[1] G varies as the fourth power of d. Like the corresponding result for the thermal conductivity, [Eq. (2.16)], the above result [Eq. (3.5)] for G is coincident with that of a non-interacting Fermi gas and is actually the Fermi gas analog of the result originally obtained by Knudsen[6] for a rarefied classical gas.[12]

4. INTERMEDIATE REGIME AND APPLICATION TO LIQUID He³

A phenomenological description of the effects of inter-QP collisions can be included in the transport equation (2.1) by adding to the right hand side of Eq. (2.1) a suitable relaxation time term.[13] When boundary effects are negligible the solution of this transport equation leads to expressions (2.18) for D_0 and K_0. In the intermediate regime where the inter-QP mean free paths λ_D and λ_K are comparable in magnitude to the channel diameter d, the methods used in the previous sections can be easily extended to give solutions for K and D that are of the same form as (2.18) but with mean free paths Λ_K and Λ_D which, although in detail are rather complicated functions of ν and d/λ_K and ν and d/λ_D, respectively,[2] can be well approximated by[14]

$$\Lambda_K^{-1}=\Lambda_B^{-1}+\lambda_K^{-1} \tag{4.1}$$

$$\Lambda_D^{-1}=\Lambda_B^{-1}+\lambda_D^{-1}, \tag{4.2}$$

where Λ_B is the geometrical mean free path encountered previously for boundary scattering. Equations (4.1) and (4.2) clearly correspond to treating inter-QP scattering and boundary scattering as independent scattering mechanisms. Thus, in the intermediate regime ($d\sim\lambda$), the thermal conductivity K and spin diffusion D are approximately

$$K=K_BK_0/(K_B+K_0) \tag{4.3}$$

$$D=D_BD_0/(D_B+D_0), \tag{4.4}$$

where D_B and K_B are given by (2.14) and (2.16) respectively.

We have used (4.3) and (4.4) to perform an illustrative calculation for liquid He³, supposed contained in a

[12] Knudsen's result, (Ref. 6), which was derived on the assumption of completely diffuse scattering, is given precisely by Eq. (3.5) with $\nu=0$ and p_0/m set equal to the r.m.s. molecular velocity $\bar{c}$.

[13] Reference 7, Sec. 10.

[14] See, for example, R. B. Dingle, Proc. Roy. Soc. A, **201**, 545, (1950).

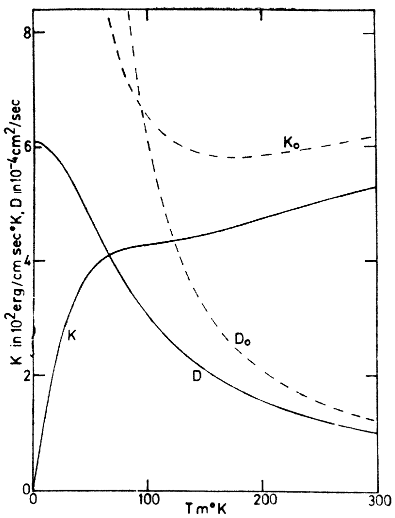

FIG. 1. K and D for liquid He^3 in a cylindrical channel of diameter 100 Å, calculated from Eq. (4.3) and (4.4) on the assumption of completely diffuse scattering. The thermal conductivity and spin diffusion, K_0 and D_0, of the infinite liquid are shown for comparison.

cylinder of radius 100 Å. Using the He^3 Fermi liquid parameters tabulated by Wheatley[11] and assuming perfectly diffuse scattering, so that $\Lambda_B = d = 100$ Å, Eqs. (2.14) and (2.16) yield $D_B = 6.1 \times 10^{-4}$ cm^2 sec^{-1} and $K_B/T = 1.29 \times 10^4$ erg cm^{-1} sec^{-1} °K^{-1} for He^3 under low pressure. We take the low-pressure data of Anderson, Connolly, Vilches and Wheatley[15] for K_0 and the low-pressure data of Anderson, Reese, Sarwinski and Wheatley[16] for D_0. The resulting computations of Eqs. (4.3) and (4.4) for K and D are shown in Fig. 1 as a function of temperature.[17] The experimental data on K_0 and D_0 is also shown for comparison.[17] The latter curves can also be considered to represent the case of completely specular scattering, $\nu = 1$, in which the boundary has zero net effect on K and D. For intermediate values of ν, the curves for K and D will lie between those shown in Fig. 1 for K and K_0 and D and D_0. The question as to how representative this calculation is for measurements of K and D on liquid He^3 in vycor glass is the subject of a present investigation.[5]

We now consider briefly the temperature dependence expected for G in the intermediate regime. As inter-QP collisions become relatively more frequent, organized hydrodynamic flow of the Poiseuille type will compete with the "collisionless" Knudsen flow discussed in Sec. 3. The competition between these two distinct types of flow will be reflected in the appearance of a "Knudsen-minimum" in the temperature dependence of G. This conclusion is based on the following qualitative argument. In the Fermi liquid region the mean free path λ decreases with increasing $T(\lambda \sim T^{-2}, T \rightarrow 0)$.[7] As T is increased slightly from absolute zero, the immediate effects of inter-QP collisions are, in view of results like Eqs. (4.1) and (4.2), to reduce the magnitude of the effective mean free path in Eq. (3.4) for G by a factor of order $(1 + \Lambda_B/\lambda)$ so that G begins to decrease with increasing temperature. G continues to decrease with increasing temperature over a temperature range for which Λ_B/λ is small by comparison to unity. In the temperature range for which $\lambda \lesssim \Lambda_B$ however, inter-QP collisions are sufficiently frequent to produce partially organized hydrodynamic flow, in which case our treatment of the transport equation in Sec. 3, which essentially assumes a spacially-independent mean drift velocity, breaks down. As T is increased further, so that $\lambda \ll \Lambda_B$ the hydrodynamic, or viscous, flow predominates and, by Poiseuille's formula,[1] G becomes inversely proportional to the viscosity coefficient η i.e., inversely proportional to λ. Thus G now increases with increasing T. G therefore has passed through a minimum $G(T_0)$ in the transition region $\lambda \sim \Lambda_B$. For liquid He^3 in vycor glass, where $d \sim 100$ Å, $T_0 \sim 100$ m°K if perfectly diffuse scattering is assumed.

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[15] A. C. Anderson, J. I. Connolly, O. E. Vilches, and J. C. Wheatley, Phys. Rev. **147**, 86 (1966); J. I. Connolly, thesis, University of Illinois, 1965 (unpublished).

[16] A. C. Anderson, W. Reese, R. J. Sarwinski, and J. C. Wheatley, Phys. Rev. Letters **7**, 220 (1961).

[17] The datas on D_0 and K_0 used here (Refs. 16, 15) were found to be well represented by the formulas $1/D_0T^2 = \alpha_D w_D(T/\theta_D)$, $1/K_0T = \alpha_k w_k(T/\theta_K)$ [where the functions $w_i(t)$ are defined explicitly in M. J. Rice, Phys. Rev. **159**, 153 (1967)], with $\alpha_D = 0.75 \times 10^6$ sec cm^{-2} °K^{-2}, $\theta_D = 0.56$°K, $\alpha_K = 2.5 \times 10^{-2}$ cm sec erg^{-1} and $\theta_K = 1.04$°K. For convenience only, these formulas were used in computing Eqs. (4.3) and (4.4) and for representing K_0 and D_0 in Fig. 1.