# Photon Bremsstrahlung from an Extreme-Relativistic Electron Gas*

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The photon bremsstrahlung spectrum from an extreme-relativistic electron gas is obtained. This gives for the total energy radiated per unit space-time volume

$$W_{ee}(\tau)=24\, n^2\, mc^3\, r_0^2\alpha\tau\, [\ln 2\tau-C+5/4],$$

where $\tau\equiv kT/mc^2\gg 1$, and $C$ is Euler's constant. The leading term is exactly *twice* that obtained when considering electron-ion bremsstrahlung and is seen to represent the fact that in electron-electron bremsstrahlung, as contrasted with electron-ion bremsstrahlung, *both* particles radiate.

## I. INTRODUCTION

THE problem of the electromagnetic radiation emitted from plasmas has been studied by several authors in the past few years.[1–5] The method of solution depends, of course, on the temperature region of interest. For example, at low temperatures ($kT\ll mc^2$, where $m$ is the mass of the electron) the most important source of radiation is that due to electron-ion interactions. However, at higher temperatures, the radiation due to electron-electron interactions becomes of considerable importance. It is clear that at these higher temperatures the behavior of the particles is governed by relativistic laws. Consequently, the kinematics of the particles, as well as the dynamics of the interactions between them must be treated relativistically.

In the present work the photon bremsstrahlung due to extreme-relativistic electron-electron interactions is studied. In order to derive the photon spectrum emitted by a high-temperature ($kT\gg mc^2$) electron gas represented by a Maxwell-Boltzmann equilibrium distribution, one must obtain first the bremsstrahlung cross section for high-energy electrons in an *arbitrary* frame of reference. The existing cross sections for electron-electron bremsstrahlung are usually given in specific frames of reference (center-of-mass system or the frame of reference where one of the electrons is at rest) and are, thus, of no avail.

## II. BREMSSTRAHLUNG CROSS SECTION FOR HIGH-ENERGY ELECTRONS

As will become evident in Sec. III, the derivation of the photon spectrum emitted by an electron gas at high temperatures requires a knowledge of the bremsstrahlung cross section for energetic electrons in an arbitrary frame of reference. This result is not available in the literature but can readily be derived from existing works.

The differential cross section for electron-electron bremsstrahlung, to lowest order in perturbation theory, has been obtained; but, owing to its complexity, integrations of it to obtain cross sections are rather cumbersome and have not been done.[6] However, nonrelativistic and extreme-relativistic limits of the result may be obtained[7] and are more tractable and, therefore, amenable to work.

Baier, Fadin, and Khoze[7] have derived the differential cross section for electron-electron bremsstrahlung, in the extreme-relativistic case, directly in terms of invariants. Therefore, their result is suitable for different reference frames. The contribution to the differential cross section, due to the radiation of *one* of the electrons (diagrams shown in Fig. 1, Ref. 7), is given, for $\nu$ large, by

$$d\sigma=d\sigma_l+d\sigma_{nl} \tag{1}$$

with

$$d\sigma_l=\frac{r_0^2\alpha}{\nu^3}\frac{dK_1}{K_1^2}dK_2L\left\{2\nu(\nu-K_2)+K_2^2+\frac{2}{K_1}K_2(K_2-\nu)+\frac{K_2^2}{K_1^2}\left(1-\frac{K_2}{\nu}\right)\right\}, \tag{2}$$

$$d\sigma_{nl}=\frac{r_0^2\alpha}{\nu^3}\frac{dK_1}{K_1^2}dK_2\left\{\frac{1}{K_1^2}\left(1-\frac{K_2}{\nu}\right)\times[-3K_1^2\nu^2+K_1K_2\nu(8+K_1)-4K_2^2]-\nu^2\right\}, \tag{3}$$

$$L=2\ln\left\{2\nu\left[\frac{\nu}{(K_1+K_2)}-1\right]\right\}, \tag{4}$$

and

$$K_1=k\cdot p_1,\quad K_2=k\cdot p_2,\quad \nu=p_1\cdot p_2, \tag{5}$$

where $r_0$ is the classical electron radius and $\alpha$ is the fine-structure constant. The metric is such that the product

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[1] J. Kvasnica, Czech. J. Phys. **B10**, 14 (1960).
[2] J. Kvasnica, Czech. J. Phys. **B10**, 261 (1960).
[3] J. Stickforth, Z. Physik **164**, 1 (1961).
[4] M. S. Maxon and E. G. Corman, Phys. Rev. (to be published).
[5] Further references may be found in Refs. 1–4.

[6] There are some recent attempts of numerical integrations. S. M. Swanson, Phys. Rev. **154**, 1601 (1967); K. J. Mork, *ibid.* **164**, 1065 (1967). Reference to earlier work may be found in these references.
[7] V. N. Baier, V. S. Fadin, and V. A. Khoze, Zh. Eksperim. i Teor. Fiz. **51**, 1135 (1966) [English transl.: Soviet Phys.—JETP **24**, 760 (1967)].

of two 4-vectors $a$ and $b$ is defined by $a\cdot b\equiv a_0b_0-\mathbf{a}\cdot\mathbf{b}$. Henceforth, the system of units is used in which the velocity of light, the mass of the electron, and Planck's constant divided by $2\pi$ are taken equal to unity, i.e., $c=m=\hbar=1$.

The angular distribution of the radiation is strongly peaked in the neighborhood of $K_1=0$, i.e., in the direction of motion of the radiating particle. Now, from Eq. (5)

$$K_2=\omega E_2(1-\hat{k}\cdot\mathbf{v}_2)$$

$$\approx\omega E_2\left(1-\frac{\mathbf{p}_1}{E_1}\cdot\frac{\mathbf{p}_2}{E_2}\right)=\frac{\omega}{E_1}\nu \qquad (6)$$

since $K_1\approx 0$ and $|\mathbf{p}_1|\approx E_1$ implies $\mathbf{v}_1=\mathbf{p}_1/E_1\approx\hat{k}$. Therefore, the main contribution to the integral with respect to the variable $K_1$ for fixed frequency $\omega$, i.e., fixed $K_2$ according to (6), comes from its lower limit. This lower limit is determined from the inequality[7]

$$K_2\leq K_1[\nu+(\nu^2-1)^{1/2}]\approx 2\nu K_1 \qquad (7)$$

which together with (6) gives

$$K_1\geq\omega/2E_1. \qquad (8)$$

On performing the integration over $K_1$ with only the contribution from the lower limit (8), one obtains for the cross section for *both* particles radiating

$$\frac{d\sigma}{d\omega}d\omega=4r_0^2\alpha\frac{d\omega}{\omega}\frac{E_1-\omega}{E_1}\left(\frac{E_1}{E_1-\omega}+\frac{E_1-\omega}{E_1}-\frac{2}{3}\right)\times\left(\ln 2\nu\frac{E_1-\omega}{\omega}-\frac{1}{2}\right)+(1\leftrightarrow 2). \qquad (9)$$

Expression (9) gives the electron-electron bremsstrahlung cross section in an arbitrary frame of reference, provided *both* particles have extreme-relativistic energies. One such frame of reference is the center-of-mass system (c.m.s.) in which case $\nu=2E_1^2-1\approx 2E_1^2$, where $E_1$ is the energy of the electron in the c.m.s., and the result, Eq. (9), reduces to that obtained by other authors.[7,8]

The laboratory system (lab.), where one of the electrons is initially at rest, does not satisfy the criterion set above in deriving Eq. (9). However, the first term in the right-hand side of (9), which gives the contribution due to one of the (high-energy) electrons, does agree with the contribution due to the fast particle in the lab. system. For this frame of reference, $\nu=E_1$ and the first term on the right-hand side of (9) agrees with that given by Eq. (17) in Ref. 7.

[8] G. Altarelli and F. Buccella, Nuovo Cimento 34, 1337 (1964).

## III. RADIATION SPECTRUM FROM AN EXTREME-RELATIVISTIC ELECTRON GAS

The elementary-particle process of high-energy electron-electron bremsstrahlung was studied in the preceding section. In an arbitrary frame of reference, where *both* electrons have high energies, the cross section is given by Eq. (9). This result can be used to obtain the radiation spectrum from an aggregate of electrons due to such processes.

Suppose the electron gas is described by the distribution function

$$F(E)dE=\frac{n}{\tau e^{1/\tau}K_2(1/\tau)}E(E^2-1)^{1/2}e^{-(1/\tau)(E-1)}dE, \qquad (10)$$

where $E$ is the energy of an electron and $\tau=kT$ ($\tau=kT/mc^2$ in the usual units). The function $K_2$ is the modified Bessel function of the second kind with subscript 2 and Eq. (10) is normalized according to

$$\int_1^\infty F(E)dE=n \qquad (11)$$

with $n$ representing the particle density of electrons.

Let $\sigma$ denote the total cross section per unit scattering center, and $\rho_1$ and $\rho_2$ the particle densities of the incident and target particles (at rest), respectively. Then,

$$d^4N/d^4x=\rho_1\rho_2 v\sigma, \qquad (12)$$

where $v$ is the velocity of the incident particle, gives the number of scattered particles per unit space-time volume. The quantity $d^4N/d^4x$ is clearly a relativistic invariant. However, Eq. (12) is not in covariant form, i.e., both sides of the equation do not have the same transformation properties under Lorentz transformations. Consider an arbitrary frame of reference. The velocity $v$ of the incident particle becomes, in an arbitrary frame of reference, the relative velocity between the colliding particles. Since the relative velocity is a relativistic invariant [see Eq. (14)], one sees that if the total cross section is *required* to be a relativistic invariant, then the covariant form of Eq. (12) is given by[9]

$$d^4N/d^4x=j_1\cdot j_2 v\sigma, \qquad (13)$$

with

$$v=\left[\frac{(p_1\cdot p_2)^2-m_1^2m_2^2}{(p_1\cdot p_2)^2}\right]^{1/2}, \qquad (14)$$

where $j_\mu$ is the particle-current 4-vector with components $(\rho,\mathbf{v}\rho)$ and $p_\mu$ is the ordinary energy-momentum 4-vector. This manner of generalizing equations, which refer to the rest frame, to arbitrary frames of reference is a usual procedure in relativity theory.

[9] This result is the same as that given in J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Chap. 8.

The number of particles scattered per unit space-time volume into the momentum volume element $d^3p$ may be obtained from Eq. (13) and is

$$(d^6\mathfrak{N}/d^4x\,dp\,d\Omega_p)dp\,d\Omega_p = j_1\cdot j_2 v(d^2\sigma/d\Omega_p dp)dp\,d\Omega_p\,, \quad (15)$$

where $d^2\sigma/d\Omega_p dp$ is the differential cross section. Note that (15) is a relativistic invariant, thus making $(d^2\sigma/d\Omega_p dp)dp\,d\Omega_p$ also an invariant. However, $(d\sigma/dp)dp$, where $d\sigma/dp=\int(d^2\sigma/d\Omega_p dp)d\Omega_p$, is *not* an invariant so that its knowledge in a specific frame of reference (center-of-mass or lab. system) cannot be used to obtain its value in an arbitrary frame of reference.

For an electron gas with a distribution function given by Eq. (10), the spectrum of the emitted radiation per unit space-time volume is obtained from

$$P_{ee}(\omega,\tau)d\omega = \frac{1}{4[\tau e^{1/\tau}K_2(1/\tau)]^2} \times\int_\omega^\infty\int_0^\infty dp_1dp_2\,p_1^2p_2^2 \times\exp\{-[(1+p_1^2)^{1/2}+(1+p_2)^{1/2}-2]/\tau\} \times\int_{-1}^{1}d\xi\, j_1j_2v\frac{d\sigma}{d\omega}d\omega\,, \quad (16)$$

with

$$E_i=(1+p_i^2)^{1/2}\,, \quad (i=1,2) \quad (17)$$

and

$$\xi=\mathbf{p}_1\cdot\mathbf{p}_2/|\mathbf{p}_1|\,|\mathbf{p}_2|\,. \quad (18)$$

In Eq. (16), one has properly divided by a factor of 2 to take into account the double counting contained in the integrations.

In the extreme-relativistic case ($E\approx|\mathbf{p}|$, $E\gg1$), with the aid of Eq. (9), Eq. (16) becomes, after some straightforward calculations,

$$P_{ee}(\omega,\tau)d\omega = 2n^2r_0^2\alpha e^{-\omega/\tau}\frac{d\omega}{\omega} \times\left\{\frac{28}{3}+2\frac{\omega}{\tau}+\frac{1}{2}\frac{\omega^2}{\tau^2}+2\left(\frac{8}{3}+\frac{4}{3}\frac{\omega}{\tau}+\frac{\omega^2}{\tau^2}\right)\ln\frac{2\tau}{\gamma} -e^{\omega/\tau}\,\mathrm{Ei}\left(-\frac{\omega}{\tau}\right)\left(\frac{8}{3}-\frac{4}{3}\frac{\omega}{\tau}+\frac{\omega^2}{\tau^2}\right)\right\}, \quad (\tau\gg1)\,, \quad (19)$$

where Ei $(-x)$ is the exponential-integral function defined by

$$\mathrm{Ei}(-x)=-\int_x^\infty\frac{e^{-t}}{t}dt \quad (20)$$

and $\gamma=e^C$. ($C$ = Euler's constant.)

One noticeable difference between the spectrum (19) and that due to electron-electron bremsstrahlung in the nonrelativistic case ($\tau\ll1$) [1,4] is that in the latter case once the spectrum is known at one temperature and *all* frequencies, the spectrum is known for *any* value of the frequency and temperature.[10] This is no longer true in the extreme-relativistic case ($\tau\gg1$) obtained above.

## IV. PARTIAL AND TOTAL ENERGY RADIATED

The extreme-relativistic electrons of the gas considered in the preceding section undergo several types of collisions. Once the radiation is generated, the most important processes are electron-electron elastic scattering, electron-photon scattering (Compton scattering), and electron-electron bremsstrahlung (together with its inverse). Of these three, the first two are assumed, on the average, to cause no net change of the gas, i.e., the electron and photon distributions remain unchanged. Hence, the net effect of these interactions is the gradual loss of energy by the electron gas, this due to the creation and subsequent loss of radiation.

Consider the energy loss per unit space-time volume due to photons with energy below frequency $\omega$. This is given by

$$E_{ee}(\omega,\tau)=\int_0^\omega \omega' P_{ee}(\omega',\tau)d\omega'\,. \quad (21)$$

With the aid of (19) and after some lengthy but straightforward calculations one obtains

$$E_{ee}(\omega,\tau)=2n^2r_0^2\alpha\tau\left[15+12\ln\frac{2\tau}{\gamma} -e^{-\omega/\tau}\left(15+3\frac{\omega}{\tau}+\frac{5}{6}\frac{\omega^2}{\tau^2}\right) -e^{-\omega/\tau}\left(12+\frac{20}{3}\frac{\omega}{\tau}+2\frac{\omega^2}{\tau^2}\right)\ln\frac{2\tau}{\gamma} -\frac{\omega}{\tau}\,\mathrm{Ei}\left(-\frac{\omega}{\tau}\right)\left(\frac{8}{3}-\frac{2}{3}\frac{\omega}{\tau}+\frac{1}{3}\frac{\omega^2}{\tau^2}\right)\right]. \quad (22)$$

The total energy loss per unit space-time volume follows readily from Eq. (22) in the limit $\omega\to\infty$ and gives

$$W_{ee}(\tau)=24n^2r_0^2\alpha\tau[\ln(2\tau/\gamma)+5/4]\,, \quad (\tau\gg1)\,. \quad (23)$$

The leading term in Eq. (23) is precisely *twice* that due to the leading term in the electron-ion radiation energy $W_{ei}(\tau)$.[3] *A posteriori*, this result is to be expected since the nuclear bremsstrahlung cross section for unit charge is a good high-energy limit to the radiation cross section for electron-electron bremsstrahlung due to the *primary* electron. The additional factor of 2 corresponds to *both* particles radiating in electron-electron bremsstrahlung, whereas only the electron radiates in the electron-ion case.

[10] Provided, of course, that $\tau\ll1$.

Result (23) for the total energy emitted per unit space-time volume differs from the results derived by other authors.[2,3] In the extreme-relativistic case, Kvasnica[2] obtained

$$W_{ee}(\tau)=12n^2r_0^2\alpha\tau[\ln(2\tau/\gamma)+\tfrac{5}{6}], \quad (\tau\gg1) \quad (24)$$

by averaging over the electron distribution the expected total energy radiated for a specific energy of the incident electron. This calculation is done in the frame of reference where one of the electrons is at rest and, needless to say, is appropriate to the electron-ion problem where the ion is assumed to be nearly at rest and the particles concerned (ions and electrons) are different. However, in the electron-electron case both of these features are no longer true and a more careful analysis is required. (See below.)

A second result for $W_{ee}(\tau)$ differs even more drastically from the one given in the present work. Stickforth[3] finds

$$W_{ee}(\tau)=36n^2r_0^2\alpha\tau^2[\ln(2\tau/\gamma)+0.967], \quad (\tau\gg1). \quad (25)$$

Besides a difference in an over-all numerical factor, this result differs from Eq. (23) in the functional dependence of $W_{ee}(\tau)$ on the temperature. At very high temperatures, Eq. (25) gives a radiation loss which is far greater than that due to electron-ion bremsstrahlung. This behavior differs considerably from that of Eq. (23) which yields

$$\frac{W_{ee}(\tau)}{W_{ei}(\tau)}\xrightarrow[\tau\to\infty]{}2. \quad (26)$$

Consider now a direct calculation of $W_{ee}(\tau)$. The derivation of the photon spectrum emitted by an electron gas requires a knowledge of the electron-electron bremsstrahlung cross section in an arbitrary frame of reference. With the aid of this spectrum, result (23) for $W_{ee}(\tau)$ was obtained. However, for a direct calculation of $W_{ee}(\tau)$ one can use the cross section in the c.m.s. and perform the calculation more directly. This will serve as a double check of our result for $W_{ee}(\tau)$ and will confirm the correctness of our result, Eq. (19), for the photon spectrum.

Let $d^2\sigma/d\Omega d\omega$ denote the differential cross section in the c.m.s. for photon emission in electron-electron bremsstrahlung. The energy of the photon in an arbitrary frame of reference is given, in terms of center-of-mass variables, by

$$\omega'=\frac{\omega}{(1-v_{\text{c.m.}}^2)^{1/2}}(1+\mathbf{v}_{\text{c.m.}}\cdot\hat{k}), \quad (27)$$

where $\hat{k}$ is a unit vector in the direction of the emitted photon in the c.m.s. and $\mathbf{v}_{\text{c.m.}}$ is the velocity of the center-of-mass system as seen from the arbitrary frame of reference and is given by

$$\mathbf{v}_{\text{c.m.}}=(\mathbf{p}_1+\mathbf{p}_2)/(E_1+E_2), \quad (28)$$

where $\mathbf{p}_i(E_i)$ is the momentum (energy) of the $i$th particle in the arbitrary frame of reference. The expected total energy radiated per unit space-time volume in the arbitrary frame of reference is

$$R_{ee}(E_1,E_2,p_1\cdot p_2)=v(j_1\cdot j_2)\int\int\omega'\frac{d^2\sigma}{d\Omega d\omega}d\Omega d\omega, \quad (29)$$

where all the unintegrated quantities in the right-hand side are referred to the arbitrary frame of reference.

As discussed in Sec. II, the differential cross section is deduced from a scattering amplitude which consists of a certain set of Feynman diagrams together with the set obtained by interchanging the electron 4-momenta ($p_1\leftrightarrow p_2$). In the center of mass, this interchange results in ($\mathbf{p}\leftrightarrow-\mathbf{p}$), where $\mathbf{p}$ is the momentum of one of the electrons in the c.m.s., so that

$$\frac{d^2\sigma(\mathbf{k})}{d\Omega d\omega}=|f(\mathbf{k}\cdot\mathbf{p},\cdots)+f(-\mathbf{k}\cdot\mathbf{p},\cdots)|^2 =\frac{d^2\sigma(-\mathbf{k})}{d\Omega d\omega}. \quad (30)$$

This symmetry in the c.m.s., together with Eq. (27), simplifies the integration in Eq. (29) to

$$R_{ee}(E_1,E_2,p_1\cdot p_2)=(j_1\cdot j_2)v\int\frac{\omega d\omega}{(1-v_{\text{c.m.}}^2)^{1/2}}\times\int\frac{d^2\sigma}{d\Omega d\omega}d\Omega =\frac{(j_1\cdot j_2)v}{(1-v_{\text{c.m.}}^2)^{1/2}}\int_0^\epsilon\omega\frac{d\sigma}{d\omega}d\omega, \quad (31)$$

where $\epsilon$ is the energy of the electron in the c.m.s. and is related to the 4-momenta in the arbitrary frame by

$$\epsilon=[\tfrac{1}{2}(p_1\cdot p_2+1)]^{1/2}. \quad (32)$$

The bremsstrahlung cross section in the c.m.s. has already been derived and is[7,8]

$$\frac{d\sigma}{d\omega}d\omega=8r_0^2\alpha\frac{d\omega}{\omega}\frac{\epsilon-\omega}{\epsilon}\left[\frac{\epsilon-\omega}{\epsilon}+\frac{\epsilon}{\epsilon-\omega}-\frac{2}{3}\right]\times\left\{\ln\left[\frac{4\epsilon^2(\epsilon-\omega)}{\omega}\right]-\frac{1}{2}\right\}. \quad (33)$$

Substituting in Eq. (31) and after using Eq. (28),

$$R_{ee}(E_1,E_2,p_1\cdot p_2)=16r_0^2\alpha\epsilon\frac{(j_1\cdot j_2)v}{(1-v_{\text{c.m.}}^2)^{1/2}}[\ln 2\epsilon-\tfrac{1}{6}]$$

$$=8r_0^2\alpha(E_1+E_2)\rho_1\rho_2\frac{p_1\cdot p_2}{E_1E_2}v \times[\ln 2\epsilon-\tfrac{1}{6}]. \quad (34)$$

It now remains to average Eq. (34) over the Maxwellian distributions of the electrons:

$$W_{ee}(\tau)=\frac{1}{4[\tau e^{1/\tau}K_2(1/\tau)]^2}\int_0^\infty dp_1\int_0^\infty dp_2\, p_1^2p_2^2 \times\exp\{-[(1+p_1^2)^{1/2}+(1+p_2^2)^{1/2}-2]/\tau\} \times\int_{-1}^{1}d\xi\, R_{ee}(E_1,E_2,p_1\cdot p_2). \quad (35)$$

After some calculations, one obtains

$$W_{ee}(\tau)=24n^2r_0^2\alpha\tau[\ln(2\tau/\gamma)+5/4], \quad (\tau\gg 1), \quad (36)$$

which agrees with our previous result, Eq. (23).

In the derivation of $W_{ee}(\tau)$ [Eq. (25)] in Ref. 3, the author is aware of the symmetry (30) for the differential cross section in the c.m.s. However, in the actual calculation, the differential cross section used is that due to Garibyan[11] which refers to the frame of reference where one of the electrons is at rest, First, this symmetry is not satisfied in this frame of reference. Secondly, the flux in such a reference frame differs considerably from that in the arbitrary frame of reference.

[11] G. M. Garibyan, Izv. Akad. Nauk. Arm. S.S.R. **5**, No. 3 (1952).

## V. CONCLUSIONS

A systematic study has been presented of electron-electron bremsstrahlung for high-energy electrons. The cross section is derived in an arbitrary frame of reference where *both* electrons have extreme-relativistic energies. When evaluated in the center-of-mass system, the obtained cross section reduces to that derived by other authors. Also, when properly evaluated, it agrees with that, also derived in the literature, given in the lab. system of reference.

With the aid of this newly obtained cross section, the problem of the radiation emitted by an electron gas in a Maxwell-Boltzmann equilibrium distribution at high temperature ($kT\gg mc^2$) is solved. It is found that the energy emitted per unit space-time volume at high temperatures is twice that due to electron-ion radiation. This makes the process of electron-electron bremsstrahlung of considerable importance at high temperatures and essential to the proper study and understanding of radiation emission from plasmas. This value for the radiated energy is found to disagree with results given by other authors. Some weak points in the analysis of these authors are presented and are the source of the disagreement.

This study of the radiation emitted by an extreme-relativistic electron gas in equilibrium, together with the work of others in the nonrelativistic region, gives a partial answer to the general problem of radiation emission from plasmas.

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