Theory of a Turbulent Stationary State of a Plasma

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We first establish a self-consistent scheme to determine the fluctuation spectrum for a class of turbulent plasmas in which a conventional linear theory predicts an exponential growth of the density-fluctuation excitations, or instability. This theoretical approach is based on the fundamental postulate that a proper inclusion of the correlation effects or the existence of the fluctuations in the stationary state should be able to remove the instability and lead to a description of the most stable state for the turbulent plasma. A dielectric response function $\epsilon(\mathbf{k},\omega)$ for such a turbulent plasma is calculated within the validity of the hydrodynamic description. In this calculation, there are involved various polarization processes associated with the interaction between the external test charge (or the induced fluctuations) and the turbulent fluctuations existing in the plasma; the nature of those processes is clarified with the aid of diagrammatic considerations. The fluctuations of the internal electric field give rise to an additional mechanism for particle diffusion; the effective diffusion coefficient in the turbulent plasma is obtained by investigating the behavior of $\epsilon(\mathbf{k},\omega)$ in the limit of long wavelengths and low frequencies. The effects of turbulence upon the properties of the ion acoustic wave are studied. Following the self-consistent scheme, an integral equation is derived for the fluctuation spectrum associated with the acoustic mode; it is then solved for values of the electron drift velocity $\mathbf{V}_d$ above and below the critical one, V_c. We thus find that the results indeed support our original postulate, and the dielectric response function remains stable for the entire range of the drift velocity. The over-all structure of the fluctuation spectrum is investigated. In terms of the small plasma parameter, $g\equiv 1/n_0\lambda_D^3$, the energy $\epsilon(\mathbf{k})$ in the acoustic mode with wave vector $\mathbf{k}$ is of the order of g (i.e., around the thermal level) in the stable region; as the plasma enters the transition region, $\hat{k}\cdot\mathbf{V}_d\simeq V_c$, the order of $\epsilon(\mathbf{k})$ goes up to $g^{1/2}$; in the turbulent region, $\epsilon(\mathbf{k})$ contains a part of the order of g^0. It is also shown that a certain domain of the turbulence spectrum can be explained with the aid of a dimensional argument. The results of the calculations are compared with a fluctuation spectrum measured by a microwave scattering experiment.

I. INTRODUCTION

SEVERAL years ago, Pines, Rostoker, and one of the authors[1,2] developed a theory of critical fluctuations in a plasma. According to this theory, an enormous enhancement of density fluctuations is predicted in the plasma, when it approaches from a region of stability a critical point corresponding to the onset of an instability. Such an enormous increase of the density fluctuations has been subsequently observed by means of microwave scattering experiments.[3]

When an instability sets in, the plasma goes over to a turbulent state; in this domain, the above theory of critical fluctuations is no longer applicable. The principal problems now will be, for example: How does the turbulence develop and approach equilibrium in a plasma?—or—what is the turbulent state of a plasma? In this connection, it may be significant to note the fundamental distinction which one must make between the following two cases of turbulence problems.

One is what may be called an *initial value problem.* Here, the plasma system is *isolated* from the external energy source; initially, however, it is characterized by physical conditions such that a certain collective mode can grow exponentially. Experimentally, a situation corresponding to this case may, for example, be realized in the initial stage of a beam-plasma interaction experiment when a pulsed beam of charged particles is injected into a quiescent plasma. The theoretical problem then is to analyze the time development, or approach to equilibrium, of the combined system of particles and oscillations, starting from such unstable initial conditions. An important progress has been made toward the solution of the problems in this category by the advent of the quasilinear theory of plasma oscillations[4]; this approach takes explicit account of the feedback action of the growing oscillations upon the single-particle distribution function and treats the wave-wave interaction in a perturbation-theoretical way.

Consider now a second class of turbulence problems, which arise when a system maintains connection with an external source and a sink of energy. Under these circumstances, if we wait long enough, the plasma may reach a new kind of stationary state: a *turbulent stationary state.* Experimental examples pertaining to this case may be found in various plasma phenomena, including the positive column of a glow discharge and

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[1] S. Ichimaru, D. Pines, and N. Rostoker, Phys. Rev. Letters **8**, 231 (1962).

[2] S. Ichimaru, Ann. Phys. (N. Y.) **20**, 78 (1962).

[3] V. Arunasalam and S. C. Brown, Phys. Rev. **140**, A471 (1965).

[4] W. E. Drummond and D. Pines, Nucl. Fusion Suppl. **3**, 1049 (1962); A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, *ibid.* **2**, 465 (1962).

solid-state plasma experiments. In such a state, a steady flow of energy is established through interactions among particles and oscillations. We may then ask: What is the characteristic feature of such a turbulent stationary state, particularly if the state is characterized by those physical parameters for which a conventional linear theory predicts an exponential growth of oscillations, or instability? The purpose of the present paper is a theoretical study of such a turbulent stationary state for the plasma.[5]

The general structure of the energy flow pattern established in such a turbulent stationary plasma may be understood in the following way. First of all, we consider a plasma which is kept in the so-called unstable physical conditions by a certain external means (e.g., by application of constant electric field above its critical value); the external source thus feeds energy constantly into the charged particles. A part of this energy goes to the excitation of oscillations via the wave-particle coupling mechanism which causes the instability; the remainder will be lost to the environment through collisions. The large amplitude oscillations thus built up in the plasma interact frequently with each other so that a continuous flow of energy toward the large wave-number region is established; the oscillation energy is eventually dissipated into heat by collision damping when the wave number steps out of the domain of instability. A stationary state may thereby be set up in the plasma and a continuous flow of energy is maintained in it. In this paper we wish to take up this sort of stationary-state problem; we shall see in the following analysis that all those physical features are borne out in our calculations. In particular we shall find that a certain domain of the turbulence energy spectrum may be interpreted with the aid of a dimensional argument of Kolmogorov-Obuknov type.

Existence of frequent collisions among large amplitude oscillations may also make it possible to consider a new mode of wave propagation of second-sound type, as first suggested by Pines. We shall show in the subsequent paper[6] that propagation of a second-sound wave is indeed possible in a turbulent plasma.

In Sec. II, we discuss in some detail the fundamental considerations which lead us to establish a self-consistent scheme of determining the fluctuation spectrum in a turbulent stationary plasma. As a first step along the self-consistent approach we consider in Sec. III the dielectric response function $\epsilon(\mathbf{k},\omega)$ for a single-component turbulent plasma within the hydrodynamic description; the various polarization processes which lead to the dielectric response function are studied with the aid of diagrammatic techniques in Sec. IV. Section V extends the calculation of $\epsilon(\mathbf{k},\omega)$ to the cases of two-component plasmas. The behavior of $\epsilon(\mathbf{k},\omega)$ in the limit of long wavelengths and low frequencies is studied and the effective diffusion coefficient in a turbulent plasma is obtained in Sec. VI. The nature of ion acoustic waves in the turbulent plasma is investigated in Sec. VII; an integral equation for fluctuations associated with the acoustic mode is derived and solved in Sec. VIII. Section IX contains physical discussions on the assumptions involved and an interpretation of the spectrum in terms of the dimensional consideration; a comparison of our result with the fluctuation spectrum measured by a microwave scattering experiment is given in Sec. X. The dielectric response function for a turbulent plasma in a magnetic field is treated in Appendix A; some of the calculational details are given in Appendices B and C.

II. FUNDAMENTAL CONSIDERATIONS

For a theoretical study of the stability of a plasma, a first step that one naturally takes is to specify the unperturbed stationary state of the plasma. In a hydrodynamic analysis, this is usually accomplished by assigning specific values to hydrodynamic or thermodynamic parameters such as the density, flow velocity, and temperature; in a kinetic theoretical treatment, one ordinarily uses a more refined description of the plasma by means of single-particle distribution functions. One then applies a weak external disturbance and studies the characteristic response of the system; if a certain growing disturbance is found possible in the system, the plasma is said to be unstable against that particular kind of disturbance. If the state of the plasma is kept unchanged by a suitable external means, this would imply that the plasma would collapse by endless development of such disturbances in itself.[7]

In reality, however, we encounter various examples of plasmas in which a stationary state is maintained even under the so-called unstable circumstances; although such a plasma is generally "noisy" and frequently accompanied by anomalous transport phenomena,[8] we must regard such a state as *stable* because it is realized and sustained in a stationary way. We therefore seem to be faced with a gap between physical reality and what theory would indicate about its stability.

In order to resolve this gap, it may be instructive to go back and recall the physical significance of the stability analysis; it is clear that all that a stability analysis can tell us is whether or not the particular state *originally specified* is stable against external disturbances; if a different state is chosen, a different stability criterion will result. The seeming discrepancy between physical reality and a theoretical analysis may therefore be

[5] A preliminary account of the present theory was reported in S. Ichimaru and T. Nakano, Phys. Letters **25A**, 163 (1967).

[6] S. Ichimaru, following paper, Phys. Rev. **165**, 251 (1968).

[7] We remark here on an important difference between the quasilinear calculation and the present theory: In the quasilinear theory, the state of plasma, being isolated from the external energy source, is subsequently modified toward stability by the feedback action of the growing oscillations.

[8] See, for example, F. C. Hoh and B. Lehnert, Phys. Fluids, **3**, 600 (1960).

traced simply to the inappropriateness of one's original selection of the stationary state; a better choice would lead to a stable description of the plasma, in accord with the actual physical observations.

How do we then find a new stationary state which should be appropriate to describe the true situation? A clue to this problem is already apparent if we look more closely at what the theory of critical fluctuations[1,2] indicates. In this analysis we started with the given velocity distribution functions to characterize the state of the plasma; fluctuations, or space-time correlations between the physical variables, are implicitly neglected in this description of the plasma state. When the plasma is in thermodynamic equilibrium, it is well known that the fluctuations are so small that one can legitimately disregard their effects on the properties of the system; under these circumstances a stability analysis which ignores the presence of fluctuations in the stationary state can be well justified. However, when the plasma approaches from the region of stability a critical point corresponding to the onset of an instability, there occurs an enormous increase of fluctuations above the thermal level; the amplitude of fluctuations would seem to diverge to infinity at the critical point. It is in this divergent behavior of density fluctuation that we sense a danger signal which points to the inadequacy of our original specification of the stationary state by means of the single-particle distribution functions only; the tremendous enhancement of the fluctuations should be regarded as a signal which demands that one must take a proper account of the existence of fluctuations (or correlations between the physical variables) in order to describe correctly the properties of the plasma in the vicinity of the critical point and in the turbulent region.

In fact we may find a number of examples in statistical physics in which inadequacy of a theoretical treatment is signaled by the onset of an instability; a proper inclusion of correlation effects can in many cases remove such an instability and lead to a correct description of the ground state of the system, an aspect which has been so clearly demonstrated in the theory of superconductivity,[9] for example.

We now make a fundamental assumption: To the extent that a stationary plasma may be realized, it should be possible to determine a state which is *stable* against weak external perturbations even though the plasma be *turbulent*. With the aid of this fundamental assumption, we may then establish the following self-consistent scheme for calculating spectral functions of fluctuations in a turbulent plasma: (1) Assume a stationary state of a uniform turbulent plasma which may be characterized by the existence of finite fluctuation spectra superposed on an ordinary quiescent stationary state; the amplitude of the spectral function $S(\mathbf{k},\omega)$ is left undetermined at this stage. (2) Apply a weak perturbing field to this turbulent state, and calculate various linear response functions. (3) Make use of the dielectric superposition principle for a nonequilibrium plasma,[10,11] to write the fluctuation spectrum in terms of the above response functions. (4) Those response functions in turn contain the spectral functions of fluctuations; our final step is to solve the resulting self-consistent equation for $S(\mathbf{k},\omega)$.

We remark that the present approach is nonperturbative in its nature; time secularities involved in the individual terms of the perturbation solution[12] are simply absent at the onset of our treatment. In this scheme, we can pass smoothly from a stable region to a turbulent region. In the stable region, the $S(\mathbf{k},\omega)$ thus calculated will turn out to be so small in magnitude that the resulting corrections are negligible; the fluctuation spectrum will be essentially equivalent to the one obtained from a quiescent calculation. In the turbulent region, the stationary state is characterized by macroscopic intensities of fluctuations associated with certain modes of oscillation.

The entire analysis contained in this paper is based upon a certain set of hydrodynamic equations, representing macroscopic moment equations of conserved quantities; the analysis therefore fails to describe such delicate microscopic phenomena as the resonant interaction between the particles and the oscillations. Instead, the basic equations contain phenomenological constants which measure the rates of momentum relaxation for the charged particles; growth or damping of the collective oscillations due to their interaction with other particles is thus described in our analysis in terms of those relaxation rates.

Such a hydrodynamic treatment may be contrasted with a kinetic theoretical approach starting from the Vlasov equation. We remark that, while a kinetic theoretical analysis is capable of handling much detailed information concerning the microscopic properties of the plasma, the theoretical basis of the Vlasov equation becomes rather questionable when it is applied to the description of plasma turbulence. It is well known[13] that the BBGKY hierarchy of plasma kinetic equations can be truncated if one makes use of an expansion procedure with respect to the discreteness parameters (e, m, and $1/n$), or equivalently, the small plasma parameter, $g\equiv 1/n\lambda_D{}^3$, i.e., the reciprocal of average number of

[9] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[10] This principle has been established for quite some time in the literature (Ref. 11), and will be briefly discussed later in this section. The name "dielectric superposition principle" is first proposed here.

[11] P. Nozières and D. Pines, Phys. Rev. **109**, 762 (1958); Nuovo Cimento **9**, 470 (1958); W. B. Thompson and J. Hubbard, Rev. Mod. Phys. **32**, 714 (1960); S. Ichimaru, Phys. Rev. **140**, B226 (1965); see, in particular, D. Pines and P. Nozières, *The Theory of Quantum Liquids* (W. A. Benjamin, Inc., New York, 1966), pp. 204–215.

[12] E. Frieman and P. Rutherford, Ann. Phys. (N. Y.) **28**, 134 (1964).

[13] N. Rostoker and M. N. Rosenbluth, Phys. Fluids **3**, 1 (1960).

charged particles contained in a Debye cube; the Vlasov equation is then obtained from the lowest-order equations in g by making an ansatz, equivalent to the Hartree factorization, that a many-particle distribution function is expressed as a simple product of single-particle distribution functions. Implicit in this ansatz is an assumption that the sizes of the higher-order correlation functions are of higher order in g as compared with that of the single-particle distribution function and thus negligible when only the lowest-order contributions are retained. This assumption can be well justified for a quiescent plasma, in which the fluctuations remain in the vicinity of the thermal level; the fluctuation spectrum and the space-time pair correlation are connected with each other via the Fourier transformation. When the plasma goes over to a turbulent state, however, there is no general guarantee that the level of fluctuations will remain small; on the contrary, we may expect that certain modes of fluctuations may grow to such an extent that the Hartree factorization of the many-particle distribution functions breaks down. It is perhaps only after imposing careful limitation on the sizes of higher-order correlation functions[12] that one can make meaningful use of an expansion procedure.

In connection with the plasma parameter expansion of the hierarchy equations, it may also be important to remark on the applicabilities and the differences in the two kinds of superposition principles which may be used for the calculation of the fluctuations in the plasma.

Starting from the hierarchy equations which take account of the first-order contributions in g, Rostoker[14] has given an elegant proof that there exists a superposition principle for the calculation of the pair correlation function which involves the single-particle distribution functions and conditional probabilities determined from the Vlasov equation. This proof depends on the assumption concerning the ordering of the correlation functions which we have just pointed out in connection with the use of the Vlasov equation; it therefore follows that the superposition principle in the form proved by Rostoker (we shall call it the "hierarchy" superposition principle) is not strictly applicable for a turbulent plasma.

In our subsequent calculations, we shall evoke another superposition principle, which superficially looks similar to the above in some cases but differs significantly in its origin and therefore in its applicability; we shall call it the "dielectric" superposition principle. Historically, this was proposed earlier[11] than the hierarchy superposition principle. Nevertheless, the difference between the two has not been well recognized hitherto in the literature. It is particularly in the treatment of a turbulent plasma that the difference plays an essential role.

[14] N. Rostoker, Nucl. Fusion **1**, 101 (1961); Phys. Fluids **7**, 479, 491 (1964).

The dielectric superposition principle is based on the observation that for each many-particle system consisting of charged particles, one can imagine a fictitious neutral counterpart which may be constructed by adiabatically turning off the long-range part of Coulomb interaction between the particles in the real system; the turning off can be achieved by subtracting the average self-consistent field of each particle, and thus the screened short-range Coulomb forces remain in the resulting fictitious system. The matrix elements of the density fluctuation excitations in the fictitious system are then given by the product of those in the real charged system and the dielectric response function [see Eq. (4.45) in the book by Pines and Nozières[11]]. We remark that both the density fluctuations in the real system and the dielectric response function are always physically well defined quantities; the above statement may thus be regarded as a mathematical definition of the fictitious system. We thereby separate the calculation of fluctuations in the Coulomb interacting many-particle system artificially into two parts: the calculation of the fluctuation spectrum in the fictitious system and that of the dielectric response function in the real system.

So far it might appear that we have been successful only in dividing a difficult problem into two difficult ones. We now wish to illustrate the significant gain which may be achieved when we apply the dielectric superposition principle to a plasma turbulence problem.

First of all we note that while the real plasma system may be turbulent, the fictitious counterpart cannot be so, because by its construction the latter system is devoid of the long-range Coulomb interaction which is responsible for the instability. The fluctuation spectrum in the fictitious system should therefore be insensitive to the anomalies which generally accompany the onset of turbulence; although we know of no rigorous theoretical method to calculate it for a given nonequilibrium stationary plasma, we may well expect that any reasonable evaluation of the fluctuation spectrum for the fictitious system should provide a sufficient basis for the calculation of the turbulence spectrum.

The complexity arising from the plasma turbulence greatly affects the calculation of the dielectric response function. We emphasize here an important difference involved in this connection between the hierarchy superposition principle and the dielectric superposition principle: In the former scheme, the dielectric response function should inevitably be the one determined from the solution of the Vlasov equation; in the latter, it must be the one which describes the ***true*** density response of the system against an external test charge. For a turbulent plasma, an external test charge introduced in the system will induce fluctuations not only from the average quiescent background but also from the turbulent fluctuations, owing to the nonlinear coupling. The induced fluctuations can couple again with the background or the turbulence; such polarization processes thus proceed endlessly, and the central problem involved in the calculation of the dielectric response function is to find a way to take account of those higher-order interaction processes in as meaningful a way as possible. In the following treatments we shall show that it is possible to sum an important subset of all those higher-order interaction processes with the aid of relatively simple techniques; we shall thereby find a stable dielectric response function which describes the turbulent stationary state of the plasma.

III. DIELECTRIC RESPONSE FUNCTION FOR A TURBULENT ELECTRON GAS

As a first step in the self-consistent approach described in the previous section, let us consider the dielectric response function for a weakly ionized turbulent plasma within the region of validity of the hydrodynamic description.[15,16] For the sake of simplicity, we begin with a single component plasma, or the electron gas, in the absence of the magnetic field; the dielectric response function for a turbulent plasma in a uniform external magnetic field will be considered in Appendix A.

The basic equations are the equation of continuity, the equation of diffusion, and Poisson's equation:

$$(\partial/\partial t)n(\mathbf{r},t)+\nabla\cdot\Gamma(\mathbf{r},t)=0, \tag{3.1}$$

$$(\partial/\partial t)\mathbf{\Gamma}(\mathbf{r},t)=-(1/\tau)[\mathbf{\Gamma}(\mathbf{r},t)+D\nabla n(\mathbf{r},t)+\mu n(\mathbf{r},t)\mathbf{E}(\mathbf{r},t)], \tag{3.2}$$

$$\nabla\cdot\mathbf{E}(\mathbf{r},t)=-4\pi e[n(\mathbf{r},t)-n_0]. \tag{3.3}$$

Here, $n(\mathbf{r},t)$ and $\mathbf{\Gamma}(\mathbf{r},t)$ represent the local density and flux of the electrons, $\mathbf{E}(\mathbf{r},t)$ is the total electric field, D and μ are the diffusion coefficient and mobility, τ is the relaxation time of the electrons due to short-range collisions, and n_0 denotes the constant mean value of $n(\mathbf{r},t)$. We also note the following basic relations:

$$\mu=e\tau/m, \tag{3.4}$$

$$D=\mu\kappa T/e=\tau\kappa T/m, \tag{3.5}$$

where m is the mass of an electron, T is the temperature, and κ is the Boltzmann constant. We eliminate $\mathbf{\Gamma}(\mathbf{r},t)$ from (3.1) and (3.2) to obtain

$$(\partial/\partial t)(\partial/\partial t+1/\tau)n(\mathbf{r},t)-(D/\tau)\nabla^2 n(\mathbf{r},t)-(e/m)\nabla\cdot[n(\mathbf{r},t)\mathbf{E}(\mathbf{r},t)]=0. \tag{3.6}$$

Let us now expand $n(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$ in Fourier series with periodic boundary conditions for a cube of unit volume and for a period of unit time interval:

$$n(\mathbf{r},t)=n_0+\sum_{\mathbf{k},\omega}n(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)],\quad \mathbf{E}(\mathbf{r},t)=\sum_{\mathbf{k},\omega}\mathbf{E}(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]. \tag{3.7}$$

In terms of the Fourier components, Eqs. (3.6) and (3.3) are expressed as

$$[-\omega^2-i\omega/\tau+(D/\tau)k^2]n(x)-i(en_0/m)\mathbf{k}\cdot\mathbf{E}(x)-i(e/m)\mathbf{k}\cdot\sum_{x'}n(x')\mathbf{E}(x-x')=0, \tag{3.8}$$

$$\mathbf{E}(x)=i\mathbf{k}(4\pi e/k^2)n(x), \tag{3.9}$$

where we have used four-vector notation

$$x\equiv(\mathbf{k},\omega),\quad x'\equiv(\mathbf{k}',\omega') \tag{3.10}$$

for brevity.

In order to calculate the dielectric response function $\epsilon(\mathbf{k},\omega)$, we introduce a test-charge field $-eQ(\mathbf{k},\omega)$ in the system; there then arise induced density fluctuations $n'(\mathbf{k},\omega)$ of the electrons, and the dielectric response function[17] is defined in terms of their statistical average $\langle n'(\mathbf{k},\omega)\rangle$ as

$$1/\epsilon(\mathbf{k},\omega)=1+\langle n'(\mathbf{k},\omega)\rangle/Q(\mathbf{k},\omega). \tag{3.11}$$

Linearizing (3.8) with respect to the test-charge field and the induced fluctuations, and noticing that the additional electric field $\mathbf{E}'(x)$ is given by

$$\mathbf{E}'(x)=i\mathbf{k}(4\pi e/k^2)[Q(x)+n'(x)], \tag{3.12}$$

we find

$$[-\omega^2-i\omega/\tau+(D/\tau)k^2]n'(x)+\omega_p^2[n'(x)+Q(x)]+(\omega_p^2/n_0)\sum_{x'}C(x,x-x')[n(x')n'(x-x')+n(x')Q(x-x')+n'(x')n(x-x')]=0. \tag{3.13}$$

Here, $C(x,x')$ is a coupling constant defined by

$$C(x,x')\equiv\mathbf{k}\cdot\mathbf{k}'/|\mathbf{k}'|^2, \tag{3.14}$$

and $\omega_p=(4\pi n_0e/m)^{1/2}$ is the plasma frequency of the electron gas.

For a quiescent plasma, one can ignore the presence of fluctuations $n(x)$ in its unperturbed stationary state. There are no contributions arising from the last convolution terms of (3.13). The dielectric response function $\epsilon^{(0)}(x)$ appropriate to such a system is readily obtained from (3.11) and (3.13) as[15]

$$\epsilon^{(0)}(x)=1+4\pi\alpha^{(0)}(x), \tag{3.15}$$

$$4\pi\alpha^{(0)}(x)=\tau\omega_p^2/[Dk^2-i\omega(1-i\omega\tau)]. \tag{3.16}$$

For a turbulent plasma, it is important to take account of the fluctuations in the stationary state; since we do not expect any correlations between the external test charge $Q(x-x')$ and the fluctuations $n(x')$, i.e.,

$$\langle Q(x-x')n(x')\rangle=0, \tag{3.17}$$

[15] S. Ichimaru, J. Phys. Soc. Japan **19**, 1207 (1964); **21**, 996 (1966).

[16] Y. H. Ichikawa, Phys. Fluids, **9**, 111 (1966).

[17] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **28**, No. 8 (1954); for a review and further references on the dielectric approach, see e.g., A. A. Rukhadze and V. P. Silin, Usp. Fiz. Nauk **76**, 79 (1962).

the statistical average of (3.13) becomes

$$[-\omega^2-i\omega/\tau+(D/\tau)k^2]\langle n'(x)\rangle+\omega_p{}^2[\langle n'(x)\rangle+Q(x)] +(\omega_p{}^2/n_0)\sum_{x'}C(x,x-x')[\langle n(x')n'(x-x')\rangle +\langle n'(x')n(x-x')\rangle]=0. \quad (3.18)$$

To evaluate the statistical average $\langle n(x')n'(x-x')\rangle$, we shall make use of an iteration and truncation procedure similar to that employed by Ichikawa.[16] There are, however, crucial differences between Ichikawa's treatment and ours which will play a critical role in determining the nature of the analysis.

In order to see this difference, let us explore the structure of the iteration calculation in some detail. For this purpose we first write a direct solution of (3.13) for $n'(x)$:

$$n'(x)=-\frac{4\pi\alpha^{(0)}(x)}{\epsilon^{(0)}(x)}\left\{Q(x)+\frac{1}{n_0}\sum_{x'}C(x,x-x') \times[n(x')n'(x-x')+n(x')Q(x-x') +n'(x')n(x-x')]\right\}. \quad (3.19)$$

We then immediately note that, when this expression is inserted in the average of $\langle n(x')n'(x-x')\rangle$, the first term, $-[4\pi\alpha^{(0)}(x)/\epsilon^{(0)}(x)]Q(x)$, makes no contribution to that average simply because of (3.17); generally, owing to the fact that there are no correlations between $Q(x)$ and $n(x)$, a statistical average involving $Q(x)$ can be factored into a form which requires no further iteration. Typical terms involved in the calculation of $\langle n(x')n'(x-x')\rangle$ may thus be expressed as

$$\langle n(x_1)n'(x_2)\rangle \sim\sum\nolimits_{x_3+x_4=x_2}A(x_3,x_4)\langle n(x_1)n(x_3)n'(x_4)\rangle. \quad (3.20)$$

In view of the fact that

$$\langle n(x)\rangle=0,\quad \langle n'(x)\rangle\neq 0, \quad (3.21)$$

the statistical average of the triple product can now be factored as

$$\langle n(x_1)n(x_3)n'(x_4)\rangle=\langle n(x_1)n(x_3)\rangle_c\langle n'(x_4)\rangle +\langle n(x_1)n(x_3)n'(x_4)\rangle_c, \quad (3.22)$$

where $\langle\ \rangle_c$ means the *correlated* average of the product of the statistical variables involved.

In Ichikawa's treatment,[16] the last term of (3.22) is neglected. Truncation is thus complete at this stage and one can substitute (3.20) with $x_1+x_2=x$ in (3.18) to calculate the dielectric response function.

It is our conviction, however, that the contribution from $\langle n(x_1)n(x_3)n'(x_4)\rangle_c$ must be retained partially at least, in order to describe the essential nature of a turbulent stationary state for the plasma. We therefore proceed to investigate its structure by substituting (3.19) in place of $n'(x_4)$; we find that the typical terms are

$$\langle n(x_1)n(x_3)n'(x_4)\rangle_c \sim\sum\nolimits_{x_5+x_6=x_4}A(x_5,x_6)\{\langle n(x_1)n(x_3)n(x_5)\rangle_c\langle n'(x_6)\rangle +\langle n(x_1)n'(x_6)\rangle_c\langle n(x_3)n(x_5)\rangle_c+\langle n(x_1)n(x_5)\rangle_c \times\langle n(x_3)n'(x_6)\rangle_c+\langle n(x_1)n(x_3)n(x_5)n'(x_6)\rangle_c\}. \quad (3.23)$$

The first term in the curly bracket involves the intrinsic ternary correlation $\langle n(x_1)n(x_3)n(x_5)\rangle_c$ of the turbulence; the next two terms contain the pair correlation of the turbulence and the correlated average of the product between the turbulence and the induced fluctuation; the last term is the correlated average of the product of four density variables.

We can go on to decompose the last term of (3.23) by means of another substitution of (3.19) for $n'(x_6)$; we will find that it consists of the sum of the terms like

$$\langle n(x_1)n(x_3)n(x_5)n(x_7)\rangle_c\langle n'(x_8)\rangle,$$
$$\langle n(x_1)n(x_3)n'(x_8)\rangle_c\langle n(x_5)n(x_7)\rangle_c,$$
$$\langle n(x_1)n(x_5)n'(x_8)\rangle_c\langle n(x_3)n(x_7)\rangle_c,$$
$$\langle n(x_3)n(x_5)n'(x_8)\rangle_c\langle n(x_1)n(x_7)\rangle_c,$$
$$\langle n(x_1)n(x_3)n(x_7)\rangle_c\langle n(x_5)n'(x_8)\rangle_c,$$
$$\langle n(x_1)n(x_5)n(x_7)\rangle_c\langle n(x_3)n'(x_8)\rangle_c,$$
$$\langle n(x_3)n(x_5)n(x_7)\rangle_c\langle n(x_1)n'(x_8)\rangle_c,$$

and

$$\langle n(x_1)n(x_3)n(x_5)n(x_7)n'(x_8)\rangle_c,$$

where $x_7+x_8=x_6$. The last term can then be decomposed into the sum of still higher-order products, and so on. The iteration procedure thus proceeds endlessly, a general feature of a perturbation theoretical analysis for a nonlinear problem.

Based on simple physical considerations, however, we can take into account a certain and important subset of all those correlated averages of higher-order products in our calculation. The size of the subset is rather small. Specifically, we neglect the ternary and the higher-order correlations in the turbulence; the first term in the right-hand side of (3.23), for example, is therefore outside the scope of our consideration. We shall later discuss in Sec. IX how the neglect of those higher-order correlations may be reasoned from our basic considerations. The second term in (3.23) will be fully taken into account, but the third term cannot be summed within our scheme; finally the fourth term will be partly retained through the terms $\langle n(x_1)n(x_3)n'(x_8)\rangle_c\langle n(x_5)n(x_7)\rangle_c$ and a part of $\langle n(x_1)n(x_3)n(x_5)n(x_7)n'(x_8)\rangle_c$; the latter contains still higher-order terms.

In order to carry out such a partial summation of higher-order products, we first write a response relation:

$$n'(x)=-K(x)[Q(x)+\tilde{Q}(x)], \quad (3.24)$$

where

$$\tilde{Q}(x)\equiv(1/n_0)\sum_{x'}C(x,x-x')[n(x')n'(x-x') +n(x')Q(x-x')+n'(x')n(x-x')]. \quad (3.25)$$

Equation (3.24) would become identical to (3.19) if $K(x)$ is given by

$$K_0(x)=4\pi\alpha^{(0)}(x)/\epsilon^{(0)}(x). \qquad (3.26)$$

This would then require infinite series of iterations, as we have just indicated. Instead, we find it possible to go around this circumstance by means of a simple technique: We modify $K(x)$ from the form given by (3.26) in such a way as to take into account the effects of turbulence; the expression (3.24) is then used for the calculation of $\langle n(x')n'(x-x')\rangle$. We *must* now disregard the contributions arising from the terms $\langle n(x')n(x''-x')\times n'(x-x'')\rangle_c$ in order to prevent overcounting of the effects of turbulence. The truncation is thus complete at this stage and the result includes a partial summation of all the correlated averages of higher-order products. Discussions in the following paragraphs are devoted to clarify certain physical aspects in connection with this method of calculation.

We first remark that by virtue of (3.17) we may forget about the first term in (3.24), $-K(x)Q(x)$, insofar as this iteration procedure is concerned. Secondly, we observe that the term (3.25) plays the role of another *external* test charge to the system; this observation stems from the fact that (3.26), which is the originally suggested form of the response function $K(x)$ from (3.19), assumes an expression appropriate to the density response function against the *external* charges (not against, e.g., the total charges[18]). Although we shall have to modify (3.26) to take account of the effects of turbulence, the modification will be made in such a way that the validity of the above argument remains unaffected.

Let us explore the latter aspect further. As is clear from its expression, the response function (3.26) characterizes the density response of a *quiescent* plasma against an external test charge. We have been arguing that a quiescent stationary state does not correspond to physical reality for a plasma in turbulent domain; an external test charge introduced in the system can induce fluctuations not only from the quiescent background but also from the turbulent components already present in its stationary state. We are therefore led to conclude that the response function $K(x)$ appearing in (3.24) may be the one describing the density response of the *turbulent* stationary plasma against external test charge. The above argument, at this stage, may still sound a little intuitive; in the following section, we shall clarify, with the aid of diagrammatic considerations, what additional sum of higher-order polarization diagrams is involved in the propagation processes of the test charge when we replace $K_0(x)$ of (3.26) by a true response function $K(x)$ appropriate to the turbulent system. We shall thereby show that our method of calculation indeed enables us to carry out partial summation of all correlated averages of higher order products as indicated before.

[18] S. Ichimaru, Ref. 11.

The poles of $K(x)$ in the complex ω plane determine the nature of the density-fluctuation excitations in the system. It has been our fundamental postulate that the turbulent stationary state should be a stable one; it then follows that $K(x)$ can have poles only in the lower or the upper half of the complex ω plane depending upon the choice of the boundary conditions for Fourier transformation. If $K(x)$ is calculated with the retarded boundary conditions, we find the poles only in the lower half plane; if the advanced boundary conditions are chosen, in the upper half plane only. It now becomes necessary to determine which of the two boundary conditions we must choose for the response function $K(x)$. For this purpose, let us take another look at (3.24), with $Q(x)$ now being omitted. Formally, $K(x)$ is the density response of the system against the disturbance $\tilde{Q}(x)$; it is the character of the disturbance which imposes the boundary condition on the response function. As is clear from (3.25), $\tilde{Q}(x)$ is expressed as the convolution sum of products like $n(x')n'(x-x')$, $n'(x')n(x-x')$, and $n(x')Q(x-x')$. In the real time domain, $n'(t)$ will behave in accordance with the external test charge, which may be a simple sinusoidal variation. $n(t)$, however, has nothing to do with the behavior of the test charge; it represents the fluctuations already present in the system. By virtue of our fundamental assumption that the turbulent stationary state be stable, those fluctuations, once created by the motion of charged particles, must subsequently decay in time, because the system does not sustain growing oscillations. We must therefore interpret $\tilde{Q}(t)$, which essentially consists of the products between the above two kinds of density variations, as a *decaying* disturbance; this observation leads us to the choice of the advanced boundary conditions for the calculation of $K(x)$.

All the discussions in the preceding paragraphs indicate that we may choose

$$K(x)=4\pi\alpha^*(x)/\epsilon^*(x) \qquad (3.27)$$

in (3.24) for the iteration calculation based on the turbulent stationary state; in (3.27), $\epsilon(x)$ and $4\pi\alpha(x)$ are the dielectric response function and the polarizability of the *turbulent* plasma, respectively, and the asterisk implies that the function should be calculated with the advanced boundary conditions.

We now proceed to calculate $\langle n(x')n'(x-x')\rangle$; with the aid of (3.24), (3.25), and (3.27), we find

$$\begin{aligned}\langle n(x')n'(x-x')\rangle &= -\frac{1}{n_0}\frac{4\pi\alpha^*(x-x')}{\epsilon^*(x-x')}\sum_{x''}C(x-x',x-x'-x'')\\ &\times[\langle n(x')n(x'')Q(x-x'-x'')\rangle\\ &+\langle n(x')n(x'')n'(x-x'-x'')\rangle\\ &+\langle n(x')n'(x'')n(x-x'-x'')\rangle]. \qquad (3.28)\end{aligned}$$

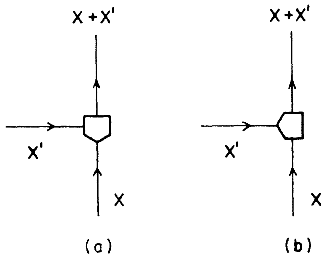

FIG. 1. Fundamental diagrams of interaction processes.

The first term may be simplified as

$$\langle n(x')n(x'')Q(x-x'-x'')\rangle = \langle n(x')n(-x')\rangle Q(x)\delta_{x'',-x'}, \quad (3.29)$$

while we must truncate the ensemble average of the last two terms by neglecting the contributions from the correlated average of the product of three density variables:

$$\langle n(x')n(x'')n'(x-x'-x'')\rangle = \langle n(x')n(-x')\rangle\langle n'(x)\rangle\delta_{x'',-x'}, \quad (3.30)$$

$$\langle n(x')n'(x'')n(x-x'-x'')\rangle = \langle n(x')n(-x')\rangle\langle n'(x)\rangle\delta_{x'',x}. \quad (3.31)$$

In these expressions $\delta_{x,x'}$ represents a four-dimensional Kronecker's delta.

With the aid of all these calculations, it is now possible to calculate the dielectric response function in

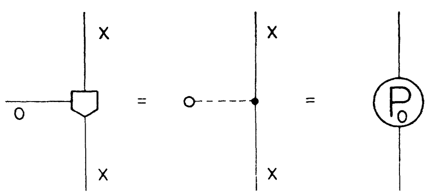

FIG. 2. Polarization of the quiescent background; $P_0(x) = -4\pi\alpha^{(0)}(x)$.

accordance with the definition (3.11). We find

$$\frac{1}{\epsilon(x)} = 1 - \frac{4\pi\alpha^{(0)}(x) - [P_1(x)+P_2(x)]}{\epsilon^{(0)}(x) - [P_1(x)+P_2(x)+P_3(x)+P_4(x)]} \quad (3.32)$$

or

$$\epsilon(x) = 1 + \frac{4\pi\alpha^{(0)}(x) - [P_1(x)+P_2(x)]}{1-[P_3(x)+P_4(x)]}, \quad (3.33)$$

where

$$\begin{Bmatrix} P_1(x) \\ P_2(x) \end{Bmatrix} = \frac{2\pi}{n_0^2} 4\pi\alpha^{(0)}(x) \sum_{x'} \frac{4\pi\alpha^*(x-x')}{\epsilon^*(x-x')} \begin{Bmatrix} C(x, x-x') \\ C(x,x') \end{Bmatrix} \times C(x-x', x)S(x'), \quad (3.34)$$

$$\begin{Bmatrix} P_3(x) \\ P_4(x) \end{Bmatrix} = \frac{2\pi}{n_0^2} 4\pi\alpha^{(0)}(x) \sum_{x'} \frac{4\pi\alpha^*(x-x')}{\epsilon^*(x-x')} \begin{Bmatrix} C(x, x-x') \\ C(x,x') \end{Bmatrix} \times C(x-x', -x')S(x'), \quad (3.35)$$

and the spectral function $S(x)$ of the density fluctuations is defined by

$$S(x) \equiv \frac{1}{2\pi}\langle n(x)n(-x)\rangle = \frac{1}{2\pi}\langle |n(x)|^2\rangle. \quad (3.36)$$

We may regard (3.32) or (3.33) as a self-consistent equation for $\epsilon(x)$.

IV. DIAGRAMMATIC CONSIDERATIONS

Polarization processes of the plasma associated with the introduction of test charges can be analyzed by means of diagrammatic considerations. Such considerations, supplementing the calculations of the previous section, enable us to understand or visualize more clearly the nature of the physical processes involved.

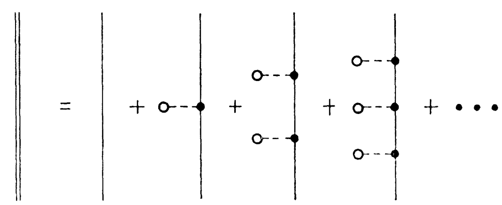

FIG. 3. Sum of the polarization processes for the quiescent background.

The fundamental diagrams are shown in Fig. 1. A vertical line stands for propagation of either the test charge or the induced fluctuations; a line entering the pentagon-shaped vertex from a horizontal direction expresses the contribution of the turbulent fluctuations existing in the plasma; and the vertex represents a process in which an incoming test charge, or induced fluctuation (with frequency and wave vector x), interacting with the turbulence (with x') induces density fluctuations (with $x+x'$) in the plasma. This is the coupling process originating from the $\nabla\cdot(n\mathbf{E})$ term of (3.6); one of the two coupling components of fluctuations must therefore play the role of the oscillating electric field which acts to joggle the other density-fluctuation component. The distinction between Figs. 1(a) and 1(b) stems from this consideration; the component which enters the vertex at a *corner* of the pentagon acts as the oscillating electric field.

Each incoming line carries its own strength corresponding to the amplitude of the fluctuations: $Q(x)$,

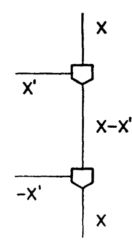

FIG. 4. Second-order interaction with turbulence.

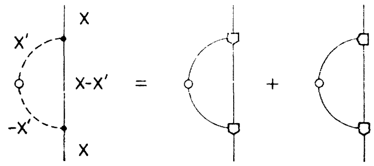

FIG. 5. Polarization process arising from second-order interactions with turbulence.

$n(x)$, or $n'(x)$. In addition, the line entering at the corner with wave vector $\mathbf{k}$ is attributed a factor $i\mathbf{k}(4\pi e/k^2)$ characterizing its contribution as the electric field [see Eqs. (3.9) and (3.12)]. For a vertex with an outgoing line with x, we attach a factor $(i\mathbf{k}/n_0e)\alpha^{(0)}(x)$. Later we shall have occasion to consider another kind of vertex (of blacked-out type—see the lower vertices of Fig. 7), in which case we use a factor $(i\mathbf{k}/n_0e)\alpha(x)$ instead. The boundary conditions for those polarizabilities must be determined from a physical consideration, as was done in the previous section.

Let us apply the above rule to calculate the induced fluctuations, $n'(x+x')$, due to Fig. 1(a). We find

$$n'(x+x')=-4\pi\alpha^{(0)}(x+x')C(x+x',x)\times[n(x')/n_0]Q(x). \quad (4.1)$$

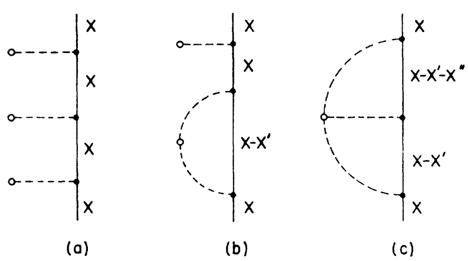

FIG. 6. Examples of third-order polarization processes. Diagram (c) represents the contribution of the intrinsic ternary correlation in turbulence, which is not taken into consideration in our treatment.

In view of the fact that

$$\langle n(x')\rangle=n_0\delta_{x',0}, \quad (4.2)$$

it is significant to single out from the processes of Fig. 1 the contribution arising from the cases that $x'=0$. We depict such a polarization process as in Fig. 2; its value is

$$n'(x)=-4\pi\alpha^{(0)}(x)Q(x), \quad (4.3)$$

that is, the first-order polarization process of the quiescent background. It is apparent that Fig. 1(b) does not contribute to the process of Fig. 2 because of the neutralizing action of the positive charge background. We may sum those processes as in Fig. 3 to arrive at an ordinary dielectric propagator in the quiescent background.

$$\begin{aligned}1/\epsilon^{(0)}(x)&=1-4\pi\alpha^{(0)}(x)+[4\pi\alpha^{(0)}(x)]^2-\cdots\\&=1/[1+4\pi\alpha^{(0)}(x)].\end{aligned} \quad (4.4)$$

Within the first order in the polarization processes of Fig. 1, the only real contribution arises from Fig. 2, because all other terms with $x'\neq 0$ vanish as soon as the statistical average is carried out. The processes of

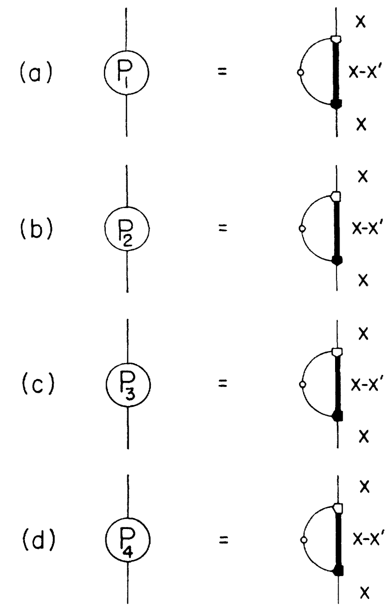

FIG. 7. Calculations of $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$, given by (3.34) and (3.35).

the latter type can make a contribution if we proceed to consider a second-order process like Fig. 4. Here, the test-charge with frequency and wave vector x, interacting with a turbulent fluctuation with $-x'$, creates a density fluctuation with $x-x'$; this fluctuation then interacts with a turbulence with x' to induce another fluctuation with x. If we calculate the induced density

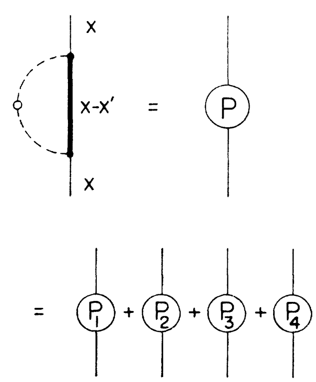

FIG. 8. Definition of $P(x)$.

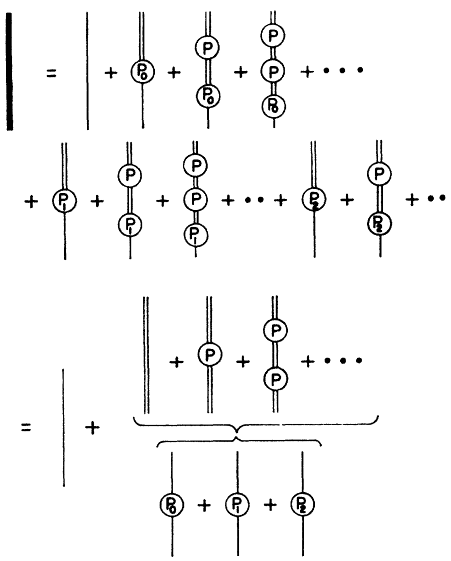

FIG. 9. Summation of polarization diagrams. This calculation yields

$$1/\epsilon(x)=1+[-4\pi\alpha^{(0)}(x)+P_1(x)+P_2(x)]\times[\epsilon^{(0)}(x)]^{-1}[1-P(x)/\epsilon^{(0)}(x)]^{-1}.$$

fluctuation $n'(x)$ through these processes and average it with respect to the states of the turbulent plasma, we find a nonvanishing contribution,

$$n'(x)=(1/n_0^2)4\pi\alpha^{(0)}(x)\sum_{x'}4\pi\alpha^{(0)}(x-x')C(x,x-x')\times C(x-x',x)\langle|n(x')|^2\rangle Q(x). \quad (4.5)$$

One can, in fact, consider four kinds of such second-order processes arising from the intercharge of Fig. 1(a) and 1(b) at each vertex. For brevity, we introduce Fig. 5, where the small circle connecting the two turbulent fluctuation lines means taking a statistical average for their product.

If we go to the third-order processes, in addition to diagrams like Figs. 6(a) and 6(b), one can also consider such a process as described by Fig. 6(c); this represents the contribution of an intrinsic ternary correlation part.

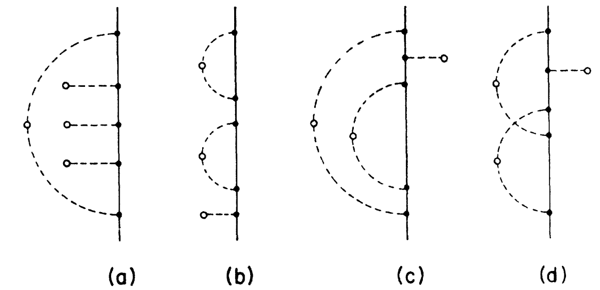

FIG. 11. Examples of fifth-order diagrams: (a) and (b) are summed both in Fig. 9 and in the treatment based on (3.26); (c) is summed in the former, but not in the latter; (d) is summed in neither of them.

We have neglected this contribution to simplify the treatment. In our calculation, therefore, there will be no higher-order terms involved than Fig. 5, as far as the statistical average of turbulent components is concerned.

Let us now consider the processes of Fig. 7. A bold vertical line in the intermediate state represents a true propagator $[1/\epsilon(x-x')]$ of density fluctuations in the turbulent plasma, which we shall presently investigate. It is clear that Fig. 7 amounts to the calculations of $P_1(x)$–$P_4(x)$ given by (3.34) and (3.35), if we choose the advanced boundary conditions for the propagators of the intermediate state, following the discussion in the previous section. For brevity, we also introduce Fig. 8.

The dielectric response function is by definition the dynamic screening factor of the medium against an externally applied electric field. The test charge is introduced in the plasma to produce electric field disturbance; for the calculation of the dielectric response function, therefore, a first polarization process must always be such that the test charge acts as an electric field, that is, either $P_0(x)$ (see Fig. 2), $P_1(x)$, or $P_2(x)$ (see Fig. 7) can enter as a first polarization diagram. With this little precaution in mind, we now calculate the dielectric response function by summing all those polarization processes as depicted in Fig. 9; the result of this calculation reproduces (3.32).

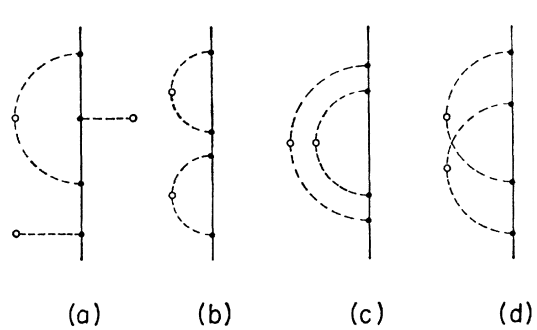

FIG. 10. Examples of fourth-order diagrams: (a) and (b) are summed both in Fig. 9 and in the treatment based on (3.26); (c) is summed in the former, but not in the latter; (d) is summed in neither of them.

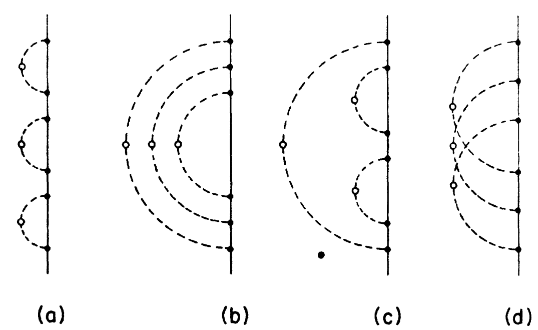

FIG. 12. Examples of sixth-order diagrams: (a) is summed both in Fig. 9 and in the treatment based on (3.26); (b) and (c) are summed in the former, but not in the latter; (d) is summed in neither of them.

It may be instructive to examine more closely what kind of diagrams we have actually summed in Fig. 9. In the fourth order, such diagrams as Figs. 10(a), 10(b), and 10(c) are included, but not those of Fig. 10(d). In the fifth and the sixth order, we were able to sum those diagrams like Figs. 11(a)–11(c) and Figs. 12(a)–12(c), but not those like Figs. 11(d) and 12(d). In short, our calculation depicted in Fig. 9 sums all those polarization processes in which no two turbulent fluctuation lines intersect each other (when drawn on one side only).

Finally, let us investigate what kind of difference is involved in the summation of the polarization processes if we are to use $K_0(x)$ of (3.26) in (3.24) instead of $K(x)$ of (3.27). It is clear that the use of $K_0(x)$ essentially amounts to replacing $P(x)$ of Fig. 8 by $P^{(0)}(x)$ of Fig. 13 [and a similar replacement for $P_1(x)$ and $P_2(x)$] in the summation of Fig. 9. A close examination then reveals that diagrams like Figs. 10(a), 10(b), 11(a), 11(b), and 12(a) are still included, but those like Figs. 10(c), 11(c), 12(b), and 12(c) are now excluded from the summation. The difference starts to appear only at the fourth order in the polarization processes, yet the modification caused by the inclusion of the diagrams of the latter type turns out to be so drastic that it affects the essential nature of the analysis of a turbulent plasma.

V. DIELECTRIC RESPONSE FUNCTION FOR A TWO-COMPONENT TURBULENT PLASMA

It is straightforward to extend the calculations of Sec. III to the cases of a turbulent plasma consisting of the electrons, the singly charged ions, and the neutrals. The degree of complexity, however, increases substantially. We quote only the results here, leaving the details of the calculation to Appendix B.

In the following we use the subscripts $+$ and $-$ to distinguish between the quantities associated with the ions and the electrons. In particular,

$$\omega_\pm=(4\pi n_0e^2/m_\pm)^{1/2},\quad k_\pm=(4\pi n_0e^2/\kappa T_\pm)^{1/2} \tag{5.1}$$

are the plasma frequency and the Debye wave number for each charged constituent. We shall assume that the relations $m_-\ll m_+$, $(\kappa T_+/m_+)^{1/2}\ll(\kappa T_-/m_-)^{1/2}$ are well satisfied for the cases of interest.

The dielectric response function $\epsilon(x)$ is now expressed as

$$\epsilon(x)=1+4\pi\alpha_-(x)+4\pi\alpha_+(x), \tag{5.2}$$

where

$$4\pi\alpha_\pm(x)=\frac{4\pi\alpha_\pm^{(0)}(x)[1-X_\pm(x)][1\pm Y_\mp(x)]\mp4\pi\alpha_\mp^{(0)}(x)[1-X_\mp(x)]Z_\pm(x)}{[1+Y_-(x)][1-Y_+(x)]+Z_+(x)Z_-(x)} \tag{5.3}$$

are the ion and electron polarizabilities of the turbulent plasma,

$$4\pi\alpha_\pm^{(0)}(x)=\tau_\pm\omega_\pm^2/[D_\pm k^2-i\omega(1-i\omega\tau_\pm)] \tag{5.4}$$

are the polarizabilities of the quiescent background of the ions and the electrons, and

$$X_\pm(x)\equiv\frac{2\pi}{n_0^2}\sum_{x'}\frac{C(x-x',x)}{\epsilon^*(x-x')}\{C(x,x-x')[4\pi\alpha_-^*(x-x')S_{\pm-}(x')+4\pi\alpha_+^*(x-x')S_{\pm+}(x')]\mp C(x,x')4\pi\alpha_\pm^*(x-x')[S_{\rho\pm}(x')\mp4\pi\alpha_\mp^*(x-x')S_{\rho\rho}(x')]\}, \tag{5.5a}$$

$$Y_\pm(x)\equiv\frac{2\pi}{n_0^2}4\pi\alpha_\pm^{(0)}(x)\sum_{x'}\frac{4\pi\alpha_\pm^*(x-x')}{\epsilon^*(x-x')}C(x-x',x')\{C(x,x-x')S_{\pm\rho}(x')\mp C(x,x')[1+4\pi\alpha_\mp^*(x-x')]S_{\rho\rho}(x')\}, \tag{5.5b}$$

$$Z_\pm(x)\equiv\frac{2\pi}{n_0^2}4\pi\alpha_\pm^{(0)}(x)\sum_{x'}\frac{4\pi\alpha_\mp^*(x-x')}{\epsilon^*(x-x')}C(x-x',x')\{C(x,x-x')S_{\pm\rho}(x')\pm C(x,x')4\pi\alpha_\pm^*(x-x')S_{\rho\rho}(x')\}. \tag{5.5c}$$

Various correlation functions $S(x')$ are involved in the definition of (5.5); the distinction is made by combinations of the subscripts $+$, $-$, and ρ, where ρ stands for the difference between the densities of the electrons and the ions. For example,

$$S_{-+}(x)\equiv(1/2\pi)\langle n_-(x)n_+(-x)\rangle,\quad S_{\rho-}(x)\equiv(1/2\pi)\langle[n_-(x)-n_+(x)]n_-(-x)\rangle, \tag{5.6}$$

and the rest may be defined similarly.

In the limit of high frequencies, $Y_\pm(x)$ and $Z_\pm(x)$ decrease as ω^{-4}, and $X_\pm(x)$ as ω^{-2}; it thus follows that the dielectric response function (5.2) guarantees the correct high-frequency asymptotic form

$$\epsilon(x)\to1-\omega_p^2/\omega^2,\quad(\omega\to\infty) \tag{5.7}$$

where

$$\omega_p^2=\omega_-^2+\omega_+^2. \tag{5.8}$$

The zeros of $\epsilon(\mathbf{k},\omega)$ determine the frequency–wave-vector dispersion relations of the density-fluctuation

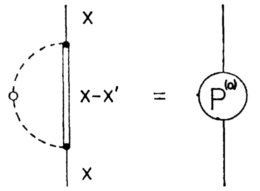

FIG. 13. The diagram which would enter in the summation of Fig. 9 in place of P (defined by Fig. 8), if (3.26) is to be used as $K(x)$ in (3.24).

excitations of the system. Our entire analysis has been based upon the hydrodynamic equations (3.1) and (3.2); it is well known that such formulation provides a correct description of the plasma in the low-frequency and long-wavelength region. We may then neglect the first term, unity, in (5.2) as compared with the magnitudes of other terms (i.e., electron and ion polarizabilities) under these circumstances. Physically, this means that the space-charge fluctuations are much smaller in amplitude than the density fluctuations of the electrons and the ions themselves in the low-frequency and long-wavelength region; in particular, the ever-stable solutions corresponding to the high-frequency plasma oscillations ($\omega \simeq \pm\omega_p$) will be lost by the neglect of unity in (5.2).

The remaining low-frequency branch of the dispersion relations represents the ion acoustic mode of oscillation; for a quiescent plasma with $T_- \gg T_+$, its propagation velocity is given by

$$s = (\kappa T_- / m_+)^{1/2}. \tag{5.9}$$

There exists, however, a lower wave-number limit,

$$k_1 \simeq \frac{1}{2s\tau_+}\left(1+\frac{m_-\tau_+}{m_+\tau_-}\right), \tag{5.10}$$

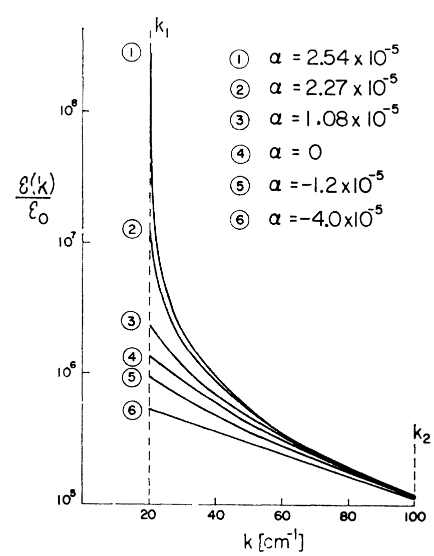

FIG. 14. Fluctuation spectrum associated with the ion acoustic mode in a turbulent helium plasma for various values of $\alpha = (\hat{k}\cdot \mathbf{V}_d - V_c)/s$ in the very vicinity of the critical point, $\alpha = 0$. $\epsilon(\mathbf{k})$ is the energy contained in the fluctuations at the mode with wave vector $\mathbf{k}$; ϵ_0 is its value at $V_d = 0$ (ϵ_0 is independent of $\mathbf{k}$). For the particular physical parameters which we have chosen for this computation, see the text.

below which the acoustic mode no longer represents a zero of the dielectric response function[15]; in this region, the acoustic mode turns into a diffusion mode.

$$\omega \simeq -ik^2 D_a, \tag{5.11}$$

where D_a is the ambipolar diffusion coefficient for the quiescent plasma,

$$D_a = (\mu_+ D_- + \mu_- D_+)/(\mu_+ + \mu_-). \tag{5.12}$$

In the following sections we shall apply a similar means of investigation to the turbulent plasma. We thereby calculate an effective diffusion coefficient and study the nature of the ion acoustic wave by investigating the behavior of the zeros of $\epsilon(\mathbf{k},\omega)$ associated with the low-frequency modes.

VI. AMBIPOLAR DIFFUSION COEFFICIENT FOR A TURBULENT PLASMA

The fluctuations of internal electric fields, which act to joggle the charged particles, can be additional mechanisms for particle diffusion.[19] In this section we investigate such an effect of turbulence upon the diffusion process of the test-charge field by looking into the behavior of the dielectric response function in the limit of long wavelengths and low frequencies.

In the light of the discussion made in the last part of the previous section, we first note that the dielectric response function can be expressed as

$$\epsilon(x) = 4\pi\alpha_-(x) + 4\pi\alpha_+(x) \tag{6.1}$$

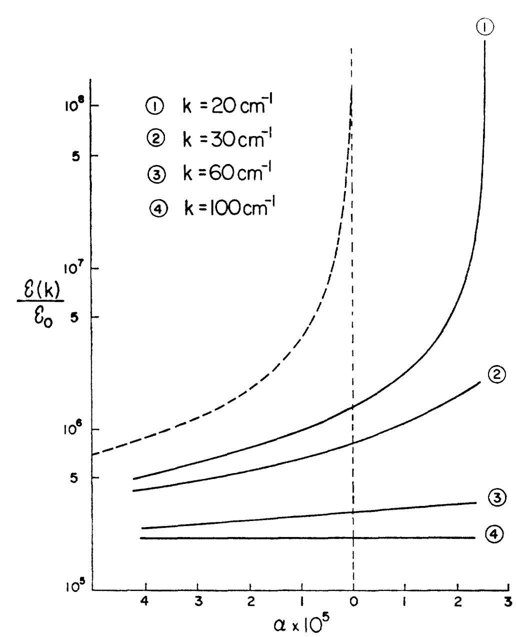

FIG. 15. Variation of fluctuation amplitude as a function of α. The case treated here is the same as Fig. 14. The dashed line represents the result based on a quiescent stationary state.

[19] L. Spitzer, Jr., Phys. Fluids 3, 659 (1960).

within the validity of the hydrodynamic description; the equation for the dispersion relation thus takes a simplified form:

$$4\pi\alpha_-^{(0)}(x)[1-X_-(x)][1-Y_+(x)-Z_+(x)]+4\pi\alpha_+^{(0)}(x)\times[1-X_+(x)][1+Y_-(x)+Z_-(x)]=0. \quad (6.2)$$

This equation contains in its structure various kinds of interaction processes between the test charge and the turbulence. Among them we must now select those which describe the effects of turbulent electric fields on the propagation of the test-charge field only.

For a single-component plasma, as may be clear from the considerations in Sec. IV, we have been taking into account those elementary interaction processes which are depicted in the right-hand side of Fig. 5; the point of distinction among them is which of the two interacting fluctuations plays the part of the oscillating electric field. This sort of distinction can easily be made among the terms consisting of the dielectric response function, because the wave vector of the oscillating electric field always enters as the latter argument of the coupling constant, (3.14). Therefore, if we are only interested in the interaction of the turbulent electric field (with x') with the test charge, we may simply retain the terms involving the product, $C(x-x',x')C(x,x')$.

This feature prevails, without any essential change, in the cases of a two-component plasma as well. We notice from (5.5) that only the last halves of $Y_\pm(x)$ and $Z_\pm(x)$ represent such contributions, and that there are no such terms involved in $X_\pm(x)$. Dividing (6.2) by $4\pi\alpha_+^{(0)}(x)4\pi\alpha_-^{(0)}(x)$, we thus find

$$[4\pi\alpha_+^{(0)}(x)]^{-1}+[4\pi\alpha_-^{(0)}(x)]^{-1}+(2\pi/n_0^2)\sum_{x'}C(x-x',x')C(x,x')S_{\rho\rho}(x')=0. \quad (6.3)$$

In the calculation of the third term on the left-hand side of (6.3), we have made a simplification arising from (6.1).

In the limit of long wavelengths and low frequencies, we seek a solution of (6.3) in the form

$$\omega=-ik^2D_a[1+\beta(\hat{k})], \quad (6.4)$$

where $\hat{k}$ and similar notations occurring later represent unit vectors in the corresponding directions. Substituting (6.4) in (6.3) and going to the limit, $k\to 0$, we find

$$\beta(\hat{k})=\frac{2\pi}{n_0^2}\lim\frac{k_-^2}{k^2}\sum_{\mathbf{l},\omega}\frac{(\mathbf{k}-\mathbf{l})\cdot\mathbf{l}}{l^2}\frac{\mathbf{k}\cdot\mathbf{l}}{l^2}S_{\rho\rho}(\mathbf{l},\omega)=\frac{k_-^2}{(2\pi)^3n_0^2}\int d\omega\int dl\int_{-1}^{1}\mu^2d\mu\int_0^{2\pi}d\phi S_{\rho\rho}(\mathbf{l},\omega), \quad (6.5)$$

where μ is the direction cosine between $\hat{l}$ and $\hat{k}$; ϕ is the azimuthal angle of $\hat{l}$ with respect to $\hat{k}$. In the final step of (6.5), we have changed the summation into integration via the relation

$$\sum_{\mathbf{k},\omega}\to\frac{1}{(2\pi)^4}\int d\omega\int k^2dk\int d\mu\int d\phi, \quad (6.6)$$

and have made use of the symmetry property

$$S_{\rho\rho}(\mathbf{k},\omega)=S_{\rho\rho}(-\mathbf{k},-\omega). \quad (6.7)$$

The fluctuation spectrum associated with the space-charge density fluctuations, $\rho(\mathbf{k},\omega)\equiv n_-(\mathbf{k},\omega)-n_+(\mathbf{k},\omega)$, is directly related to that of the internal electric field fluctuations, $S_E(\mathbf{k},\omega)$, through

$$S_E(\mathbf{k},\omega)=(4\pi e/k)^2S_{\rho\rho}(\mathbf{k},\omega). \quad (6.8)$$

With the aid of this relation, (6.5) can be re-expressed as

$$\beta(\hat{k})=\frac{k_-^2}{(2\pi)^3n_0^2(4\pi e)^2}\int d\omega\int l^2dl\int_{-1}^{1}\mu^2d\mu\int_0^{2\pi}d\phi S_E(\mathbf{l},\omega). \quad (6.9)$$

Equation (6.9) thus offers an explicit expression for the enhancement caused by the presence of fluctuating electric field in the turbulent plasma. The spectral function $S_E(\mathbf{k},\omega)$ associated with the ion acoustic mode of oscillations will be determined in the following sections in accordance with the self-consistent scheme of calculation.

VII. ION ACOUSTIC WAVES IN A TURBULENT PLASMA

Let us now consider the ion acoustic waves in a weakly ionized plasma in which the electrons as a whole move under the action of an applied electric field with the velocity $\mathbf{V}_d$ relative to the ions and the neutrals; the effects of such uniform drift motion can be incorporated in the formalism of Sec. V by replacing ω in the polarizability of the quiescent background of the electrons by a displaced frequency variable

$$\tilde{\omega}=\omega-\mathbf{k}\cdot\mathbf{V}_d. \quad (7.1)$$

That is, the polarizability of the quiescent electrons is now given by $4\pi\alpha_-^{(0)}(\mathbf{k},\tilde{\omega})$.

The ion acoustic mode in the turbulent plasma may be obtained from a solution of (6.2) by going to the low-frequency expression for the electronic polarizability and the high-frequency expression for the ionic polarizability. We now remark from (5.5) that $X_\pm(x)$, $Y_\pm(x)$, and $Z_\pm(x)$ all involve factors which may be symbolically written as $2\pi S(x)/n_0^2$. If there exists a component of turbulent fluctuations with definite frequency and wave vector which amounts to, say, a 10% modulation of the average density n_0, then $2\pi S(x)/n_0^2$ will take on a value near 10^{-2} at that ω and $\mathbf{k}$. As long as we disregard the possibility of having an excessive degree of modulation by a single mode of turbulence, we may thus neglect the contributions of terms involving $[2\pi S(x)/n_0^2]^2$ in favor of those of linear terms.

Dividing (6.2) by $4\pi\alpha_-^{(0)}(x)4\pi\alpha_+^{(0)}(x)$, we now find

$$[1-X_-(x)]/4\pi\alpha_+^{(0)}(x)+[1-X_+(x)]/4\pi\alpha_-^{(0)}(x)+U(x)=0, \quad (7.2)$$

where

$$U(x)\equiv[Y_-(x)+Z_-(x)]/4\pi\alpha_-^{(0)}(x)-[Y_+(x)+Z_+(x)]/4\pi\alpha_+^{(0)}(x)$$
$$=(2\pi/n_0^2)\sum_{x'}C(x-x',x')[C(x,x-x')+C(x,x')]S_{\rho\rho}(x'). \quad (7.3)$$

In the final expression of (7.3), there is again a simplification arising from (6.1).

We now write

$$\omega=\omega(\mathbf{k})+i\gamma(\mathbf{k}) \quad (7.4)$$

and substitute it in (7.2). Assuming that $|\omega(\mathbf{k})|\gg|\gamma(\mathbf{k})|$, we find $\omega(\mathbf{k})$ from the real part of Eq. (7.2). For $T_-\gg T_+$, we obtain

$$\omega(\mathbf{k})=\pm sk\{1+(k_-^2/k^2)U(\mathbf{k},\pm sk)+\mathrm{Re}[X_-(\mathbf{k},\pm sk)-X_+(\mathbf{k},\pm sk)]\}^{1/2}, \quad (7.5)$$

where s is the sound velocity defined by (5.9), and the choice of sign in the frequency variable of X_-, X_+, and U should agree with that in front of (7.5).

It is clear from (7.5) that the presence of fluctuations acts to change the propagation velocity of the acoustic wave from its quiescent value s to

$$s_\pm=s\{1+(k_-^2/k^2)U(\mathbf{k},\pm sk)+\mathrm{Re}[X_-(\mathbf{k},\pm sk)-X_+(\mathbf{k},\pm sk)]\}^{1/2}. \quad (7.6)$$

Although this modification certainly leads to a different quantitative evaluation, we do not expect that it will result in any drastic change in the *nature* of the phenomena involved. For this reason we shall not pay much attention to this sort of modifications in this section; we shall be mainly concerned with the effects of turbulence which act to stabilize the ion acoustic oscillations under those circumstances for which a conventional linear theory predicts instability.

One of the two solutions, Eq. (7.5), represents an ever-damped branch of the acoustic modes. As will soon become clear from (7.10), this branch is given by the choice of sign opposite that of $\mathbf{k}\cdot\mathbf{V}_d$. The fluctuations associated with such heavily damped oscillations can be treated by ordinary means without consideration of the effects of turbulence; their magnitude remains in the vicinity of the thermal level. We shall not be interested in this branch in the following analysis.

With the aid of the considerations enumerated above, we may write for the acoustic mode of our interest

$$\omega(\mathbf{k})=(\mathbf{k}\cdot\mathbf{V}_d/|\mathbf{k}\cdot\mathbf{V}_d|)sk, \quad (7.7)$$

where we may interpret s as a renormalized velocity of the acoustic wave which contains effects of turbulence.

Given the real frequency, we can calculate the growth rate $\gamma(\mathbf{k})$ from the imaginary part of (7.2). Within the same order as foregoing calculations, we find

$$\gamma(\mathbf{k})=-\frac{1}{2\tau_+}\frac{m_-}{2m_+\tau_-}\frac{\omega(\mathbf{k})-\mathbf{k}\cdot\mathbf{V}_d}{\omega(\mathbf{k})}\times\{1+\mathrm{Re}[X_-(\mathbf{k},\omega(\mathbf{k}))-X_+(\mathbf{k},\omega(\mathbf{k}))]\}+\tfrac{1}{2}\omega(\mathbf{k})\,\mathrm{Im}[X_-(\mathbf{k},\omega(\mathbf{k}))-X_+(\mathbf{k},\omega(\mathbf{k}))]. \quad (7.8)$$

The curly bracket in the second term represents a correction factor on the wave-electron coupling term due to the presence of turbulence. We shall not be concerned with this sort of correction, for the reason given before.

By evoking (6.1) again for our hydrodynamic analysis, we may calculate the last term of (7.8) with the aid of (5.5a) as

$$\frac{\pi}{n_0^2}\omega(\mathbf{k})\sum_{x'}\mathrm{Im}\left\{\frac{C(x-x',x)}{\epsilon^*(x-x')}[C(x,x-x')+C(x,x')][4\pi\alpha_-^*(x-x')S_{\rho-}(x')+4\pi\alpha_+^*(x-x')S_{\rho+}(x')]\right\}$$
$$=\frac{\pi}{n_0^2}\omega(\mathbf{k})\sum_{x'}\mathrm{Im}\frac{4\pi\alpha_-^*(x-x')}{\epsilon^*(x-x')}C(x-x',x)[C(x,x-x')+C(x,x')]S_{\rho\rho}(x')$$
$$+\frac{\pi}{n_0^2}\omega(\mathbf{k})\sum_{x'}C(x-x',x)[C(x,x-x')+C(x,x')]\,\mathrm{Im}S_{\rho+}(x'), \quad (7.9)$$

where x now takes on values $[\mathbf{k},\omega(\mathbf{k})]$. We show in Appendix C that the second term of (7.9) amounts to another turbulent correction to the first two terms of (7.8); we leave out such relatively unimportant contributions of the turbulence. We can thus re-express (7.8) in a simplified form as

$$\gamma(\mathbf{k})=-\frac{1}{2\tau_+}\frac{m_-}{2m_+\tau_-}\frac{\omega(\mathbf{k})-\mathbf{k}\cdot\mathbf{V}_d}{\omega(\mathbf{k})}+\frac{1}{16\pi^3n_0^2(4\pi e)^2}\int d\omega'\int d^3l\,\mathrm{Im}\frac{4\pi\alpha_-^*[\mathbf{k}-\mathbf{l},\omega(\mathbf{k})-\omega']}{\epsilon^*[\mathbf{k}-\mathbf{l},\omega(\mathbf{k})-\omega']}\times l^2\frac{(\mathbf{k}-\mathbf{l})\cdot\mathbf{k}}{k^2}\left[\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{l})}{|\mathbf{k}-\mathbf{l}|^2}+\frac{\mathbf{k}\cdot\mathbf{l}}{l^2}\right]S_E(\mathbf{l},\omega'), \quad (7.10)$$

where we have made use of the interchange (6.6) and the identity (6.8).

If we consider only the vicinity of the acoustic pole, (7.7) and (7.10), we may write[20]

$$\mathrm{Im}\frac{4\pi\alpha_-^*[\mathbf{k}-\mathbf{l},\omega(\mathbf{k})-\omega']}{\epsilon^*[\mathbf{k}-\mathbf{l},\omega(\mathbf{k})-\omega']}\simeq\tfrac{1}{2}\pi\omega(\mathbf{k}-\mathbf{l})\delta[\omega(\mathbf{k})-\omega'-\omega(\mathbf{k}-\mathbf{l})], \tag{7.11}$$

$$S_E(\mathbf{k},\omega)=S_E(k,\hat{k})\delta[\omega-\omega(\mathbf{k})], \tag{7.12}$$

and the amplitude function $S_E(k,\hat{k})$ defined via (7.12) satisfies a symmetry relation,

$$S_E(k,\hat{k})=S_E(k,-\hat{k}). \tag{7.13}$$

The last term of (7.10) is then explicitly calculated as

$$-\frac{\omega(\mathbf{k})}{16\pi n_0^2(4\pi e)^2}\left\{\int_{k_1}^{k}2kl^3S_E(l,\hat{k})dl+\int_{k}^{k_2}(k^2+l^2)l^2S_E(l,\hat{k})dl\right\}, \tag{7.14}$$

where k_1 and k_2 are lower and upper limits of wave number between which the acoustic wave may be well defined. It is clear that (7.14) represents a nonlinear damping of an acoustic wave, due to the presence of fluctuations of acoustic type in the stationary state, i.e., a mode-coupling effect.

VIII. FLUCTUATION SPECTRUM ASSOCIATED WITH THE ION ACOUSTIC MODE

According to the dielectric superposition principle,[11] the fluctuation spectrum for the internal electric field is given by

$$S_E(\mathbf{k},\omega)=(4\pi e/k)^2[S_-^{(0)}(\mathbf{k},\tilde{\omega})+S_+^{(0)}(\mathbf{k},\omega)]/|\epsilon(\mathbf{k},\omega)|^2$$
$$=-\left(\frac{4\pi e}{k}\right)^2[S_-^{(0)}(\mathbf{k},\tilde{\omega})+S_+^{(0)}(\mathbf{k},\omega)]\frac{1}{\mathrm{Im}\epsilon(\mathbf{k},\omega)}\mathrm{Im}\left[\frac{1}{\epsilon(\mathbf{k},\omega)}\right]. \tag{8.1}$$

Here $S_\pm^{(0)}(\mathbf{k},\omega)$ represent the density fluctuation spectra of the ions and the electrons in a fictitious system in which the effects of long-range Coulomb interactions are suppressed. Within the hydrodynamic description, they are given by[15]

$$S_\pm^{(0)}(\mathbf{k},\omega)=\frac{k^2\kappa T_\pm}{4\pi^2e^2\omega}\mathrm{Im}4\pi\alpha_\pm^{(0)}(\mathbf{k},\omega)$$
$$=\frac{n_0}{\pi}\frac{k^2D_\pm}{\omega^2+(k^2D_\pm-\tau_\pm\omega^2)^2}. \tag{8.2}$$

Note that the frequency variable for $S_-^{(0)}(\mathbf{k},\omega)$ in (8.1) has been replaced by the displaced frequency $\tilde{\omega}$ of (7.1) to take account of the uniform drift motion of the electrons as a whole.

The existence of a long-lived collective mode at $\omega\simeq\omega(\mathbf{k})$ gives rise to a sharp peak in the fluctuation spectrum (8.1); $\delta[\omega-\omega(\mathbf{k})]$ arises from $-\mathrm{Im}[1/\epsilon(\mathbf{k},\omega)]$, and the strength of the peak is inversely proportional to $\mathrm{Im}\epsilon[\mathbf{k},\omega(\mathbf{k})]$ or $-\gamma(\mathbf{k})$.[1,2] If we integrate (8.1) across the resonance pole (7.7) by taking account of the expressions (7.10), (7.12), (7.14), and (8.2), we obtain the following self-consistent equation for $S_E(k,\hat{k})$:

$$S_E(k,\hat{k})=2\pi\kappa T_-(m_-/m_+\tau_-)(k^2/k_-^2)\Bigg/\left[\frac{m_-}{m_+\tau_-}\frac{V_c-\hat{k}\cdot\mathbf{V}_d}{s}+\frac{sk}{8\pi n_0^2(4\pi e)^2}\left\{\int_{k_1}^{k}2kl^3S_E(l,\hat{k})dl+\int_{k}^{k_2}(k^2+l^2)l^2S_E(l,\hat{k})dl\right\}\right]. \tag{8.3}$$

Here,

$$V_c=[1+(m_+\tau_-/m_-\tau_+)]s \tag{8.4}$$

is a renormalized critical velocity and we have assumed $\hat{k}\cdot\mathbf{V}_d>0$, knowing that the other cases can be taken care of by the symmetry relation (7.13).

[20] The sharp resonant line structures of Eqs. (7.11) and (7.12) are valid only in the limit of $|\gamma(\mathbf{k})|\rightarrow 0$. Although this assumption may certainly be incorrect for a turbulent plasma and the broadening of the resonance line due to the finiteness of $\gamma(\mathbf{k})$ should be taken into consideration for a rigorous treatment, we also note that (7.11) and (7.12) appear in the integrand of (7.10) only. As long as the remaining factor in the integrand is a slowly varying function of the integration variable (in fact, it is independent of ω'), the primary importance is not in the minute detail of the resonance structure, but in the total strength represented by the area under the resonance curve. Modification of (7.11) and (7.12) to Lorentzian line shapes, for example, may be subsequently carried out with the values of $\gamma(\mathbf{k})$ which are determined from (7.10) by first assuming (7.11) and (7.12).

It is sometimes more convenient to consider the energy $\epsilon(k,\hat{k})$ contained in the **k** mode of acoustic oscillation, rather than the strength of internal field fluctuations. The transition between the two may be achieved by noting the following points: (1) The field energy density is given by $\mathbf{E}\cdot\mathbf{D}/8\pi$, where $\mathbf{D}$ is the electric displacement vector; (2) the dielectric shielding factor for the acoustic wave is k_-^2/k^2 for $k^2 \ll k_-^2$; (3) there is another contribution of equal amount from the kinetic energy. We thus find a relation[21]

$$\epsilon(k,\hat{k})=(k_-^2/k^2)S_E(k,\hat{k})/4\pi\,, \tag{8.5}$$

and we may rewrite (8.3) in terms of $\epsilon(k,\hat{k})$ as

$$\epsilon(k,\hat{k})=\kappa T_-(m_-/2m_+\tau_-)\Bigg/\left[\frac{m_-}{m_+\tau_-}\frac{V_c-\hat{k}\cdot\mathbf{V}_d}{s}+\frac{sk}{2n_0^2(4\pi e)^2k_-^2}\left\{\int_{k_1}^{k}2kl^5\epsilon(l,\hat{k})dl+\int_k^{k_2}(k^2+l^2)l^4\epsilon(l,\hat{k})dl\right\}\right]. \tag{8.6}$$

When $V_d \ll V_c$, $\epsilon(k,\hat{k})$ is so small (of the order of κT_-) that the mode-coupling term in the denominator is indeed negligible. We find from (8.6)

$$\epsilon(k,\hat{k})=\tfrac{1}{2}\kappa T_-\frac{s}{V_c-\hat{k}\cdot\mathbf{V}_d}. \tag{8.7}$$

This is identical to what one obtains from the theory of critical fluctuations,[1,2] or from a steady solution of Pines-Schrieffer collective kinetic equations.[2,22] When $V_d=0$, (8.7) exhausts its equipartition contribution $\frac{1}{2}\kappa T_-$ in the limit of $T_+/T_-\to 0$; the remaining $\frac{1}{2}\kappa T_-$ comes from $\epsilon(k,-\hat{k})$.[21]

As V_d enters the transition region $\hat{k}\cdot\mathbf{V}_d\simeq V_c$, $\epsilon(k,\hat{k})$ grows substantially so that the mode coupling term becomes significant. We have carried out a numerical solution of (8.6) in this transition region by choosing the following set of parameters for a helium plasma: $n_0=3\times10^{11}$ cm^{-3}, $T_-=100T_+=6.6\times10^4$ °K, $V_c=4\times10^7$ cm/sec, $s=1.2\times10^6$ cm/sec, $\tau_-=10^{-10}$ sec, and $\tau_+=2\times10^{-8}$ sec (the density of the neutral atoms $=2\sim3\times10^{17}$ cm^{-3}). The results are summarized in Figs. 14 and 15. We observe a substantial enhancement of $\epsilon(\mathbf{k})$ as compared with its equipartition value ϵ_0 at $V_d=0$, and an enormous piling up of fluctuations in the long-wavelength region as soon as $\hat{k}\cdot\mathbf{V}_d$ exceeds V_c. The cutoff at $k=k_1$ in Fig. 14 appears as a result of our assumption that the transition between the acoustic mode and the diffusion mode take place sharply at $k=k_1$. The transition is in fact rather a mild one[15]; we may expect that such a gradual transition acts to round off the peak and produce a tail of the spectrum toward the small k region in the vicinity of $k=k_1$.

Finally, if we pass to the region where $\hat{k}\cdot\mathbf{V}_d>V_c$, then $\epsilon(k,\hat{k})$ increases almost proportionally to $\hat{k}\cdot\mathbf{V}_d-V_c$, as may be determined from the denominator of (8.6). Let us investigate the over-all structure of the solution to (8.6) in this domain. In order to secure a systematic means to estimate the relative magnitude of the terms involved, we feel it convenient to consider the order of each term with respect to the plasma parameter, $g\equiv k_-^3/n_0(\ll 1)$. Since the discreteness parameters[13] e, m, and $1/n_0$ are all regarded to be of the same order as g, we find that the numerator is first order in g, while the first term in its denominator remains zeroth-order unless $\hat{k}\cdot\mathbf{V}_d$ takes on a value very close to V_c.

Let us now split $\epsilon(k,\hat{k})$ into two parts:

$$\epsilon(k,\hat{k})=\epsilon_0(k,\hat{k})+\epsilon_1(k,\hat{k})\,, \tag{8.8}$$

where $\epsilon_0(k,\hat{k})$ represents that part of $\epsilon(k,\hat{k})$ which is zeroth order in g, and $\epsilon_1(k,\hat{k})$ is the remaining higher-order part. If we susbsitute (8.8) in (8.6) and retain only the lowest-order contributions in g, we find an equation of the following structure:

$$\epsilon_0(k,\hat{k})=g\frac{A}{-B+F(k,\hat{k})}, \tag{8.9}$$

where

$$F(k,\hat{k})\equiv Ck\left[\int_{k_1}^{k}2kl^5\epsilon_0(l,\hat{k})dl+\int_k^{k_2}(k^2+l^2)l^4\epsilon_0(l,\hat{k})dl\right] \tag{8.10}$$

and A, B, C are positive parameters whose magnitudes are of the zeroth order in g. It is clear that $F(k,\hat{k})$ is a monotonically increasing function of k. We first observe that $\lim_{k\to k_1}F(k,\hat{k})-B$ must approach a small positive value of at least first order in g in order that the right-hand side of (8.9) can produce a zeroth-order contribution. This condition being satisfied, we also find from the structure of (8.10) that $F(k,\hat{k})-B$ can remain of the order g only in such a close neighborhood of k_1 that $(k-k_1)/k_1\sim g$; outside that, $F(k,\hat{k})-B$ goes up to larger values so that the right-hand side of (8.9) cannot produce values which belong to $\epsilon_0(k,\hat{k})$. All these indications suggest that $\epsilon_0(k,\hat{k})$ should exhibit a sharp peak of singular character as $g\to 0$, which may then be approximately expressed by the δ function, i.e.,

$$\epsilon_0(k,\hat{k})=\frac{(4\pi e)^2n_0^2m_-k_-^2}{m_+s\tau_-k_1^7}\frac{\hat{k}\cdot\mathbf{V}_d-V_c}{s}\delta(k-k_1). \tag{8.11}$$

[21] $\epsilon(k,\hat{k})$ in this calculation means the energy contained in the mode $[\mathbf{k},\omega(\mathbf{k})]$; we also note that there exists $\epsilon(k,-\hat{k})$ from the mode $[-\mathbf{k},-\omega(\mathbf{k})]$ whose amount is equal to $\epsilon(k,\hat{k})$. The use of the same notation ϵ for both the dielectric response function and the energy in the collective mode should not cause any confusion; the distinction can be made easily from the arguments.

[22] D. Pines and J. R. Schrieffer, Phys. Rev. 125, 804 (1962).

Again we must add a quick warning that this δ-function behavior should not be taken literally, because the appearance of the δ function depends critically on our assumption of the sharp transition at $k=k_1$. We may, however, well conclude from this calculation that, when $\hat{k}\cdot\mathbf{V}_d>V_c, \epsilon(k,\hat{k})$ should contain a peak structure around $k=k_1$ whose strength is proportional to $\hat{k}\cdot\mathbf{V}_d-V_c$.

Substituting (8.8) into (8.6), where $\epsilon_0(k,\hat{k})$ is now given by (8.11), we find the following equation for $\epsilon_1(k,\hat{k})$:

$$\epsilon_1(k,\hat{k})=\kappa T_-(m_-/2m_+\tau_-)\Big/\left[\frac{m_-}{m_+\tau_-}\frac{\hat{k}\cdot\mathbf{V}_d-V_c}{s}\left(\frac{k^2}{k_1^2}-1\right)+\frac{sk}{2n_0^2(4\pi e)^2k_-^2}\left\{\int_{k_1}^{k}2kl^5\epsilon_1(l,\hat{k})dl+\int_{k}^{k_2}(k^2+l^2)l^4\epsilon_1(l,\hat{k})dl\right\}\right]. \tag{8.12}$$

As we see in this equation, there are no more singularities involved for $\epsilon_1(k,\hat{k})$. In the vicinity of k_1, an approximate expression for $\epsilon_1(k,\hat{k})$ is

$$\epsilon_1(k,\hat{k})\simeq(4\pi)^3n_0^3e^4m_-/m_+\tau_-\Big/ sk\int_{k_1}^{k_2}l^6\epsilon_1(l,\hat{k})dl, \tag{8.13}$$

while in the vicinity of k_2, we find

$$\epsilon_1(k,\hat{k})\simeq\kappa T_-(m_-/2m_+\tau_-)\Big/\left[\frac{m_-}{m_+\tau_-}\frac{\hat{k}\cdot\mathbf{V}_d-V_c}{s}\left(\frac{k_-}{k_1}\right)^2+\frac{s}{n_0^2(4\pi e)^2}\int_{k_1}^{k_2}l^5\epsilon_1(l,\hat{k})dl\right]\frac{k^2}{k_-^2}. \tag{8.14}$$

In both (8.13) and (8.14), we simplified the equations by assuming $k_2^2\gg k_1^2$.

IX. DISCUSSION

For the theoretical study of a turbulent plasma, we first introduced our fundamental assumption by postulating a stable turbulent state for a plasma. The dielectric response function for such a turbulent plasma was then calculated, and with the aid of it the fluctuation spectrum was determined self-consistently. As should be clear from (7.10), (7.14), and (8.3), the results of our calculations indeed support our original assumption; the dielectric response function remains stable for the whole range of physical parameters.

We may thus pass rather smoothly from a stable region to a turbulent region. In terms of the small expansion parameter g of the plasma, the energy $\epsilon(\mathbf{k})$ in the acoustic wave with wave vector $\mathbf{k}$ is estimated to be of the order g (i.e., the thermal level) in the stable region; as the plasma enters the transition region, $\hat{k}\cdot\mathbf{V}_d\simeq V_c$, the order of $\epsilon(\mathbf{k})$ goes up to $g^{1/2}$; finally in the turbulent region we have seen that $\epsilon(\mathbf{k})$ contains a part of the order g^0. A conclusion from this observation is that, if one considers the limit of $g\rightarrow 0$ by neglecting the thermal fluctuations, the onset of instability, or passage through the critical point, $\hat{k}\cdot\mathbf{V}_d=V_c$, may still be regarded as an abrupt process.

In the course of calculations, we introduced a couple of simplifying assumptions. Let us now examine physical implications of those assumptions.

Firstly, we have simplified the iteration procedure by disregarding the contributions from the ternary and the higher-order correlation parts. We cannot of course justify the neglect of those higher order correlations. Based upon our calculations, however, we may certainly note that this neglect has not led to a dangerous or catastrophic behavior of the fluctuation spectrum, as did the neglect of pair correlations in the vicinity of the critical point. It was indeed in the divergent behavior of the fluctuation spectrum at the critical point that we sensed a signal which pointed to the inappropriateness of ignoring the effects of pair correlations in the description of the stationary state. As long as we do not see such a signal in the neglect of the ternary and higher-order correlation functions, we may content ourselves with the perspective that inclusion of such higher-order correlations would not produce a drastic change in the nature of the entire analysis, although it might possibly result in quantitative corrections to the final results.

Scondly, in the solution of the dispersion relation for the acoustic mode, we have neglected the contribution from $[2\pi S(x)/n_0^2]^2$ terms in favor of that from linear terms. It might therefore be said that by so doing we have shifted ourselves toward the consideration of weak turbulences. Nonetheless, we would like to emphasize that our original approach is equally valid for both strong and weak turbulences; the above concession had to be made only for the sake of analytical accessibility. Furthermore, even in the weak turbulence domain, our approach differs in a significant way from a perturbation-theoretic calculation starting from a quiescent stationary state.

To see this difference, let us go back to Eq. (7.10): Since the zeros of $\epsilon^*(\mathbf{k},\omega)$ remain in the upper half of the complex ω plane, calculation of the last term always leads to a nonlinear *damping* of the wave, as we have seen in (7.14). On the other hand, if we were to start from a quiescent state and carry out a perturbation calculation with respect to the fluctuations, we would have obtained the following equation in place of

(7.10):

$$\gamma(\mathbf{k})=-\frac{1}{2\tau_+}\frac{1}{2m_+\tau_-}\frac{\omega(\mathbf{k})-\mathbf{k}\cdot\mathbf{V}_d}{\omega(\mathbf{k})}+\frac{1}{16\pi^3 n_0^2(4\pi e)^2}\times\int d\omega'\int d^3l\,\mathrm{Im}\frac{4\pi\alpha_-^{(0)}[\mathbf{k}-\mathbf{l},\omega(\mathbf{k})-\omega']}{\epsilon^{(0)}[\mathbf{k}-\mathbf{l},\omega(\mathbf{k})-\omega']}\times l^2\frac{(\mathbf{k}-\mathbf{l})\cdot\mathbf{k}}{k^2}\left[\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{l})}{|\mathbf{k}-\mathbf{l}|^2}+\frac{\mathbf{k}\cdot\mathbf{l}}{l^2}\right]S_E(\mathbf{l},\omega'). \quad (9.1)$$

When the plasma is in the so-called unstable region, $\epsilon^{(0)}(\mathbf{k},\omega)$ possesses a zero in the upper-half plane; if one picks up the contribution of this pole in the calculation of the last term in (9.1), it will result in an ordinary nonlinear damping term of a perturbation calculation. If, however, we stop to think about a stable case, we may easily find that this term now changes its sign because all the zeros are located in the lower-half plane; this should then indicate that in the stable domain the mode coupling term acts to push the system toward instability. Furthermore, one cannot pass smoothly from the stable region to the turbulent region within this scheme because a zero of $\epsilon^{(0)}(\mathbf{k},\omega)$ in (9.1) cuts the real axis of frequency integration at the critical point.

All those unsatisfactory features have been cured in (7.10) by the choice of a correct dielectric response function for the turbulent plasma and appropriate boundary conditions. We point out that our results cannot be obtained by a straightforward extension of perturbation calculations starting from a quiescent state, since the turbulent dielectric response function involves additional sum of infinite number of ploarization diagrams as we have seen in Sec. IV.

It is instructive to investigate the spectrum (8.13) in terms of a dimensional argument of the Kolmogorov-Obukhov type.[23] In the region where $k\gtrsim k_1$, the principal decay mechanism is the successive energy-transfer process toward a higher wave-number region due to mode coupling with waves which possess relatively small wave numbers [mainly with those contained in $\epsilon_0(k,\hat{k})$]. Yet, the wave numbers in that region are much smaller than k_2 so that the wave energy cannot be effectively dissipated into heat. We may therefore argue that the flow rate of energy $\partial\epsilon_\mathbf{k}/\partial t$ toward large k region should be continuous and independent of k. To maintain a steady state, it must then be equal to the rate of energy transferred down from the mode with $k=k_1$. Since its strength [see Eq. (8.11)] is significantly large as compared with $\epsilon_1(k,\hat{k})$ and proportional to $(\hat{k}\cdot\mathbf{V}_d-V_c)$, it may be natural to assume that the rate of energy transfer is also proportional to $(\hat{k}\cdot\mathbf{V}_d-V_c)$; this estimation is also supported by the fact that the time rate of wave production by the drifting charged particles is proportional to $(\hat{k}\cdot\mathbf{V}_d-V_c)$. Indeed, contrary to the cases of isotropic turbulence in ordinary fluid, where the critical parameter is determined by the dimensionless Reynolds number, the onset of an instability for the acoustic plasma wave is characterized by the critical drift velocity. For the plasma, we may consequently regard $(\hat{k}\cdot\mathbf{V}_d-V_c)^{-1}\partial\epsilon_\mathbf{k}/\partial t\equiv A$ as a constant parameter over the wave-number range of interest. The constant must be expressible in terms of k and $\epsilon_\mathbf{k}$ only; the only dimensionally correct combination is

$$A\sim k\epsilon_\mathbf{k},$$

and thus we find

$$\epsilon_\mathbf{k}=C/k, \quad (9.2)$$

where C is a constant independent of the drift velocity. Comparing equations (8.13) and (9.2), we see that the result of our calculations satisfies (9.2) in the region where $k\gtrsim k_1$.

[23] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Ltd., London, 1959), Chap. III.

X. COMPARISON WITH EXPERIMENT

Let us now compare our calculation with an experimental result. Arunasalam and Brown[3] measured the density fluctuation spectrum in a weakly ionized turbulent plasma by means of microwave scattering experiments. The result of their measurement may be summarized by the expression

$$\epsilon_k\simeq\beta\kappa T_-(k_-^2/k^2), \quad (10.1)$$

where the enhancement factor β varies for the helium plasma from 460 in one case (case I) to 1010 in another (case II). They also pointed out that the theory of critical fluctuations[1,2] could explain the shape (i.e., the k^{-2} dependence) of (10.1) provided that (1) the state of the plasma was *strictly* at the critical point, and that (2) the critical wave number k_c was so small that the waves in its vicinity could not be detected by the scattering experiment. The former requirement seems too stringent to suppose; the latter assumption may be deemed unphysical because in particular it requires that the effects of collisions be negligible. Those rather unsatisfactory aspects were of course recognized by Arunasalam and Brown; a theoretical explanation of (10.1) has thus remained an open question.

Our present calculation, based upon the hydrodynamic description, has the advantage of taking almost full account of the effects of collisions; it suffers a severe drawback of hydrodynamics in that it loses its theoretical basis when applied to short-wavelength phenomena. We note that the region in which the measurement of (10.1) was actually carried out falls rather in the short-wavelength region, $0.1k_-\lesssim k$, so that a hydrodynamic analysis would be of doubtful validity in this domain. As far as the acoustic waves are concerned, however, we know from a kinetic-theoretical analysis that oscillations of the same character can exist down to a short-wavelength region comparable to the electron Debye length. With the help of this consideration, we may extend our calculation and choose the upper limit k_2 in the neighborhood of k_-.

We thus find that (8.14) offers a theoretical explanation of (10.1) without relaying upon such dubious assumptions as the exact critical conditions and the absence of collisional effects. Indeed, the k^{-2} dependence of the energy spectrum in the short-wavelength region ($k \simeq k_2$) is obtained in (8.14), not only at the critical conditions, but also above them; the effects of collisions are of course included in (8.14).

The peak structure of (8.11) and the k^{-1} part of (8.13) cannot be observed by their experiment because k_1 takes on a value much smaller than k_-. For case I, we compute[24] $k_1 \simeq 1.2$ cm^{-1} and $k_- \simeq 1.6\times10^2$, while for case II, $k_1 \simeq 4.0$ cm^{-1} and $k_- \simeq 90$ cm^{-1}; in both cases, $k_1/k_- < 0.05$, so that the oscillations with wave numbers in the vicinity of k_1 are outside the experimental detectability.

Finally, we note that (8.14) offers a theoretical means to estimate β. Comparing equations (8.14) and (10.1), we find

$$\beta \simeq \left[2\frac{\hat{k}\cdot \mathbf{V}_d - V_e}{s}\left(\frac{k_-}{k_1}\right)^2 + \frac{2m_+ s\tau_-}{m_- n_0^2(4\pi e)^2}\int_{k_1}^{k_2} l^5 \epsilon_1(l,\hat{k})dl \right]^{-1}. \tag{10.2}$$

Since there are many ambiguous factors involved in the application of this formula to the practical cases, we had to make a number of attempts of computations to allow for various possible choices of the parameters. All the attempts produced values of β considerably smaller than the experimental values. For example, when we assume $\hat{k}\cdot \mathbf{V}_d - V_e = 0$ and $k_2 = \frac{1}{2}k_-$, the computed values of β are about 101 for case I and about 315 for case II; these are to be compared with the experimental values, 460 and 1010, respectively. In view of the fact that there are many additional complications involved, such as the external magnetic field and the scattering from a finite plasma, which may all affect the estimation of the absolute value β, the above discrepancy may not be too surprising. We may maintain that there still remains some room to make further critical analysis concerning the absolute estimation of the enhancement factor.

Note added in proof. Recently our attention was called to the microwave scattering experiments carried out by J. T. Coleman [J. Appl. Phys. **38**, 2655 (1967)]. The significance of his experimental findings appears to be twofold: (1) A possibility of switching on and off the turbulence by controlling the electron temperature was demonstrated; (2) the existence of pronounced peaks in the turbulence spectrum at $k \simeq 0.02k_-$ was detected. We remark that the present theory is consistent with those experimental findings. The theory describes quite manifestly the appearance of turbulence as the plasma passes the critical conditions; it also predicts sharp peaks at about $k = k_1$ in the turbulence spectrum whose strengths, given by (8.11), are conspicuously larger than those associated with the other parts of the spectrum, as Fig. 8 in Coleman's paper seems to demonstrate experimentally. Furthermore, if we use numerical values, $n_0 = 3\times10^{10}$ cm^{-3}, $T_- = 5\times10^4$°K, and $1/\tau_+ = 7\times10^6$ sec^{-1}, which may be appropriate to characterize his experimental conditions for a hydrogen arc discharge at the pressure of 0.2 mm Hg [J. T. Coleman (private communication)], then we find $k_1 = 0.016k_-$; this value is to be compared with the experimentally found location of the peaks, $k \simeq 0.02k_-$. We may interpret this comparison as an indication that those peaks could be explained by Eq. (8.11). We are grateful to Dr. V. Arunasalam for calling our attention to Coleman's work.

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APPENDIX A: DIELECTRIC RESPONSE FUNCTION OF A TURBULENT PLASMA IN A MAGNETIC FIELD

For the sake of simplicity, we consider the cases of the electron gas; the basic equations are

$$\partial n/\partial t + \nabla\cdot\mathbf{\Gamma} = 0, \tag{A1}$$

$$\partial\mathbf{\Gamma}/\partial t = -(1/\tau)[\mathbf{\Gamma} + D\nabla n + \mu n\mathbf{E} + \mu(\mathbf{\Gamma}/c)\times\mathbf{B}], \tag{A2}$$

$$\nabla\cdot\mathbf{E} = -4\pi e(n - n_0). \tag{A3}$$

We Fourier expand everything in sight and eliminate $\mathbf{\Gamma}(x)$ and $\mathbf{E}(x)$; the elimination of $\mathbf{\Gamma}(x)$ can be simply done by following the procedure in the latter work of Ref. 15. Introducing a tensor $\mathsf{B}(x)$ by

$$\mathsf{B}(x) \equiv \frac{1-i\omega\tau}{(1-i\omega\tau)^2+(\Omega\tau)^2}\begin{bmatrix} 1-i\omega\tau & -\Omega\tau & 0 \\ \Omega\tau & 1-i\omega\tau & 0 \\ 0 & 0 & \dfrac{(1-i\omega\tau)^2+(\Omega\tau)^2}{1-i\omega\tau} \end{bmatrix}, \tag{A4}$$

[24] In all the computations, we have assumed $J=1$ in Table I of Ref. 3.

and defining a new coupling constant $\tilde{C}(x,x')$ by

$$\tilde{C}(x,x')=\mathbf{k}\cdot\mathbf{B}(x)\cdot\mathbf{k}'/|\mathbf{k}'|^2, \tag{A5}$$

we obtain an equation

$$\left[-\omega^2-\frac{i\omega}{\tau}+\left(\frac{D}{\tau}k^2+\omega_p^2\right)\tilde{C}(x,x)\right]n(x)+\frac{\omega_p^2}{n_0}\sum_{x'}\tilde{C}(x,x-x')n(x')n(x-x')=0. \tag{A6}$$

On comparing (A6) with a combined result of (3.8) and (3.9), we find that all the remaining calculations of Sec. III can be transformed into a form valid for the cases with magnetic field if we make the following replacement:

$$\begin{aligned}&C(x,x')\rightarrow\tilde{C}(x,x')/\tilde{C}(x,x),\\&4\pi\alpha^{(0)}(x)\rightarrow4\pi\tilde{\alpha}^{(0)}(x)=\tau\omega_p^2\tilde{C}(x,x)/[Dk^2\tilde{C}(x,x)-i\omega(1-i\omega\tau)].\end{aligned} \tag{A7}$$

APPENDIX B: CALCULATION OF $\epsilon(x)$ FOR A TWO-COMPONENT PLASMA

In this Appendix we outline some of the intermediate steps toward the calculation of the dielectric response function (5.2). The fundamental equations for the Fourier components, $n_\pm'(x)$, of the induced density fluctuations are

$$\left(-\omega^2-\frac{i\omega}{\tau_\pm}+\frac{D_\pm}{\tau_\pm}k^2\right)n_\pm'(x)\mp\omega_\pm^2\left\{\rho'(x)+\frac{1}{n_0}\sum_{x'}C(x,x-x')[n_\pm(x')\rho'(x-x')+n_\pm'(x')\rho(x-x')]\right\}=0, \tag{B1}$$

where

$$\rho(x)\equiv n_-(x)-n_+(x),\quad \rho'(x)\equiv Q(x)+n_-'(x)-n_+'(x), \tag{B2}$$

and $-eQ(x)$ is the strength of the external test charge. We carry out the statistical averaging of (B1) with the aid of the iteration and truncation procedure described in Sec. III. We find

$$\begin{aligned}\langle n_\pm(x')\rho'(x-x')\rangle=&-\frac{2\pi}{n_0}C(x-x',x)\left[\frac{4\pi\alpha_-^*(x-x')}{\epsilon^*(x-x')}S_{-\pm}(x')+\frac{4\pi\alpha_+^*(x-x')}{\epsilon^*(x-x')}S_{\pm+}(x')\right]\langle\rho'(x)\rangle\\&+\frac{2\pi}{n_0}C(x-x',x')\frac{4\pi\alpha_+^*(x-x')}{\epsilon^*(x-x')}S_{\pm\rho}(x')\langle n_+'(x)\rangle+\frac{2\pi}{n_0}C(x-x',x')\frac{4\pi\alpha_-^*(x-x')}{\epsilon^*(x-x')}S_{\pm\rho}(x')\langle n_-'(x)\rangle,\end{aligned} \tag{B3}$$

$$\begin{aligned}\langle n_\pm'(x-x')\rho(x')\rangle=&\pm\frac{2\pi}{n_0}C(x-x',x)\frac{4\pi\alpha_\pm^*(x-x')}{\epsilon^*(x-x')}\{[1+4\pi\alpha_\mp^*(x-x')]S_{\rho\pm}(x')-4\pi\alpha_\mp^*(x-x')S_{\rho\mp}(x')\}\langle\rho'(x)\rangle\\&\mp\frac{2\pi}{n_0}C(x-x',x')\frac{4\pi\alpha_\pm^*(x-x')}{\epsilon^*(x-x')}[1+4\pi\alpha_\mp^*(x-x')]S_{\rho\rho}(x')\langle n_\pm'(x)\rangle\\&\pm\frac{2\pi}{n_0}C(x-x',x')\frac{4\pi\alpha_+^*(x-x')4\pi\alpha_-^*(x-x')}{\epsilon^*(x-x')}S_{\rho\rho}(x')\langle n_\mp'(x)\rangle.\end{aligned} \tag{B4}$$

With the aid of those calculations, $\epsilon(x)$ can be obtained from (B1) as

$$\epsilon(x)=1+\frac{\langle n_+'(x)\rangle}{\langle\rho'(x)\rangle}-\frac{\langle n_-'(x)\rangle}{\langle\rho'(x)\rangle}. \tag{B5}$$

APPENDIX C: CALCULATION OF THE SECOND TERM IN EQ. (7.9)

With the aid of the dielectric superposition principle,[11] one finds

$$\mathrm{Im}S_{\rho+}(x')=-\mathrm{Im}[4\pi\alpha_+(x')]\frac{S_-^{(0)}(x')}{|\epsilon(x')|^2}+\mathrm{Im}[4\pi\alpha_-(x')]\frac{S_+^{(0)}(x')}{|\epsilon(x')|^2}. \tag{C1}$$

We have been solving the dispersion relation by retaining contributions of linear terms in $2\pi S(x)/n_0^2$. In (C1), we may therefore replace the imaginary parts of $4\pi\alpha_\pm(x')$ by those of $4\pi\alpha_\pm^{(0)}(x')$. It may then be clear that the second term in (7.9) can be decomposed into terms which have the same structure as the first two terms of (7.8).