Ion Acoustic Waves in a Collisional Plasma

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The damping of the ion acoustic waves by Coulomb collisions is studied by using the Fokker-Planck equation of Rosenbluth *et al.* for both the species constituting the plasma, namely, the electrons and the ions. In a plasma with weak collisions and in the absence of any external field, one finds that irrespective of the ratio T of the ion temperature to the electron temperature, the characteristic freqeuency gets affected only by the electron-ion collisions. However, as far as the collisional damping is concerned, both the ion-ion and the electron-ion collisions play a somewhat equally important role; the electron-electron and the ion-electron contributions are negligible compared to the other two. The damping increases with increase in the collision frequency but decreases with increase in T.

I. INTRODUCTION

ION acoustic waves, which were first predicted by Tonks and Langmuir[1] using the fluid analysis, have been studied by a number of authors[2–5] on the basis of the collisionless Boltzmann equation. The collisionless theory shows that when the electrons and the ions have the same temperatures, these waves are very heavily Landau-damped. Experimentally, the ion waves were first observed by Revans,[6] and their Landau damping was measured by Wong *et al.*[7] Wong *et al.* studied the space damping rather than the time damping in cesium and potassium and observed that the damping constant depends on the magnitude of the ion drift.

The collisional damping of the ion waves was studied by Bhadra and Varma[8]; they used Krook's model[9] to describe the ion-ion collisions. Kulsrud and Shen[10] used a slightly more realistic model for the collisional process. In their model, the ions were described by the Fokker-Planck equation of Rosenbluth *et al.*[11]; however, the electrons were treated by a fluid equation with the further assumption that the electrons were isothermal. This assumption of electrons being isothermal is equivalent to considering the Vlasov equation for the electrons, thereby neglecting the electron-electron and the electron-ion collisions. We remove this restriction on the electrons and treat both the electrons and the ions by the corresponding Fokker-Planck equations. We indeed find that the electron-ion contribution to the collision term can not be neglected as compared to the ion-ion contribution.

Similar calculations for high-frequency plasma waves have been done by Comisar[12] and Buti.[13–15] The effect of strong collisions on the low-frequency electrostatic plasma oscillations with and without an external magnetic field has been studied by Kuckes[16] by using fluid equations. Following Comisar and Buti, we solve the Fokker-Planck equation on the assumption that the Coulomb collisions are weak, i.e., $\omega\tau \gg 1$, where ω is the characteristic frequency of the wave and τ is the mean collision time for the collision process under consideration. We find that in the absence of any external field, irrespective of the ratio of the ion temperature to the electron temperature, the characteristic frequency gets affected only by the electron-ion collisions; however, as far as the collisional damping is concerned, both the electron-ion and the ion-ion collisions play a role: The electron-electron and the ion-electron contributions are negligible compared to the other two.

II. GENERAL THEORY

Consider an unbounded fully ionized hot plasma consisting only of electrons and ions without any external field. In equilibrium, both the electrons and the ions obey Maxwellian distribution of velocities (for the validity of the Fokker-Planck equation, see the Appendix), i.e.,

$$f_{0a}(v) = (2\pi v_a^2)^{-3/2} e^{-v^2/(2v_a^2)}, \tag{1}$$

where the subscript a stands either for the electron or for the ion and $v_a^2 = KT_a/m_a$ with $m_e = m$ and $m_i = M$. Nf_{0a} is the equilibrium distribution function for species a. For small perturbations, the linearized Fokker-

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Planck equation for longitudinal oscillations is given by

$$\partial f_a/\partial t+\mathbf{v}\cdot\partial f_a/\partial\mathbf{x}+(Ne_a/m_a)\mathbf{E}\cdot(\partial f_{0a}/\partial\mathbf{v}) =(\partial f_a/\partial t)_c, \quad (2)$$

where $f_a(\mathbf{x},\mathbf{v},t)$ is the perturbed distribution function and $(\partial f_a/\partial t)_c$ takes into account the collisions between like and unlike particles and is represented by

$$\left(\frac{\partial f_a}{\partial t}\right)_c=-\frac{\partial}{\partial\mathbf{v}}\cdot[f_{0a}\langle\mathbf{\Delta}\rangle_a+f_a\langle\mathbf{\Delta}\rangle_{0a}] +\frac{1}{2}\frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}}:[f_{0a}\langle\mathbf{\Delta\Delta}\rangle_a+f_a\langle\mathbf{\Delta\Delta}\rangle_{0a}], \quad (3)$$

with

$$\langle\mathbf{\Delta}\rangle_a=N\Gamma_a\sum_{J=e,i}\left(1+\frac{m_a}{m_J}\right)\frac{\partial}{\partial\mathbf{v}}\int d\mathbf{v}'\frac{f_J(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|}, \quad (4)$$

$$\langle\mathbf{\Delta}\rangle_{0a}=N\Gamma_a\sum_{J=e,i}\left(1+\frac{m_a}{m_J}\right)\frac{\partial}{\partial\mathbf{v}}\int d\mathbf{v}'\frac{f_{0J}(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|}, \quad (5)$$

$$\langle\mathbf{\Delta\Delta}\rangle_a=N\Gamma_a\sum_{J=e,i}\frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}}\int d\mathbf{v}'f_J(\mathbf{v}')|\mathbf{v}-\mathbf{v}'|, \quad (6)$$

and

$$\langle\mathbf{\Delta\Delta}\rangle_{0a}=N\Gamma_a\sum_{J=e,i}\frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}}\int d\mathbf{v}'f_{0J}(\mathbf{v}')|\mathbf{v}-\mathbf{v}'|. \quad (7)$$

In these equations $\Gamma_a=(4\pi e_a{}^2/m_a{}^2)\ln(4\pi N\lambda_D{}^3)$, with $\lambda_D{}^2=KT_{\mathrm{av}}/(4\pi Ne^2)$; T_{av} is the average of the electron and the ion temperatures. The perturbed electric field in Eq. (2) is given by the Poisson's equation,

$$\mathrm{div}\mathbf{E}=4\pi\sum_J e_J\int d\mathbf{v}\, f_J. \quad (8)$$

To solve the pair of coupled equations (2) and (8), we take the Fourier transforms in space and Laplace transforms in time of all the perturbed quantities; Eq. (2) then becomes

$$(s+i\mathbf{k}\cdot\mathbf{v})f_a'(\mathbf{k},\mathbf{v},s)-g_a(\mathbf{k},\mathbf{v})+(\mathrm{N}e_a/m_a)\mathbf{E}_k' \cdot(\partial f_{0a}/\partial\mathbf{v})=(\partial f_a'/\partial t)_c, \quad (9)$$

where f_a' and $\mathbf{E}_k'$ are the Fourier-Laplace transforms of f_a and $\mathbf{E}$ and $g_a(\mathbf{k},\mathbf{v})$ is the Fourier transform of the initial perturbation in the distribution function. To ensure the convergence of the integrals, we can have s in Eq. (9) such that $\mathrm{Re}\, s>0$. On further taking the Fourier transforms in velocity space, thus defining

$$F_a(\mathbf{k},\boldsymbol{\sigma},s)=\int d\mathbf{v}\, e^{-i\boldsymbol{\sigma}\cdot\mathbf{v}}f_a'(\mathbf{k},\mathbf{v},s), \quad (10)$$

Eq. (9) takes the form

$$\left(\frac{s}{k}-\frac{\partial}{\partial\sigma_z}\right)F_a-\frac{1}{k}G_a(\mathbf{k},\sigma) +\frac{ie_aN\sigma_zE_z'}{m_ak}e^{-\sigma^2v_a{}^2/2}=\frac{1}{k}\left(\frac{\partial F_a}{\partial t}\right)_c, \quad (11)$$

where

$$E_z'=-\frac{4i\pi}{k}\sum_J e_JF_J(\mathbf{k},0,s). \quad (12)$$

In writing Eqs. (11) and (12), we have assumed that $\mathbf{k}$ is along the z axis. Following the procedure outlined in Refs. 12–14, we can easily evaluate the collision term and on integrating Eq. (11), we get

$$F_a(\mathbf{k},\boldsymbol{\sigma},s)=e^{s\sigma_z/k}\left[Q_a(\boldsymbol{\sigma})+\frac{ie_aNE_z'}{m_ak}P_a(\boldsymbol{\sigma})\right]+\frac{v_a{}^4}{kL_a}e^{s\sigma_z/k} \times\int_{\sigma_z}^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}[F_a(\boldsymbol{\eta})(K_1{}^b+K_2{}^a+K_3{}^a) +F_b(\boldsymbol{\eta})K_4{}^a], \quad a\neq b \quad (13)$$

where $\boldsymbol{\sigma}'=(\sigma_x,\sigma_y,\sigma_z')$,

$$Q_a(\boldsymbol{\sigma})=\frac{1}{k}\int_{\sigma_z}^{\infty}d\sigma_z'G_a(\mathbf{k},\boldsymbol{\sigma}')e^{-s\sigma_z'/k}, \quad (14)$$

$$P_a(\boldsymbol{\sigma})=-\int_{\sigma_z}^{\infty}d\sigma_z'\sigma_z'\exp\left[-\frac{s\sigma_z'}{k}-\tfrac{1}{2}v_a{}^2\sigma'^2\right], \quad (15)$$

$$K_1{}^b(\boldsymbol{\sigma},\boldsymbol{\eta})=\frac{1}{2\pi^2}\left[\left(1+\frac{m_a}{m_b}\right)\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})}{(\boldsymbol{\sigma}-\boldsymbol{\eta})^2}-\frac{\{\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})\}^2}{(\boldsymbol{\sigma}-\boldsymbol{\eta})^4}\right] \times\exp[-\tfrac{1}{2}v_b{}^2(\boldsymbol{\sigma}-\boldsymbol{\eta})^2], \quad (16)$$

$$K_2{}^a(\boldsymbol{\sigma},\boldsymbol{\eta})=\frac{1}{\pi^2}\left[\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})}{(\boldsymbol{\sigma}-\boldsymbol{\eta})^2}-\frac{\{\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\eta})\}^2}{2(\boldsymbol{\sigma}-\boldsymbol{\eta})^4}\right] \times\exp[-\tfrac{1}{2}v_a{}^2(\boldsymbol{\sigma}-\boldsymbol{\eta})^2], \quad (17)$$

$$K_3{}^a(\boldsymbol{\sigma},\boldsymbol{\eta})=\frac{1}{\pi^2}\left[\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})}{\eta^2}-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})^2}{2\eta^4}\right]\exp[-\tfrac{1}{2}v_a{}^2(\boldsymbol{\sigma}-\boldsymbol{\eta})^2], \quad (18)$$

and

$$K_4{}^a(\boldsymbol{\sigma},\boldsymbol{\eta})=\frac{1}{2\pi^2}\left[\left(1+\frac{m_a}{m_b}\right)\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})}{\eta^2}-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\eta})^2}{\eta^4}\right] \times\exp[-\tfrac{1}{2}v_a{}^2(\boldsymbol{\sigma}-\boldsymbol{\eta})^2]. \quad (19)$$

In Eq. (13), we have introduced the Coulomb mean free path $L_a=v_a{}^4/(N\Gamma_a)$. If we assume that the collisions are infrequent, i.e., $kL_a\gg1$, then to first order in $(kL_a)^{-1}$, Eq. (13) gives

$$F_a(\mathbf{k},\boldsymbol{\sigma},s)=e^{s\sigma_z/k}\left[Q_a(\boldsymbol{\sigma})+\frac{ie_aNE_z'}{m_ak}P_a(\boldsymbol{\sigma})\right]+\frac{v_a^4}{kL_a}e^{s\sigma_z/k}\int_{\sigma_z}^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}\,e^{s\eta_z/k}\left[(K_1^b+K_2^a+K_3^a)\right.$$
$$\left.\times\left\{Q_a(\boldsymbol{\eta})+\frac{ie_aE_z'N}{m_ak}P_a(\boldsymbol{\eta})\right\}+K_4^a\left\{Q_b(\boldsymbol{\eta})+\frac{ie_bE_z'N}{m_bk}P_b(\boldsymbol{\eta})\right\}\right].\quad(20)$$

On substituting Eq. (20) in Eq. (12), we obtain

$$E_z'=\Phi(k,s)/\Psi(k,s)\,,\quad(21)$$

where

$$\Phi(k,s)=+\frac{4i\pi}{k}\left[\sum_{J=e,i}e_JQ_J(0)+\frac{v_e^4e}{kL_e}\int_0^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}e^{s\eta_z/k}\left\{Q_e(\boldsymbol{\eta})(K_1^i+K_2^e+K_3^e)_{\sigma=0}\right.\right.$$
$$\left.\left.-\frac{m^2}{M^2}Q_i(\boldsymbol{\eta})(K_1^e+K_2^i+K_3^i)_{\sigma=0}+Q_i(\boldsymbol{\eta})K_4^e(\boldsymbol{\sigma}=0)-\frac{m^2}{M^2}Q_e(\boldsymbol{\eta})K_4^i(\boldsymbol{\sigma}=0)\right\}\right]\quad(22)$$

and

$$\Psi(k,s)=1-\sum_{J=e,i}\frac{\omega_{pJ}^2}{k^2}P_J(\boldsymbol{\sigma}=0)-\frac{v_e^4\omega_{pe}^2}{k^3L_e}\int_0^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}e^{s\eta_z/k}\left[P_e(\boldsymbol{\eta})(K_1^i+K_2^e+K_3^e)_{\sigma=0}\right.$$
$$\left.-\frac{m}{M}P_i(\boldsymbol{\eta})K_4^e(\boldsymbol{\sigma}=0)-\frac{m^2}{M^2}P_e(\boldsymbol{\eta})K_4^i(\boldsymbol{\sigma}=0)+\frac{m^3}{M^3}P_i(\boldsymbol{\eta})(K_1^e+K_2^i+K_3^i)_{\sigma=0}\right].\quad(23)$$

In Eq. (23), $\omega_{pJ}^2=4\pi\, Ne^2/m_J$ is the plasma frequency for the species J.[16a] We may note here that $\Phi(k,s)$ depends only on the initial perturbation. If we consider only those perturbations for which $\Phi(k,s)$ is analytic in the complex s plane, then for the Laplace inversion of Eq. (21), we have to consider only the zeros of $\Psi(k,s)$ which are given by

$$1=\sum_{J=e,i}\frac{\omega_{pJ}^2}{k^2}P_J(\boldsymbol{\sigma}=0)+\frac{v_e^3\nu_c\omega_{pe}^2}{k^3}(\alpha_{ei}+\alpha_{ii}+\alpha_{ee}+\alpha_{ie})\,,\quad(24)$$

where $\nu_c=v_e/L_e$ is the effective electron collision frequency and

$$\alpha_{ei}=\int_0^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}e^{s\eta_z/k}\left[P_e(\boldsymbol{\eta})K_1^i(\boldsymbol{\sigma}=0)-\frac{m}{M}P_i(\boldsymbol{\eta})K_4^e(\boldsymbol{\sigma}=0)\right],\quad(25)$$

$$\alpha_{ii}=\frac{m^3}{M^3}\int_0^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}\times e^{s\eta_z/k}[(K_2^i+K_3^i)_{\sigma=0}]P_i(\boldsymbol{\eta})\,,\quad(26)$$

$$\alpha_{ee}=\int_0^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}\,e^{s\eta_z/k}P_e(\boldsymbol{\eta})(K_2^e+K_3^e)_{\sigma=0}\,,\quad(27)$$

$$\alpha_{ie}=\frac{m^2}{M^2}\int_0^{\infty}d\sigma_z'e^{-s\sigma_z'/k}\int d\boldsymbol{\eta}e^{s\eta_z/k}\times\left[\frac{m}{M}P_i(\boldsymbol{\eta})K_1^e(\boldsymbol{\sigma}=0)-P_e(\boldsymbol{\eta})K_4^i(\boldsymbol{\sigma}=0)\right].\quad(28)$$

[16a] In Eq. (22) and subsequently, the subscript σ appearing should be a vector.

Equation (24) represents the required dispersion relation which gives the oscillatory behavior of the plasma under consideration.

III. ANALYSIS OF THE DISPERSION RELATION

From Eq. (15), we have

$$P_J(\boldsymbol{\sigma}=0)=-\frac{1}{v_J^2}[1-\sqrt{2}a_Je^{a_J^2/2}\,\mathrm{erf}(a_J/\sqrt{2})]\,,\quad(29)$$

where $a_J=s/(kv_J)$ and

$$\mathrm{erf}(x)=\int_x^{\infty}dy\,e^{-y^2}.\quad(30)$$

While evaluating these integrals (P_J, etc.), strictly speaking we should take into account the pole contribution which gives Landau damping but since this has been discussed earlier, we shall not repeat it here. We shall further assume that $a_i\gg1$ but $a_e\ll1$ which actually is a consequence of the fact that $v_e^2\gg v_i^2$; thus on using the proper asymptotic and series expansions for the error function,[17] we immediately get

$$P_i(\boldsymbol{\sigma}=0)=-\frac{1}{v_i^2a_i^2}\left(1-\frac{3}{a_i^2}\right)+O\left(\frac{1}{a_i^6}\right)\quad(31)$$

and

$$P_e(\boldsymbol{\sigma}=0)=-\frac{1}{v_e^2}\left(1-\frac{\pi^{1/2}}{\sqrt{2}}a_e\right)+O(a_e^2).\quad(32)$$

As in the collisionless theory, we shall neglect terms of the order of a_e^2. Moreover, in evaluating the collision integrals, we shall retain terms only up to $1/a_i^2$.

[17] E. Jahnke and F. Emde, *Tables of Higher Functions* (McGraw-Hill Book Company, Inc., New York, 1960), 6th ed.

Let us first consider the electron-ion contribution which is given by α_{ei} in Eq. (25). On substituting Eqs. (16) and (19) in Eq. (25), we obtain

$$\alpha_{ei}=I_1{}^i+I_4{}^e, \tag{33}$$

where

$$I_1{}^i=\frac{1}{2\pi^2}\int_0^\infty d\sigma\, e^{-s\sigma/k}\int d\boldsymbol{\eta}P_e(\boldsymbol{\eta})\exp\left[\frac{s\eta_z}{k}-\tfrac{1}{2}v_i^2(\sigma\hat{e}_z-\boldsymbol{\eta})^2\right]\times\left[\left(1+\frac{m}{M}\right)\frac{\sigma(\sigma-\eta_z)}{(\sigma\hat{e}_z-\boldsymbol{\eta})^2}-\frac{\sigma^2(\sigma-\eta_z)^2}{(\sigma\hat{e}_z-\boldsymbol{\eta})^4}\right] \tag{34}$$

and

$$I_4{}^e=-\frac{m}{2\pi^2M}\int_0^\infty d\sigma\, e^{-s\sigma/k}\int d\boldsymbol{\eta}P_i(\boldsymbol{\eta})\times\exp\left[\frac{s\eta_z}{k}-\tfrac{1}{2}v_e^2(\sigma\hat{e}_z-\boldsymbol{\eta})^2\right]\left(\frac{\sigma\eta_z}{\eta^2}-\frac{\sigma^2\eta_z^2}{\eta^4}\right). \tag{35}$$

On rewriting Eq. (15) as

$$P_a(\boldsymbol{\eta})=-e^{v_a^2\eta_\perp^2/2}\int_0^\infty dy(y+\eta_z)\times\exp\left[-\frac{s}{k}(y+\eta_z)-\tfrac{1}{2}v_a^2(y+\eta_z)^2\right], \tag{36}$$

and on using Eq. (36) for P_i, Eq. (35) on performing $\boldsymbol{\eta}$ integration reduces to

$$I_4{}^e=\frac{m}{4\pi^{1/2}M}\int_0^\infty\frac{d\nu}{(\nu+b^2)^{5/2}}\int_0^\infty d\sigma\,\sigma\exp\left(-\frac{s\sigma}{k}-\tfrac{1}{2}v_e^2\sigma^2\right)\times\int_0^\infty dy\exp\left[\frac{\beta^2}{4(\nu+b^2)}-\frac{sy}{k}-\tfrac{1}{2}v_i^2y^2\right]\left[1-\nu\sigma y+\beta y+\frac{\beta}{2(\nu+b^2)}\{\beta(1-\nu\sigma y)-3\nu\sigma\}-\frac{\nu\sigma\beta^3}{4(\nu+b^2)^2}\right], \tag{37}$$

where

$$\beta=(v_e^2\sigma-v_i^2y)\quad\text{and}\quad b^2=\tfrac{1}{2}(v_e^2+v_i^2). \tag{38}$$

Now, to carry out the y integration, we shall neglect terms of the order of $\delta^2=v_i^2/v_e^2$ as well as terms of the order of $(1/a_i^4)$. The resulting expression can then be put into the dimensionless form by using the variables $(v_e\sigma)=x$ and $\frac{1}{2}v_e^2(\nu+b^2)^{-1}=z^2$; Eq. (37) thus reads

$$I_4{}^e=\frac{m}{\pi^{1/2}Mv_e^6}\int_0^{(1-\delta^2/2)}dz\int_0^\infty dx\frac{x}{\xi}\times\exp\left[-a_ex-\tfrac{1}{2}x^2(1-z^2)\right]\left[b_0+\frac{b_1}{\xi}+\frac{1}{\xi^2}(b_2-\tfrac{1}{2}\delta^2b_0)+\frac{1}{\xi^3}(b_3-\tfrac{1}{2}\delta^2b_1)-\frac{\delta^2b_2}{2\xi^4}-\frac{\delta^2b_3}{2\xi^5}\right], \tag{39}$$

where

$$\xi=(a_e+\delta^2z^2x),$$
$$b_0=z^2[1-\tfrac{1}{2}x^2(3-5z^2)-\tfrac{1}{2}z^2x^4(1-z^2)],$$
$$b_1=-\tfrac{1}{2}x[1-3z^2+z^2x^2(1-z^2)],$$
$$b_2=-\delta^2z^2(1-x^2+x^2z^2),$$

and

$$b_3=-\tfrac{1}{2}\delta^4z^2x(1-z^2). \tag{40}$$

$I_1{}^i$ can be evaluated by proceeding on the lines similar to the ones for $I_4{}^e$, and we get

$$I_1{}^i=\frac{2^{1/2}}{\pi^{1/2}v_e^6}\int_0^{(1-\delta^2/2)}dy\int_0^\infty dz\,z\int_z^\infty dx\times\exp[-a_ex-\tfrac{1}{2}x^2(1-y^2)][y^2+(1-y^2)\times\{-y^2x^2+\tfrac{1}{2}xz(1+x^2y^2-3y^2-x^2y^4)\}]. \tag{41}$$

The ion-ion contribution can be similarly obtained by simplifying Eq. (26); this gives

$$\alpha_{ii}=-8/(15\pi^{1/2}v_i^6a_i^5). \tag{42}$$

This, with i replaced by e, is exactly the same as the electron-electron contribution obtained by Comisar.[12] Equation (28), however, gives, on integration,

$$\alpha_{ie}=\frac{\sqrt{2}m^2}{\pi^{1/2}v_e^3v_i^3a_i^3M^2}\left[1-\frac{10}{a_i^2}+\frac{8m}{3a_e^2M}-\frac{2\delta^2M}{3m}\left(1-\frac{6}{a_i^2}\right)-\frac{\sqrt{2}a_e}{\pi^{1/2}}S\right], \tag{43}$$

where

$$S=\int_1^\infty\frac{d\nu}{\nu^{5/2}}\int_0^\infty dy e^{-\nu^2}\int_{-\infty}^\infty dz\exp\left[-z^2-\frac{\sqrt{2}v_ey}{\nu^{1/2}b}z\right]\times\left[\left(1-\frac{6}{a_i^2}\right)\left(1-\frac{Mv_i^2}{mb^2}z^2\right)+\frac{4Mv_i^4}{\nu mb^4a_i^2}z^4\right]. \tag{44}$$

Now it easy to see that $S_{\max}$ is $1/\delta$; so we immediately find that $\alpha_{ie}\sim(a_e^3M/m)\alpha_{ii}$ and can thus be neglected. On using the similar argument, we can show that $\alpha_{ee}\sim(a_e^3\delta M/m)\alpha_{ii}$. Hence to analyze the dispersion relation, we have to consider only the ion-ion and the electron-ion collisions.

Let us now introduce the following dimensionless variables:

$\omega/\omega_{pi}=\omega^*$, $\nu_c/\omega_{pi}=B$, $T_i/T_e=T$, and $a=k\lambda_e$,

where $\lambda_e^2=KT_e/(4\pi Ne^2)$ is the Debye length for the electrons. Equation (24), with the help of Eqs. (31), (32), (39), (41), and (42) can be written as

$$1=\frac{1}{\omega^{*2}}\left(1+\frac{3a^2T}{\omega^{*2}}\right)-\frac{1}{a^2}\left(1-\frac{\pi^{1/2}}{2^{1/2}}a_e\right)+C, \tag{45}$$

where

$$C=(C_{ii}+C_{ei})B\epsilon^{1/2}/a^3, \tag{46}$$

with $\epsilon = m/M$, $C_{ii} = v_e^6 \alpha_{ii}$, and $C_{ei} = v_e^6 \alpha_{ei}$. We shall solve Eq. (45) by the method of successive approximations. To lowest order, on neglecting C and a_e, we obtain

$$\omega^{*2} = (1/2\chi)[1+(1+12\chi a^2 T)^{1/2}] \equiv \omega_0^2, \quad \text{say.} \quad (47)$$

Here χ stands for $(1+1/a^2)$. For $T \ll 1$, Eq. (47) further reduces to

$$\omega_0^2 = 1/\chi + 3k^2\lambda_i^2; \quad (48)$$

the result obtained by Bernstein and Trehan.[18] λ_i in Eq. (48) is the ion Debye length. To next order, Eq. (45) takes the form

$$\chi = \frac{1}{\omega^{*2}}\left(1+\frac{3a^2T}{\omega^{*2}}\right) - i(\tfrac{1}{2}\epsilon\pi)^{1/2}\frac{\omega_0}{a^3} + C(\omega^* = \omega_0), \quad (49)$$

which we shall solve numerically. It is worth pointing out that

$$C_{ii}(\omega^* = \omega_0) = -8ia^5/15\pi^{1/2}T^{1/2}\omega_0^5, \quad (50)$$

being pure imaginary, does not affect the real part of ω^*; however, $C_{ei}(\omega^* = \omega_0)$ is a complex quantity and thus changes both the real and the imaginary parts of ω^*. The values of C_{ii} and C_{ei} for various values of a and T are given in Table I.

According to Eq. (49), $\mathrm{Re}\,\omega^* \equiv \omega_r^*$ increases with an increase in a, T, or B; but this increase, as shown in Tables II and III, is negligible. As illustrated in Figs. 1 and 2, $\mathrm{Im}\,\omega^* \equiv \omega_i^*$ increases with an increase in a or B, but decreases with an increase in T. So two-body Coulomb collisions have a tendency to stabilize the ion acoustic waves. As T increases, Landau damping takes over the collisional damping. For the sake of completeness, in Fig. 3, we have shown the thermal effects on the ion waves.

In a system where the collisions are frequent, this model will break down; in this case one should use the kinetic equation which takes into account the correlations between the charged particles.

TABLE I. C_{ii} and C_{ei} for various values of $a = k\lambda_e$ and $T = T_i/T_e$.

a	T	$\mathrm{Im}C_{ii}$	$\mathrm{Re}C_{ei}$	$\mathrm{Im}C_{ei}$
1.0	0.10	−2.3183	3.3309	−0.0228
	0.05	−6.9959	3.6253	0.0217
	0.01	−10.7640	4.0307	0.0686
0.1	0.10	−0.5656	2.1755	0.0465
	0.05	−1.0082	2.2879	0.0648
	0.01	−2.8690	2.4265	0.0825
0.05	0.10	−0.5568	2.1659	0.0471
	0.05	−0.9915	2.2770	0.0652
	0.01	−2.8175	2.4039	0.0826
0.01	0.10	−0.5540	2.1629	0.0473
	0.05	−0.9862	2.2736	0.0653
	0.01	−2.8011	2.3999	0.0827
0.001	0.10	−0.5539	2.1627	0.0473
	0.05	−0.9860	2.2734	0.0653
	0.01	−2.8004	2.3998	0.0827

TABLE II. $\mathrm{Re}\,\omega^*$ for $a=0.1$ and $T=0.1$, 0.05 and 0.01 for various values of B.

B	$T=0.1$	$T=0.05$	$T=0.01$
0.0	0.110936	0.105914	0.100927
0.0001	0.110938	0.105916	0.100929
0.001	0.110959	0.105939	0.100953
0.01	0.111167	0.106160	0.101187
0.05	0.112108	0.107161	0.102213
0.10	0.113323	0.108447	0.103456

TABLE III. $\mathrm{Re}\,\omega^*$ for $a=1.0$ and $T=0.1$, 0.05 and 0.01 for various values of B.

B	$T=0.1$	$T=0.05$	$T=0.1$
0.0	0.883308	0.858313	0.838316
0.0001	0.883309	0.858313	0.838317
0.001	0.883315	0.858318	0.838322
0.01	0.883376	0.858370	0.838368
0.05	0.883647	0.858599	0.838566
0.10	0.883987	0.858881	0.838807

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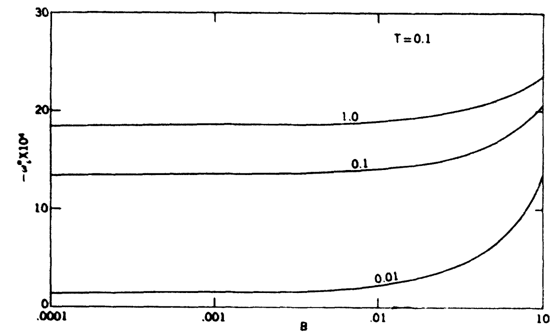

FIG. 1. Variation of $\mathrm{Im}\,\omega^*$ with B for $T=0.1$ is illustrated for $a=0.01$, 0.1, and 1.0

IV. CONCLUSIONS

Independent of the ratio of the ion temperature to the electron temperature T, the characteristic frequency of the ion acoustic waves in a plasma with weak Coulomb collisions gets affected only by the electron-

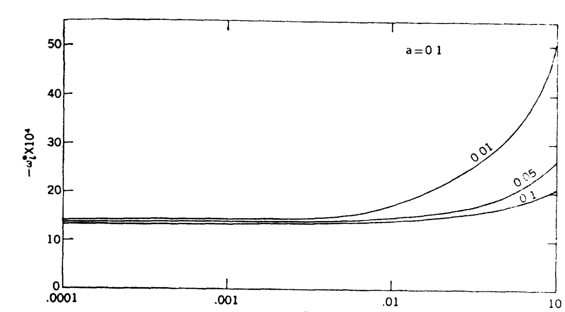

FIG. 2. Variation of $\mathrm{Im}\,\omega^*$ with B for $a=0.1$ is illustrated for $T=0.01$, 0.05, and 0.1.

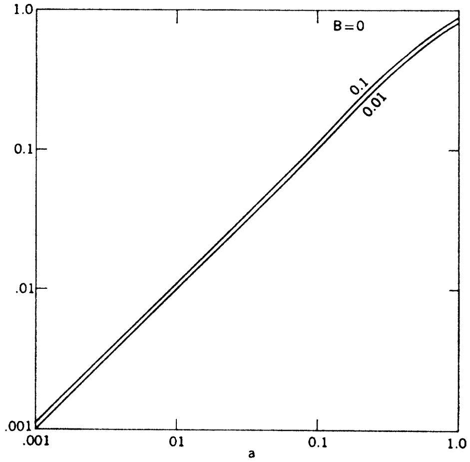

FIG. 3. Variation of Re ω^* with a for $B=0$ is illustrated for $T=0.01$ and 0.1.

ion collisions. Just as in a collisionless plasma, in this case also ω_r increases with increase in T; moreover, it increases with increase in the collision frequency ν_c as well as with $(k\lambda_e)$. The electron-electron and the ion-electron collisions are negligible compared to the ion-ion and the electron-ion collisions. The latter two play a somewhat equally important role in damping these waves. The collisional damping increases with increase in ν_c and $(k\lambda_e)$ but decreases with increase in T.

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APPENDIX

The distribution function given by Eq. (1) satisfies the Fokker-Planck equation only if the electrons and the ions have the same temperatures. In case they have different temperatures, which indeed is the case in our problem, we can show that we can have a quasi-steady state. To prove this, let us take the Fokker-Planck equation for the electrons, namely,

$$\partial f_e/\partial t+\mathbf{v}\cdot\partial f_e/\partial\mathbf{x}-(Ne/m)\mathbf{E}\cdot(\partial f_e/\partial\mathbf{v})=(\partial f_e/\partial t)_c, \quad \text{(A1)}$$

where

$$(\partial f_e/\partial t)_c=-(\partial/\partial\mathbf{v})\cdot(f_e\langle\boldsymbol{\Delta}\rangle_e)+\tfrac{1}{2}(\partial^2/\partial\mathbf{v}\partial\mathbf{v})(f_e\langle\boldsymbol{\Delta\Delta}\rangle_e); \quad \text{(A2)}$$

$\langle\boldsymbol{\Delta}\rangle_e$ and $\langle\boldsymbol{\Delta\Delta}\rangle_e$ are defined by Eqs. (4) and (6). The collision term of Eq. (A1) can be written in a more convenient form as

$$\left(\frac{\partial f_e}{\partial t}\right)_c=-\tfrac{1}{2}N\Gamma_e\frac{\partial}{\partial v_J}\sum_{a=e,i}\int\frac{d\mathbf{v}'}{|\mathbf{v}-\mathbf{v}'|}\times\left[\delta_{iJ}-\frac{(v_i-v_i')(v_J-v_J')}{(|\mathbf{v}-\mathbf{v}'|)^2}\right]\times\left[f_a(\mathbf{v}')\frac{\partial f_e}{\partial v_i}-\frac{m}{m_a}f_e(\mathbf{v})\frac{\partial f_a}{\partial v_1'}\right]. \quad \text{(A3)}$$

For a quasi-steady state, let the distribution function be

$$f_a=f_{0a}(1+\phi(t)), \quad \text{(A4)}$$

where f_{0a} is given by Eq. (1) and $\phi\ll1$. On substituting Eq. (A4) in Eq. (A1), we obtain

$$f_{0e}(\mathbf{v})d\phi/dt=(\partial f_e/\partial t)_c. \quad \text{(A5)}$$

On multiplying Eq. (A5) by v^2 and integrating over $\mathbf{v}$, we get

$$d\phi/dt=(N\Gamma_e/6v_e^4)(1+\phi)I, \quad \text{(A6)}$$

where

$$I=\int d\mathbf{v}v^2\frac{\partial}{\partial v_J}\left[f_{0e}(\mathbf{v})\int\frac{dv'}{|\mathbf{v}-\mathbf{v}'|}f_{0i}(\mathbf{v}')\left(v_i-\frac{v_i'}{T}\right)\times\left\{\delta_{iJ}-\frac{(v_i-v_i')(v_J-v_J')}{(|\mathbf{v}-\mathbf{v}'|)^2}\right\}\right], \quad \text{(A7)}$$

and $T=T_i/T_e$. On using Eq. (1) for f_{0e} and f_{0i}, after angular integrations Eq. (A7) yields

$$I=-\frac{2}{\pi v_e^3v_i^3}\int_0^\infty dv e^{-v^2/2V_e^2}v^2\int_0^\infty dv' e^{-v'^2/2V_i^2}v'^2S, \quad \text{(A8)}$$

where

$$S=\int_{-1}^1\frac{dx}{(v^2+v'^2-2vv'x)^{1/2}}\left[v^2-\frac{vv'}{T}x-\frac{v^2}{(v^2+v'^2-2vv'x)}\times\left\{v^2+\frac{v'^2}{T}-vv'\left(1+\frac{1}{T}\right)x-v'/v\left(v^2+\frac{v'^2}{T}\right)x+v'^2\left(1+\frac{1}{T}\right)x^2\right\}\right]. \quad \text{(A9)}$$

Now if we use the relation

$$\int_0^\infty dv\int_0^\infty dv' f(v,v')F(|v-v'|)=\int_0^\infty dv\int_0^v dv'[f(v,v')+f(v',v)]F(v-v'), \quad \text{(A10)}$$

Eq. (A8) can be written as

$$I=-\frac{2}{\pi v_e^3 v_i^3}\int_0^\infty dv\, v^2\int_0^v dv' v'^2[S\exp(-av^2-a'v'^2)$$
$$+S'\exp(-a'v^2-av'^2)], \quad (A11)$$

where $a=1/(2v_e^2)$, $a'=1/(2v_i^2)$, and $S'(v,v')=S(v',v)$. The result of simplifying S and S' is to take Eq. (A11) to

$$I=-(8/3\pi v_e^3 v_i^3)(1-1/T)[\alpha(1,4)+\beta(4,1)], \quad (A12)$$

where

$$\alpha(m,n)=\int_0^\infty dv\int_0^v dv' v^m v'^n \exp(-av^2-a'v'^2) \quad (A13)$$

and $\beta(m,n)$ is obtained from $\alpha(m,n)$ by simply interchanging a and a'. Finally on using the relation

$$\int_0^\infty dy\, y e^{-sy^2}\operatorname{erf}(a^{1/2}y)=\frac{(a\pi)^{1/2}}{4s(a+s)^{1/2}},$$

we obtain

$$I=-\frac{(1-1/T)}{2\pi^{1/2}v_e^3 v_i^3}\left[\frac{1}{aa'^2(a+a')^{1/2}}-\frac{1}{a'^2(a+a')^{3/2}}\right.$$
$$\left.-\frac{1}{a'(a+a')^{5/2}}+\frac{1}{aa'^{5/2}}-\frac{1}{a(a+a')^{5/2}}\right]. \quad (A14)$$

As done in the text, if we neglect terms of the order (v_i^2/v_e^2), then Eq. (A6) with the help of Eq. (A14) gives

$$\frac{d\phi}{dt}\cong-\frac{2^{3/2}N\Gamma_e v_i^2}{3v_e^5\pi^{1/2}}\left(1-\frac{1}{T}\right), \quad (A15)$$

which in terms of the electron collision frequency ν_c can be rewritten as

$$\frac{1}{\omega_{pi}}\frac{d\phi}{dt}=\frac{2^{3/2}}{3\pi^{1/2}}\left(\frac{\nu_c}{\omega_{pi}}\right)\left(\frac{m}{M}\right)(1-T). \quad (A16)$$

In case of weak collisions, $(\nu_c/\omega_{pi})\ll 1$, so $(d\phi/dt)$ is negligible and thus Eq. (1) holds good to a very good approximation.

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Computer "Experiments" on Classical Fluids. II. Equilibrium Correlation Functions*

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Equilibrium correlation functions for a dense classical fluid are obtained by integrating the equation of motion of a system of 864 particles interacting through a Lennard-Jones potential. The behavior of the correlation function at large distance, and that of its Fourier transform at large wave number, are discussed in detail and shown to be related to the existence of a strong repulsion in the potential. A simple hard-sphere model is shown to reproduce very well the Fourier transform of those correlation functions at high density, the only parameter of the model being the diameter a of the hard spheres.

I. INTRODUCTION

USING a technique directly inspired by the beautiful work of Rahman,[1] we have performed some experiments on a classical fluid composed of 864 molecules interacting through a Lennard-Jones potential $V(r)=4\epsilon[(\sigma/r)^{12}-(\sigma/r)^6]$ cut at $r_v=2.5\sigma$ or $r_v=3.3\sigma$. The details of these computations and a discussion of the thermodynamical results have been given elsewhere.[2] Here we give a discussion of the pair function $g(r)$ and of the various quantities which can be derived from it, namely its Fourier transform and the direct correlation function.

We discuss in Sec. II the pair function as given by the machine computation. Some comparisons are made with the results of the integral equations. The maximum of $g(r)$ is seen to be a compromise between the tendency of the particles to cluster around the core of the potential at high density and the attraction due to the bowl of the potential, which plays an essential role at low temperature. These results would be meager if it were not possible to extend them (Sec. III). Firstly, it is shown that the effect of the tail of the potential, for $r>r_v$, which has been neglected in the molecular-dynamics calculation, would not have changed $g(r)$ appreciably for $r<r_v$ if it had been included. The results can thus be extended to an uncut potential. Secondly, a procedure is constructed to extrapolate $g(r)$

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