Groups for Static Multispin Meson-Source Models. I

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We suggest such generalizations of the static strong-coupling group as would incorporate multispin meson sources in place of the conventional p-wave source only. A model which contains both p- and d-wave sources but no internal symmetry is considered in detail. This serves to illustrate the essential arithmetic of the problem. Also, the results of this model are directly adaptable to the corresponding group with isotopic-spin internal symmetry.

I. INTRODUCTION

THE large number of resonances that have been discovered in recent years and the expectation of an indefinite increase in this number suggest very strongly that one look for symmetries based on noncompact groups. This is because the physically interesting representations of such groups, namely, the unitary representations, are all infinite dimensional and could therefore accommodate in a natural way an indefinitely large number of particles and resonances in one single multiplet. This would enable a great deal of experimental information about all these particles and resonances to be correlated.

Such studies were initiated some time back by several authors,[1] but this was done within the context of relativistic dynamics, which makes the problem much too complicated. It would be of interest to pursue such a program within models which are simpler to deal with and which at the same time may have at least an approximate validity from the point of view of application to physical systems. This is what we wish to do in this series of papers.

Our starting point is the dynamical group of the static strong-coupling model for the pion-nucleon system, which was first obtained by Cook *et al.*[2] This group is a semidirect product of a mutually commuting set of nine operators $T_{\alpha i}$ ($\alpha=1, 2, 3; i=1, 2, 3$) and the group $SU(2)_I \otimes SU(2)_J$, where $SU(2)_I$ and $SU(2)_J$ are the usual isotopic-spin and spin groups. $T_{\alpha i}$ are essentially the source operators for the p-wave pion. They are nine in number because we are assigning the p-wave pion isopic spin 1, so that there are three degrees of freedom due to angular momentum and likewise three due to isospin. An interesting representation of this group is the one with $I=J=\frac{1}{2}, \frac{3}{2}, \cdots$. It has often been studied in the literature.[3] Unfortunately, in view of the experimental situation, this representation is rather restrictive. To illustrate the point, let us survey the data[4] on π-N scattering, in which case one has seen resonances up to a spin of 15/2 for $I=\frac{1}{2}$. For $I=\frac{3}{2}$ also, resonances up to spin 19/2 have been seen. On the other hand, the above representation can accommodate only one resonance for each of the above I values. As a first step in our approach then, we consider a larger representation of the above group. In fact, we choose the most general representation for which we have $I=\frac{1}{2}, \frac{3}{2}\cdots$ and $J=\frac{1}{2}, \frac{3}{2}, \cdots$ but without any restrictive constraints on I and J in the above representation.

The consideration of a larger representation of the above group, however, does not provide sufficient scope for getting any interesting results. There are two reasons for this. Firstly, many of the resonances we are interested in are odd parity resonances. The group which we are dealing with corresponds, on the other hand, to only p-wave pion sources (with intrinsic odd parity), so that its representations can accommodate only even parity resonances. Secondly, even if we confine ourselves to only even-parity resonances, their most interesting decay mode, namely, the decay into a nucleon and a pion, is forbidden as soon as we consider J values greater than $\frac{3}{2}$. This is because the final pion in these cases comes out in higher waves, whereas the group contains only p waves. The theory, therefore, does not have much predictive power.

To improve the situation, we therefore enlarge the group itself. This is done by enlarging the set of nine p-wave meson source operators so that one now has d and f waves, etc. In principle, it is possible to incorporate in this manner sources up to arbitrary high angular momenta, but because of computational complexities we confine ourselves to only s, p and d waves. This is enough to bring about the required "parity doubling" in the corresponding representations, and already many interesting decays modes can be calculated.

Our calculations follow the conventional method of exploitation of the commutation relations of the group.[5] This way we obtain at one stroke the representations of the group as well as the required Clebsch-Gordan

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[1] Abdus Salam and J. Strathdee, Proc. Roy. Soc. (London) **A292**, 314 (1965); Phys. Rev. **148**, 1352 (1966); J. Fischer and R. Raczka, ICTP, Trieste, Report IC/66/101, 1966 (unpublished); R. L. Anderson, J. Fischer and R. Raczka, ICTP, Trieste, Report IC/66/102 (unpublished).
[2] T. Cook, C. J. Goebel and B. Sakita, Phys. Rev. Letters **15**, 35 (1965). See also L. K. Pande, Phys. Letters **24B**, 243 (1967).
[3] See Ref. (2) and V. Singh, Phys. Rev. **144**, 1275 (1966); S. K. Bose, *ibid.* **145**, 1247 (1966).
[4] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).
[5] M. A. Naimark, *Linear Representation of the Lorentz Group* (Pergamon Press, Inc., New York, 1964).

coefficients that we need to compute various decay widths, etc.

The group that we have to consider is $SU(2)_J \otimes SU(2)_I \times T_{27}$, where the 27 T's now include 3-wave operators, 9 p-wave operators, and 15 d-wave operators. The representation that will be studied is characterized by $I=\frac{1}{2}, \frac{3}{2}, \cdots$ and $J=\frac{1}{2}, \frac{3}{2}, \cdots$ with all I and J values allowed. Besides, we shall have parity doubling. This representation will be of interest for application to the π-N system. A somewhat different representation that is characterized by $I=0, 1, \cdots$ but the same J values as above will also be considered in connection with the discussion of the hyperon isobars.[6]

The s-wave part of the problem in the above is quite trivial. It turns out that the p- and d-wave part of the problem gets essentially completely solved if the solution for the case of $SU(2)_J \times T_8$ is known. This latter group corresponds to p- and d-wave sources but without the internal symmetry $SU(2)_I$. In the present paper, we confine ourselves therefore to a detailed discussion of this group. We obtain the relevant representation and all the Clebsch-Gordan coefficients of interest. This is all contained in the following section. In the Appendix, we obtain the Casimir operators of this group, which we have used in the text.

In the following paper the results obtained here will be adapted to the case of $SU(2)_J \otimes SU(2)_I \times T_{27}$, and application will then be made to the pion-nucleon and the hyperon-pion systems.

II. THE GROUP WITH p AND d WAVES

(1) We shall here consider the case of a pseudoscalar meson interacting with a baryon, with neither of these particles having any internal quantum numbers. For the sake of definiteness, if we wish, we could take these particles to be η and Λ, respectively. Since we want to include in our discussion both p and d waves, the group of interest is $SU(2) \times T_8$, i.e., the semidirect product of the $SU(2)$ group of spin and the Abelian group of eight generators T_8, the first three of which correspond to p waves and the last five to d waves. The underlying commutation relations, in the spherical basis which we shall always employ, are as follows:

$$[J_+,J_-]=2J_z, \tag{1a}$$

$$[J_z,J_\pm]=\pm J_\pm, \tag{1b}$$

$$[J_\pm,P_\alpha]=[(1\mp\alpha)(2\pm\alpha)]^{1/2}P_{\alpha\pm1}, \tag{1c}$$

$$[J_z,P_\alpha]=\alpha P_\alpha, \tag{1d}$$

$$[J_\pm,T_\mu]=[(2\mp\mu)(3\pm\mu)]^{1/2}T_{\mu\pm1}, \tag{1e}$$

$$[J_z,T_\mu]=\mu T_\mu; \tag{1f}$$

and

$$[P_\alpha,P_\beta]=0, \tag{2a}$$

$$[T_\mu,T_\nu]=0, \tag{2b}$$

$$[P_\alpha,T_\mu]=0. \tag{2c}$$

$J+$, $J-$, and J_z here stand for the usual angular-momentum operators; P_α corresponds to the first three operators of the set T_8, and T_μ to the last five; α (and β) take the values $+1$, 0, and -1 and μ (and ν) take values $+2, +1, 0, -1, -2$. The commutation relation of P_α and T_μ with the J's follow easily from the requirement that they go as irreducible spin one and spin two operators under $SU(2)$.[7] The commutation relations between the various T's and P's are characteristic of our dynamical model. They may be classed as "dynamical" commutation relations. The remaining ones may then conveniently be labelled as "kinematical" commutation relations.

The fact that we are dealing with a pseudoscalar meson means that the operators P_α are even under parity and the operators T_μ are odd. The simultaneous presence of these even and odd operators will lead to representations with "parity-doubling," i.e., we shall find that if we have a particle in the representation with spin N and even parity, then the representation will also contain a particle with the same spin but opposite parity. To make this point clearer, we construct the three independent Casimir invariants of our group. [As the group we are dealing with is not semisimple, there presumably do not exist general expressions for the corresponding Casimir invariants in the literature; we therefore go through some details of this construction in Appendix A, following a contraction procedure[8] which enables us to obtain our group from the compact group $SU(2) \otimes SU(3)$ and likewise enables us to obtain our Casimir invariants also from the well-known Casimir invariants of this group.] Denoting these by A, B, and C, we have

$$A=\tfrac{1}{4}P_\alpha^2, \tag{3}$$

$$B=\tfrac{1}{6}[\tfrac{3}{4}P_\alpha^2+(-1)^\mu T_\mu T_{-\mu}], \tag{4}$$

$$\begin{aligned}C=&-(1/36)T_0^3-\tfrac{1}{6}T_{+2}T_{-2}T_0+(1/4\sqrt{6})T_{+2}T_-T_-\\&+(1/4\sqrt{6})T_{-2}T_+T_+-(3/32\sqrt{6})T_{+2}P_-P_-\\&-(3/32\sqrt{6})T_{-2}P_+P_++(3/16\sqrt{6})T_+P_0P_-\\&-(3/16\sqrt{6})T_-P_0P_+-\tfrac{1}{32}T_0P_-P_++\tfrac{1}{12}T_0T_+T_-\\&+\tfrac{1}{16}T_0P_0^2.\end{aligned} \tag{5}$$

From the above expressions, it is clear that A and B

[6] For the p-wave case, an isobar series for which $J=I\pm\frac{1}{2}=\frac{1}{2}, \frac{3}{2}\cdots$ has been considered by V. Singh and B. M. Udgaonkar, Phys. Rev. **149**, 1164 (1966).

[7] See, e.g., A. R. Edmond, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[8] E. Inonu and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **39**, 510 (1953). See also E. Inonu, *Group Theoretical Concepts and Methods in Elementary Particle Physics* (Gordon and Breach Science Publishers, Inc., New York, 1964).

TABLE I. The functions $a_J{}^\alpha(M)$, $b_J{}^\alpha(M)$, and $c_J{}^\alpha(M)$ in Eq. (9) of the text.

α	$+1$	0	-1
$a_J{}^\alpha(M)$	$[(J-M)(J-M-1)]^{1/2}$	$-\sqrt{2}(J^2-M^2]^{1/2}$	$[(J+M)(J+M-1)]^{1/2}$
$b_J{}^\alpha(M)$	$[(J+M+1)(J-M)]^{1/2}$	$-\sqrt{2}M$	$-[(J-M+1)(J+M)]^{1/2}$
$c_J{}^\alpha(M)$	$[(J+M+1)(J+M+2)]^{1/2}$	$\sqrt{2}[(J-M+1)(J+M+1)]^{1/2}$	$[(J-M+1)(J-M+2)]^{1/2}$

are even, but C is odd under parity. The latter immediately leads to parity doubling.[9,10]

With these preliminaries over, we now come to the problem of constructing the representations of our group. The simpler part of the problem here is essentially identical to constructing the representations of the smaller group $SU(2)\times T_3$ and has been dealt with by other authors.[2,3] We shall therefore go through summarily, though keeping the discussion self-contained. The next section is devoted to this part.

(2) Let us start by considering a state $\psi_M{}^J$, corresponding to spin J and the z component of spin M, which is a representation of the $SU(2)$ subgroup of the group $SU(2)\times T_3$. Using the $SU(2)$ commutation relations [1(a)] and [1(b)], we obtain

$$J_\pm\psi_M{}^J=[(J\mp M)(J\pm M+1)]^{1/2}\psi_{M\pm1}{}^J\,, \tag{6}$$

$$J_z\psi_M{}^J=M\psi_M{}^J\,. \tag{7}$$

We now consider the application of P_α on $\psi_M{}^J$. Since P_α is a spin-one object under $SU(2)$, acting on $\psi_M{}^J$, it can only produce states which have spin $J-1$, J, or $J+1$. Furthermore, it follows from the commutation relations (1d) that the z component of spin for all these states will be $M+\alpha$. Hence we can write

$$P_\alpha\psi_M{}^J=\tilde{a}_J{}^\alpha(M)\psi_{M+\alpha}{}^{J-1}+\tilde{b}_J{}^\alpha(M)\psi_{M+\alpha}{}^J+\tilde{c}_{J+1}{}^\alpha(M)\psi_{M+\alpha}{}^{J+1}\,. \tag{8}$$

We now apply the commutation relations (1c) on $\psi_M{}^J$ and make use of the expansion (8). On equating the coefficients of $\psi_{M+\alpha\pm1}{}^{J-1}$, $\psi_{M+\alpha\pm1}{}^J$, and $\psi_{M+\alpha\pm1}{}^{J+1}$ on both sides, we get a pair of equations each time involving, respectively, $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$. These equations can be solved easily to yield

$$\begin{aligned}\tilde{a}_J{}^\alpha(M)&=a_Ja_J{}^\alpha(M)\,,\\ \tilde{b}_J{}^\alpha(M)&=b_Jb_J{}^\alpha(M)\,,\\ \tilde{c}_{J+1}{}^\alpha(M)&=c_{J+1}c_{J+1}{}^\alpha(M)\,,\end{aligned} \tag{9}$$

with the M-dependent functions $a_J{}^\alpha(M)$, etc., given in Table I.

It remains now to determine a_J, b_J, and c_J. To do this, we first note that up to now there has been an arbitrariness in the definitions of a_J and c_{J+1}. This arbitrariness stems from the fact that all our formulas remain unchanged if we work with $\psi'_M{}^J=\phi(J)\psi_M{}^J$ instead of $\psi_M{}^J$, where $\phi(J)$ is an arbitrary function of J. The only change this brings in is that we now have $a_J\to a_{J'}$, $c_J\to c_{J'}$ with $a_Jc_J=a_J'c_J'$. As is demonstrated by Naimark,[5] this allows us to choose $\phi(J)$ such that we may have

$$a_J=c_J\,. \tag{10}$$

We shall therefore assume this equality. To proceed further, we now apply the dynamical commutation relation [2(a)] on $\psi_M{}^J$, making liberal use of Eqs. (8), (9), (10), and Table I. We choose, for instance, $\alpha=+1$ and $\beta=0$ in [2(a)]. Acting on $\psi_M{}^J$, the commutator now gives terms containing $\psi_{M+1}{}^{J-2}$, $\psi_{M+1}{}^{J-1}$, $\psi_{M+1}{}^J$, $\psi_{M+1}{}^{J+1}$, and $\psi_{M+1}{}^{J+2}$. The coefficients of the first and the last terms come out essentially to be zero. Equating the coefficients of the remaining terms to zero, we get the following equations:

$$\begin{aligned}&[(J+1)b_J-(J-1)b_{J-1}]a_J=0\,,\\ &[(J+2)b_{J+1}-Jb_J]a_{J+1}=0\,,\\ &(2J-1)a_J{}^2-(2J+3)a_{J+1}{}^2-b_J{}^2=0\,.\end{aligned} \tag{11}$$

It should be noted that J here characterizes the representation $\psi_M{}^J$ of the $SU(2)$ spin group with which we started. J could therefore only be $\geqslant J_0$, where $J_0=0$ or $\frac{1}{2}$. It is clear from the expansion (8), then, that $\tilde{a}_J{}^\alpha(M)=0$, or from Eq. (9) that

$$a_{J_0}=0\,. \tag{12}$$

The Eqs. (11) can be solved easily. The result is

$$b_J=\frac{iJ_0C}{J(J+1)}\,, \tag{13}$$

$$c_J=\frac{C}{J}\left(\frac{J^2-J_0{}^2}{4J^2-1}\right)^{1/2}\,, \tag{14}$$

C in these equations is an arbitrary constant. In order to have our representations unitary, we shall have to choose this constant to be purely imaginary. To see this we note that in a unitary representation the operator P_0 should be Hermitian and the operators P_+ and P_- should be Hermitian conjugates of each other, i.e., we should have

$$\langle\psi_M{}^J,P_0\Psi_M{}^J\rangle=\langle P_0\psi_M{}^J,\psi_M{}^J\rangle$$

and

$$\langle\psi_{M+1}{}^J,P_+\psi_M{}^J\rangle=\langle P_-\psi_{M+1}{}^J,\psi_M{}^J\rangle\,.$$

Using Eqs. (8)–(10), (13), and (14) and Table I, we can check that these conditions are satisfied only if we choose C as purely imaginary. Thus, if we take C_1

[9] See, e.g., W. Pauli, CERN Report No. 56-31, 1956 (unpublished).

[10] Dr. B. Sakita has kindly pointed out to us that the group corresponding to d-wave pion sources has recently been mentioned by C. J. Goebel in *Non-compact Groups in Particle Physics*, edited by Yutze Chow (W. A. Benjamin, Inc., New York, 1966).

to be a real constant, we have

$$C = iC_1, \quad C_1 \text{ real}. \tag{15}$$

We now proceed to the discussion of the d-wave part of our problem.

(3) To solve the d-wave part of the problem, we note that T_μ is now a spin-2 object under $SU(2)$ so that, acting on a state $\psi_M{}^J$, it leads to states ranging from $J-2$ to $J+2$. From the commutation relations [1(f)], it follows that the z component of spin for all these latter states will be $M+\mu$. We thus have

$$T_\mu \psi_M{}^J = \tilde{A}_J{}^\mu(M)\psi_{M+\mu}{}^{J-2} + \tilde{B}_J{}^\mu(M)\psi_{M+\mu}{}^{J-1} + \tilde{C}_J{}^\mu(M)\psi_{M+\mu}{}^J + \tilde{D}_{J+1}{}^\mu(M)\psi_{M+\mu}{}^{J+1} + \tilde{E}_{J+2}{}^\mu(M)\psi_{M+\mu}{}^{J+2}. \tag{16}$$

It should be noted that since T_μ is an odd-parity operator, we have actually two sets of $\tilde{A}_J{}^\mu(M)$, etc., one resulting from the operator of T_μ on $\psi_M{}^J$ with positive parity and the other resulting from this operator on a $\psi_M{}^J$ with negative parity. It is a simple matter to check, however, that because of time-reversal invariance, these two sets are identical. This result has been anticipated in the notation of expression (16). Furthermore, the argument which led to Eq. (12) in the previous section now implies.

$$\tilde{A}_{J_0}{}^\mu(M) = \tilde{A}_{J_0+1}{}^\mu(M) = 0, \quad \tilde{B}_{J_0}{}^\mu(M) = 0. \tag{17}$$

Next, we apply the commutator [1(e)] on $\psi_M{}^J$:

$$[J_\pm, T_\mu]\psi_M{}^J = [(2\mp\mu)(3\pm\mu)]^{1/2} T_{\mu\pm1}\psi_M{}^J. \tag{18}$$

We now use Eq. (16) and Eqs. (6) and (7) to solve (18). Note that both sides of the equation now involve $\psi^{J'}$ with $J' = J+2, J+1, J, J-1$, or $J-2$. Comparing the coefficients of each $\psi^{J'}$, we obtain equations the solutions of which give us the desired expressions for $\tilde{A}_J{}^\mu(M)$, $\tilde{B}_J{}^\mu(M)$, etc. These equations clearly show that we may write

$$\tilde{A}_J{}^\mu(M) = A_J{}^\mu(M) A_J \tag{19}$$

and similar expressions for $\tilde{B}_J{}^\mu(M)$, etc. A rather tedious calculation now needs to be performed to evaluate $A_J{}^\mu(M)$, etc. The results we thus obtain are contained in Table II. A point which should be noted in obtaining these results is that one should be careful to choose consistent phase convention. We do this by relating the phases of all $A_J{}^\mu(M)$, etc. for $\mu = 1, 0, -1, -2$ to that of the corresponding function for $\mu = 2$. For the latter we choose the phase to be $+1$, without, of course, any loss of generality.

Following now an argument identical to that which gave us Eq. (10), we can show that the functions A_J and B_J are related, respectively, to D_{J+1} and E_{J+2}. In fact, we have

$$E_J = A_J, \quad D_J = B_J. \tag{20}$$

TABLE II. The functions $A_J{}^\mu(M)$, $B_J{}^\mu(M)$, $C_J{}^\mu(M)$, $D_J{}^\mu(M)$, and $E_J{}^\mu(M)$ in Eq. (19).

	$\mu=+2$	$\mu=+1$
$A_J{}^\mu(M)$	$[(J-M)(J-M-1)(J-M-2)(J-M-3)]^{1/2}$	$-2[(J+M)(J-M)(J-M-1)(J-M-2)]^{1/2}$
$B_J{}^\mu(M)$	$[(J+M+1)(J-M)(J-M-1)(J-M-2)]^{1/2}$	$-[J+2M+1)[(J-M)(J-M-1)]^{1/2}$
$C_J{}^\mu(M)$	$[(J+M+1)(J+M+2)(J-M)(J-M-1)]^{1/2}$	$-(2M+1)[(J+M+1)(J-M)]^{1/2}$
$D_{J+1}{}^\mu(M)$	$[(J+M+1)(J+M+2)(J+M+3)(J-M)]^{1/2}$	$(J-2M)[(J+M+1)(J+M+2)]^{1/2}$
$E_{J+2}{}^\mu(M)$	$[(J+M+1)(J+M+2)(J+M+3)(J+M+4)]^{1/2}$	$2[(J+M+1)(J+M+2)(J+M+3)(J-M+1)]^{1/2}$

	$\mu=0$	$\mu=-1$	$\mu=-2$
$A_J{}^\mu(M)$	$\sqrt{6}[(J+M)(J+M-1)(J-M)(J-M-1)]^{1/2}$	$-2[(J+M)(J+M-1)(J+M-2)(J-M)]^{1/2}$	$[(J+M)(J+M-1)(J+M-2)(J+M-3)]^{1/2}$
$B_J{}^\mu(M)$	$\sqrt{6}M[(J+M)(J-M)]^{1/2}$	$(J-2M+1)[(J+M)(J-M+1)]^{1/2}$	$-[(J+M)(J+M-1)(J+M-2)(J-M+1)]^{1/2}$
$C_J{}^\mu(M)$	$(\sqrt{6})^{-1}(6M^2-2J^2-2J)$	$(2M-1)[(J+M)(J-M+1)]^{1/2}$	$[(J+M)(J+M-1)(J-M+1)(J-M+2)]^{1/2}$
$D_{J+1}{}^\mu(M)$	$-\sqrt{6}M[(J+M+1)(J-M+1)]^{1/2}$	$-(J+2M)[(J-M+1)(J-M+2)]^{1/2}$	$-[(J+M)(J-M+1)(J-M+2)(J-M+3)]^{1/2}$
$E_{J+2}{}^\mu(M)$	$\sqrt{6}[(J+M+1)(J+M+2)(J-M+1)(J-M+2)]^{1/2}$	$2[(J+M+1)(J-M+1)(J-M+2)(J-M+3)]^{2}$	$[(J-M+1)(J-M+2)(J-M+3)(J-M+4)]^{1/2}$

In what follows, we shall therefore write all our expressions in terms of only A_J, B_J, and C_J.

The next step in our problem is, of course, to solve for these functions A_J, B_J, and C_J. To this end, we use the dynamical commutator [2(b)]. We choose $\mu=+1$ and $\nu=0$, and apply it on the state $\psi_M{}^J$:

$$(T_+,T_0)\psi_M{}^J=0. \quad (21)$$

From the expansion (16), it is clear that the left-hand side will now contain states $\psi_{M+1}{}^{J'}$ with J' taking values $J\pm4$, $J\pm3$, $J\pm2$, $J\pm1$, and J. This gives us nine terms, the coefficients of which must all separately vanish. Consequently, we get nine equations. The solutions of these will give us the desired expressions for A_J, B_J, and C_J. It turns out that for $J'=J\pm4$, the coefficients vanish identically, so that nontrivial relations are obtained only for the remaining seven cases. We shall discuss in the following only the latter. Furthermore, the computations involved are immensely tedious and long and we shall therefore spare the reader from them and only quote the final result, which incorporates the following independent equations:

$$JB_{J+1}A_{J+3}-(J+4)A_{J+2}B_{J+3}=0, \quad (22)$$

$$(1-2J)C_JA_{J+2}+3B_{J+1}B_{J+2}+(2J+7)A_{J+2}C_{J+2}=0. \quad (23)$$

$$-3(J-1)B_JA_{J+1}-(J-3)C_JB_{J+1}+(J+5)B_{J+1}C_{J+1}+3(J+3)A_{J+2}B_{J+2}=0, \quad (24)$$

$$3J(J+1)(J-1)B_JA_{J+1}-J(J^2+J+3)C_JB_{J+1}+J(J^2+3J-1)B_{J+1}C_{J+1}-3(J+1)(J+3)(J+4)A_{J+2}B_{J+2}=0, \quad (25)$$

$$3J(J-2)(J-3)A_JB_{J-1}+(J+1)(-J^2+J+3)B_JC_{J-1}+(J+1)(J^2+J+3)C_JB_J-3J(J+1)(J+2)B_{J+1}A_{J+1}=0, \quad (26)$$

$$(2J+5)A_{J+2}{}^2+(J+3)B_{J+1}{}^2+2C_J{}^2+(2-J)B_J{}^2+(3-2J)A_J{}^2=0, \quad (27)$$

$$-2(2J+5)(J^2+3J+2)A_{J+2}{}^2-J(J+2)B_{J+1}{}^2+C_J{}^2+(1-J^2)B_J{}^2+2J(2J-3)(J-1)A_J{}^2=0. \quad (28)$$

We shall now solve this set of Eqs. (22)–(28). While doing so, we have to keep in mind the constraints given by Eq. (17), which now read,

$$A_{J_0}=A_{J_0+1}=0, \quad B_{J_0}=0. \quad (17')$$

Let us write Eq. (22) in the form

$$(J-1)B_JA_{J+2}-(J+3)A_{J+1}B_{J+2}=0; \quad J\geqslant J_0. \quad (29)$$

Furthermore, let us consider the case $J_0=\frac{1}{2}$. We can then check that the Eqs. (22)–(28) imply that if any $A_J=0$ for $J\geqslant J_0+2$, then all A_J's and B_J's, and consequently, all C_J's vanish. Likewise, if any $B_J=0$ for $J\geqslant J_0+1$, then again the same happens. We thus have either a trivial solution of our Eqs. (22)–(28), wherein

$$A_J=B_J=C_J=0; \quad J\geqslant J_0; \quad (30)$$

or we have

$$A_{J+2}\neq0, \quad B_{J+1}\neq0, \quad C_J\neq0; \quad J\geqslant J_0. \quad (31)$$

The case corresponding to the trivial solution, Eq. (30), is of no interest to us. We shall not consider it at all. By virtue of (31), we are now free to rewrite Eq. (29) as

$$J(J+1)(J+2)(J-1)\frac{B_JB_{J+1}}{A_{J+1}}=J(J+1)(J+2)(J+3)\frac{B_{J+1}B_{J+2}}{A_{J+2}}. \quad (32)$$

Denoting the left-hand side by $\phi(J)$, we can express (32) as

$$\phi(J)-\phi(J+1)=0. \quad (33)$$

This implies that $\phi(J)$ is independent of J. We may therefore write

$$(J-1)J(J+1)(J+2)\frac{B_JB_{J+1}}{A_{J+1}}=-J_0{}^2K \quad (34)$$

or simply

$$B_JB_{J+1}=\frac{-KJ_0{}^2A_{J+1}}{(J-1)J(J+1)(J+2)}; \quad J\geqslant J_0. \quad (35)$$

Let us now substitute in Eq. (23) the expression for $B_{J+1}B_{J+2}$ that we obtain from (35) and eliminate A_{J+2} from all the terms, so that we get

$$(2J+3)(1-2J)C_J+(2J+3)(2J+7)C_{J+2}+\frac{(-3J_0{}^2K)(2J+3)}{J(J+1)(J+2)(J+3)}=0. \quad (36)$$

This can be rewritten as

$$\left[(2J+3)(1-2J)C_J+\frac{(-3J_0{}^2K)}{2J(J+1)}\right]-\left[-(2J+3)(2J+7)C_{J+2}+\frac{(-3J_0{}^2K)}{2(J+2)(J+3)}\right]=0. \quad (37)$$

Denoting the first term here by $\chi(J)$, we can write this equation as

$$\chi(J)-\chi(J+2)=0. \quad (38)$$

This equation enables us to write

$$\chi(J)=(2J+3)(1-2J)C_J-\frac{3J_0{}^2K}{2J(J+1)}\equiv C_1, \quad J\geqslant J_0+2n \quad (n=0,1,2,\cdots) \quad (39)$$

and

$$\chi(J+1)=-(2J+5)(2J+1)C_{J+1}-\frac{3J_0{}^2K}{2(J+1)(J+2)}\equiv C_2, \quad J\geqslant J_0+2n \quad (n=0,1,2,\cdots), \quad (40)$$

but the two constants C_1 and C_2 are now unrelated. To relate them to each other and to the constant K introduced earlier, we exploit Eqs. (24)–(28). Let us take the difference of Eqs. (24) and (25). This gives

$$-12(J-1)(J+1)B_J A_{J+1} - (J-2)(J+2)(2J+3)B_{J+1}C_J + (J+2)^2(2J+5)B_{J+1}C_{J+1}=0; \quad J\geqslant J_0. \quad (41)$$

Substituting in this for A_{J+1} from Eq. (35) and dropping a nonzero common factor, we get

$$12J(J-1)^2(J+1)B_J^2-(J-2)(2J+3)KJ_0^2C_J + (J+2)(2J+5)KJ_0^2C_{J+1}=0 \quad J\geqslant J_0. \quad (42)$$

We now take $J=J_0$. This gives us

$$(J_0-2)(2J_0+3)C_{J_0}=(J_0+2)(2J_0+5)C_{J_0+1}. \quad (43)$$

Thus the two series of the solutions given by Eqs. (39) and (40), involving C_1 and C_2, get connected. Unfortunately, the coefficient of C_{J_0} in (39) vanishes, so that C_{J_0} is not really known yet. We therefore do the following two things. First we exploit the vanishing of the coefficient of C_{J_0} in (39) to get

$$C_1=[-3J_0^2K/2J_0(J_0+1)]; \quad (44)$$

and next, from Eqs. (27) and (28) we get for $J=J_0$

$$[2(J_0^2+3J_0+2)(J_0+3)-J_0(J_0+2)]B_{J_0+1}^2 = -[4(J_0^2+3J_0+2)+1]C_{J_0}^2 \quad (45)$$

or simply

$$25B_{J_0+1}^2=-16C_{J_0}^2. \quad (46)$$

Now we put $J=J_0+1$ in Eq. (42) and obtain B_{J_0+1} in terms of C_{J_0+1} and C_{J_0+2}; substituting for the former the expression involving C_{J_0} from Eq. (43) and using Eqs. (39) and (44) for the latter, we get

$$(9/2K)25B_{J_0+1}^2=(6/5)C_{J_0}-(2/5)K. \quad (47)$$

Substituting (46) in (47), we get a quadratic equation for C_{J_0}, the solutions of which yield

$$C_{J_0}=-K/12 \quad \text{or} \quad C_{J_0}=K/15. \quad (48)$$

From (40), (43), and (48), we now get

$$C_2=-K/2=C_1 \quad \text{for} \quad C_{J_0}=-K/12,$$
$$C_2=11K/50 \quad \text{for} \quad C_{J_0}=K/15. \quad (49)$$

It is easy to check that the latter solution is inconsistent with our Eqs. (22)–(28). We, therefore, take

$$C_1=C_2=[-3J_0^2K/2J_0(J_0+1)]. \quad (50)$$

Thus from Eqs. (39), (40), and (50) we finally get

$$C_J=\frac{3J_0K}{2(J_0+1)}\frac{(J-J_0)(J+J_0+1)}{J(J+1)(2J-1)(2J+3)}; \quad J\geqslant J_0+n \quad n=1, 2, \cdots \quad (51)$$

with C_{J_0} given by

$$C_{J_0}=\frac{(J_0+2)(2J_0+5)}{(J_0-2)(2J_0+3)}C_{J_0+1}. \quad (43')$$

From Eq. (42) we then get (fixing the arbitrary phase factor that arises from going from B_J^2 to B_J),

$$B_J=\tfrac{1}{2}iJ_0K\left(\frac{J_0}{J_0+1}\right)^{1/2}\frac{1}{J(J+1)(J-1)} \times\left[\frac{3J^2-J_0^2-J_0}{(2J+1)(2J-1)}\right]^{1/2}; \quad J\geqslant J_0+n \quad n=0, 1, 2, \cdots \quad (52)$$

Lastly, from Eq. (35) we get

$$A_{J+1}=\frac{J_0K}{4(J_0+1)}\frac{1}{J(J+1)(2J+1)} \times\left\{\frac{[3J^2-J_0^2-J_0][3(J+1)^2-J_0^2-J_0]}{(2J-1)(2J+3)}\right\}^{1/2} \quad J\geqslant J_0+n. \quad (53)$$

To these we may add the following:

$$A_{J_0}=A_{J_0+1}=B_{J_0}=0. \quad (54)$$

The Eqs. (51)–(54) along with Eqs. (8), (10), (12)–(16), and (20) now give us a representation of our group $SU(2)\times T_8$ which is characterized by $J=J_0+n$, $n=0, 1, \cdots$, with $J_0=\frac{1}{2}$. As discussed earlier, the representation contains both parities. Furthermore, it can be checked just as was done in the case of the p-wave operators, that in this representation T_0 is Hermitian and $T_{++}(T_+)$ and $T_{--}(T_-)$ are Hermitian conjugates of each other; the representation is therefore unitary. The above equations, along with Tables I and II, now furnish us with all relevant Clebsch-Gordan coefficients that occur in the coupling of the representation obtained with a p- or d-wave pion and the representation itself. In the sequel to this paper it will be seen that the knowledge of these Clebsch-Gordan coefficients is crucial in obtaining the representations of the bigger group $SU(2)_J\otimes SU(2)_I\times T_{27}$ which can be directly adapted to get the relations between various isobar widths that are experimentally known.

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APPENDIX

In this Appendix we shall obtain the Casimir operators of the group $SU(2)\times T_8$ by using the method of group contraction. We consider the Lie algebra of $SU(2)\otimes SU(3)$ defined by

$$\begin{gathered}[K_0,K_\pm]=\pm K_\pm\,,\quad [K_+,K_-]=2K_0\,,\\ [K_0,Q_\mu]=\mu Q_\mu\,,\quad [K_\pm,Q_\mu]=[(2\mp\mu)(3\pm\mu)]^{1/2}Q_{\mu+1}\,,\\ [Q_0,Q_{\pm1}]=\mp3(\sqrt{\tfrac{3}{2}})K_\pm\,,\quad [Q_+,Q_-]=-3K_0\,,\\ [Q_{+2},Q_{-2}]=6K_0\,,\quad [Q_{\pm2},Q_{\mp1}]=\pm3K_\pm\,,\quad [Q_0,Q_{\pm2}]=0\,,\\ [Q_{\pm1},Q_{\pm2}]=0\,,\quad [L_0,L_\pm]=\pm L_\pm\,,\quad [L_+,L_-]=2L_0\,,\\ [K_\alpha,L_\beta]=[L_\alpha,Q_\mu]=0\,.\end{gathered}\qquad (A1)$$

In writing down the above commutation relations we have employed the spherical rather than the Cartesian basis for both $SU(2)$ and $SU(3)$. Thus, while K_+, K_-, K_0, and Q_μ represent the generators of the group $SU(3)$, L_+, L_-, and L_0 correspond to the group $SU(2)$. The eight generators K_α and Q_μ bear the following simple relations with the usual generators F_i ($i=1$ to 8) of Gell-Mann, i.e.,

$$F_\pm=-\frac{1}{\sqrt{6}}Q_{\pm2}\,,\quad F_3=\tfrac{1}{2}K_0\,,\quad F_8=-\frac{1}{2\sqrt{3}}Q_0\,,$$

$$F_4\pm iF_5=\frac{1}{2}\left[\frac{1}{\sqrt{2}}K_\pm\pm\frac{1}{\sqrt{3}}Q_\pm\right]$$

$$F_6\pm iF_7=\frac{1}{2}\left[\frac{1}{\sqrt{2}}K_\mp\pm\frac{1}{\sqrt{3}}Q_\mp\right],$$

where F_i's satisfy the well-known commutation relations

$$[F_i,F_j]=if_{ijk}F_8\,.$$

Now we make the following transformations:

$$J_\alpha=K_\alpha+L_\alpha\,,\quad \tilde{P}_\alpha=\epsilon(K_\alpha-L_\alpha)\,,\quad\text{and}\quad T_\mu=\epsilon Q_\mu\,.\qquad (A2)$$

Substituting these in the commutation relations (A1) and taking the limit when $\epsilon\rightarrow0$, we get the following algebra:

$$\begin{gathered}[J_+,J_-]=2J_0\,,\quad [J_0,J_\pm]=\pm J_\pm\,,\\ [J_+,\tilde{P}_-]=2\tilde{P}_0\,,\quad [J_-,\tilde{P}_+]=-2\tilde{P}_0\,,\\ [J_0,T_\mu]=\mu T_\mu\,,\quad [J_\pm,\tilde{P}_0]=\mp\tilde{P}_\pm\,,\\ [J_0,\tilde{P}_\pm]=\pm\tilde{P}_\pm\,,\quad [J_\pm,T_\mu]=[(2\mp\mu)(3\pm\mu)]^{1/2}T_{\mu\pm1}\,,\end{gathered}\qquad (A3)$$

the other commutation relations reducing to zero. Further, we note that in order to describe this algebra on a completely spherical basis we require the transformation

$$\tilde{P}_+=-\sqrt{2}P_+\,,\quad \tilde{P}_-=\sqrt{2}P_-\,,\quad \tilde{P}_0=P_0\,,\qquad (A2')$$

which from (A3) finally leads to the commutation relations of the group $SU(2)\times T_8$ described in the text.

The second-order Casimir operator of the group $SU(3)$ is given in a standard form as

$$C_2=A_j{}^iA_i{}^j\quad (i,\,j=1,2,3)\,,\qquad (A4)$$

where the $A_j{}^i$'s are expressed in terms of the above generators as

$$A_1{}^1=-\tfrac{1}{2}(K_0-\tfrac{1}{3}Q_0)\,,\quad A_2{}^2=\tfrac{1}{2}(K_0+\tfrac{1}{3}Q_0)\,,\quad A_3{}^3=-\tfrac{1}{3}Q_0\,,$$

$$A_1{}^2=-\frac{1}{\sqrt{6}}Q_{+2}\,,\qquad A_2{}^1=-\frac{1}{\sqrt{6}}Q_{-2}\,,$$

$$A_3{}^1=\frac{1}{2}\left(\frac{1}{\sqrt{2}}K_--\frac{1}{\sqrt{3}}Q_{-1}\right),\quad A_1{}^3=\frac{1}{2}\left(\frac{1}{\sqrt{2}}K_++\frac{1}{\sqrt{3}}Q_{+1}\right),\qquad (A5)$$

$$A_3{}^2=\frac{1}{2}\left(\frac{1}{\sqrt{2}}K_+-\frac{1}{\sqrt{3}}Q_{+1}\right),\quad A_2{}^3=\frac{1}{2}\left(\frac{1}{\sqrt{2}}K_-+\frac{1}{\sqrt{3}}Q_{-1}\right),$$

where $A_1{}^1+A_2{}^2+A_3{}^3=0$ and $A_j{}^i$'s satisfy the commutation relations

$$[A_j{}^i,A_l{}^k]=\delta_j{}^kA_l{}^i-\delta_l{}^iA_j{}^k\,.$$

Using (A5) in (A4) we get, after simplification,

$$C_2=\tfrac{1}{6}[3K^2+(-1)^\mu Q_\mu Q_{-\mu}]\,.\qquad (A6)$$

We now multiply (A6) by ϵ^2, define $\epsilon^2C_2\equiv B$ as a second-order Casimir operator of the required group $SU(2)\times T_8$, make use of the transformations (A2) and (A2′) and take the limit when $\epsilon\rightarrow0$, which gives finally

$$B=\tfrac{1}{6}[\tfrac{3}{4}P^2+(-1)^\mu T_\mu T_{-\mu}]\,.\qquad (A7)$$

We note further that there exists also another Casimir operator of second rank corresponding to the group $SU(2)$ given by

$$C_2'=L^2\,,$$

which, on substituting (A2), gives on contraction the second-order Casimir operator

$$A=\tfrac{1}{4}P^2\,.\qquad (A8)$$

Similarly the third Casimir invariant of our group can be obtained by considering a third-order Casimir operator of the group $SU(3)$. This is given by

$$C_3=A_j{}^iA_k{}^jA_i{}^k\,.$$

Here again we use the relations (A5), employ the transformations (A2), and multiply the resulting expression by ϵ^3. Then, defining $\epsilon^3C_3\equiv C$ and taking the limit when ϵ approaches zero, we get the required expression (5) (given in text). We have further checked that the Casimir operator $(-1)^\mu T_\mu T_{-\mu}$ operating on the state vector $\psi_M{}^J$ gives

$$(-1)^\mu T_\mu T_{-\mu}\psi_M{}^J=\tfrac{1}{6}K^2\psi_M{}^J\,.$$