Connection between the Wigner Inequalities and Analyticity and Unitarity

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(Received 31 July 1967)

A lower bound on the momentum derivative of the phase shifts, $d\delta_l/dk$, is derived from the combination of unitarity, crossing, and analyticity (in the domain resulting from axiomatic field theory). As an application, a lower bound for the $\pi^0\pi^0$ scattering length is deduced.

I. INTRODUCTION

It was shown, long ago, by Wigner[1] that the derivative of the phase shifts with respect to energy must exceed a certain limit if the interaction of the scattered particle and scatterer vanishes beyond a certain distance a; for example, for the S wave,

$$d\delta_0(k)/dk > -a+(2k)^{-1}\sin 2(\delta_0+ak).$$

A similar limit has been obtained by Goebel, Karplus, and Ruderman[2] for a relativistic neutral two-particle system, again under the restriction that the interaction be of finite range.

For short-range interactions like the Yukawa or Gaussian ones, to our knowledge, no such explicit bound exists; even if one existed, it would depend on the strength parameters (or coupling constants) of the interaction. The first attempt to show that there exists in field theory an analog to Wigner's theorem was made by Bincer and Sakita.[3] Assuming the validity of the Mandelstam representation, they obtained the following bound:

$$d\delta_l/dk > -(n+\tfrac{1}{2}),$$

where n is the number of zeros of the real part of the S matrix along the unphysical cut.

In this paper, we wish to show that a lower bound for $d\delta_l/dk$ can be derived from the results of axiomatic field theory alone. Our aim is to obtain a bound which is independent of any hypothesis on the interaction or on the number of zeros of the S matrix. Our basic assumptions are as follows: (i) a certain amount of analyticity for the scattering amplitude as given by axiomatic field theory[4]; (ii) unitarity; and (iii) crossing symmetry.

The derivation proceeds in the following way: In the first step, we show that a lower bound on $d\delta_l/dk$ can be related to an upper bound of the partial waves on some curve of the analyticity domain; in the second step, this upper bound is derived from the assumptions (i), (ii), and (iii).

The definitions needed in the body of the paper as well as the precise formulation of the basic assumptions (i), (ii), and (iii) are given in Sec. II. In Sec. III, a lower bound on $d\delta_l/dk$ is expressed as a function of the partial-wave amplitude upper bound, the explicit form of which is derived in Sec. IV. As an application we show in Sec. V that the value at threshold of the bound, thus obtained, yields a lower bound on the $\pi^0\pi^0$ S-wave scattering length.

II. BASIC ASSUMPTIONS

Let us consider the $\pi^0\pi^0$ scattering amplitude $F(s,t,u)$, where $s=(\text{c.m. energy})^2$ for the first channel and t, u have analogous definitions, $s+t+u=4$. In the c.m. system for the first channel,

$$s=4(k^2+1),$$
$$t=-2k^2(1-\cos\theta),$$
$$u=-2k^2(1+\cos\theta),$$

where k is the relative momentum in the c.m. system and the π mass is taken as unity. We define

$$S_l(k)=e^{2i\delta_l(k)}=1+\frac{2ik}{\sqrt{s}}\times\int_0^1 F(s=4(1+k^2),\ t=(4-s)\lambda/2)P_l(1-\lambda)d\lambda. \quad (1)$$

(The integration goes from 0 to 1 by taking into account the Bose statistics of the $2\pi^0$ system.)

Our basic starting assumptions are the following:

(i) For any fixed t such that $|t|<4$, $F(s,t,u)$ is analytic in a cut plane in s, with two cuts from $s=4$ to $s=\infty$ and $s=-t$ to $s=-\infty$, and it is bounded by a polynomial in $|s|$; for any fixed s real $\geqslant 4$, $F(s,t,u)$ is analytic in t for $|t|<4$.

(ii) $S_l(k)$ satisfies the general unitarity requirement: $|S_l(k)|\leqslant 1$ for real k.

(iii) $F(s,t,u)$ is completely symmetric in s, t, u because of crossing symmetry.

From the assumptions made, it has been shown by Jin and Martin[5] that for $|t|<4$, $F(s,t)$ satisfies a twice-

[1] E. Wigner, Phys. Rev. 98, 145 (1955).
[2] C. J. Goebel, R. Karplus, and M. A. Ruderman, Phys. Rev. 100, 240 (1955),
[3] A. Bincer and B. Sakita, Phys. Rev. 129, 1905 (1963).
[4] A. Martin, Nuovo Cimento 44, 1219 (1966).
[5] Y. S. Jin and A. Martin, Phys. Rev. 135, B1375 (1964).

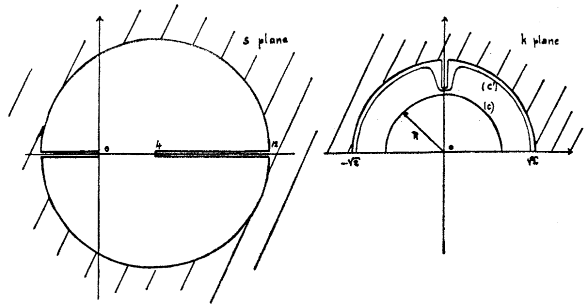

FIG. 1. The analyticity domain of $S_l(k)$ in the s plane and the k plane.

subtracted dispersion relation:

$$F(s,t)=C(t)+\frac{s^2}{\pi}\int_4^\infty \frac{A(s',t)ds'}{s'^2(s'-s)}+\frac{u^2}{\pi}\int_4^\infty \frac{A(s',t)ds'}{s'^2(s'-u)}. \quad (2)$$

This dispersion relation holds for $|t|<4$, consequently the analyticity domain of $S_l(k)$ is given by $|s-4|<8$, since $t=(4-s)\lambda/2$ with $0\leqslant\lambda\leqslant 1$ (see Fig. 1).

III. LOWER BOUND ON $d\delta_l/dk$

Let us consider the analyticity domain of $S_l(k)$ in the k plane and the semicircle (C) of radius $R<1$ (Fig. 1). Then according to a theorem due to Nevanlinna,[6] at any point k lying inside (C), one has:

$$\ln|S_l(k)|=\sum_{|k_j|\leqslant R}\ln\left|\frac{(k-k_j)(R^2-k_jk)}{(k-k_j^*)(R^2-k_j^*k)}\right| +\frac{\text{Im}k}{\pi}\int_{-R}^{R}P_1\ln|S_l(x)|dx+\frac{2R\,\text{Im}k}{\pi} \times\int_0^\pi P_2\ln|S_l(Re^{i\phi})|\sin\phi d\phi, \quad (3)$$

where

$$P_1=\frac{1}{x^2-2x\,\text{Re}k+|k|^2}-\frac{R^2}{R^4-2R^2x\,\text{Re}k+x^2|k|^2},$$

$$P_2=\frac{R^2-|k|^2}{|R^2e^{2i\phi}-2Re^{i\phi}\,\text{Re}k+|k|^2|^2},$$

and k_j are the possible zeros of $S_l(k)$. In a neighborhood of the real axis, i.e., for $k=k_1+i\epsilon$ $(\epsilon>0)$, the left-hand and right-hand sides of Eq. (3) are, respectively, left-hand side $=\ln|1-2\epsilon d\delta_l/dk_1|$ [since $|S_l(k_1)|=1$ by elastic unitarity[7]], and

$$\text{right-hand side}=2\epsilon\sum_{|k_j|\leqslant R}\frac{R^2}{(R^2-k_1\,\text{Re}k_j)^2+(k_1\,\text{Im}k_j)^2} -\frac{1}{(k_1-\text{Re}k_j)^2+(\text{Im}k_j)^2}+\frac{\epsilon}{\pi}\left\{\int_{-R}^{R}P_1\ln|S_l(x)|dx+2R \times\int_0^\pi P_2\ln|S_l(Re^{i\phi})|\sin\phi d\phi\right\}.$$

Hence, for real k, we obtain

$$d\delta_l/dk=-(\text{the contribution of the zeros}) -\frac{1}{2\pi}\int_{-R}^{R}P_1\ln|S_l(x)|dx -\frac{R}{\pi}\int_0^\pi P_2\ln|S_l(Re^{i\phi})|\sin\phi d\phi. \quad (4)$$

Now one can check easily that the first two terms are positive; therefore,

$$d\delta_l/dk\geqslant-\frac{R}{\pi}\int_0^\pi P_2\ln|S_l(Re^{i\phi})|\sin\phi d\phi. \quad (5)$$

Furthermore, if $|S_l(k)|$ has an upper bound $M_l(k)$ on the semicircle (C), since P_2 is positive, the inequality (5) implies the following one:

$$d\delta_l/dk\geqslant-\frac{R}{\pi}\int_0^\pi P_2\ln M_l(Re^{i\phi})\sin\phi d\phi, \quad (6)$$

which gives us a rigorous lower bound on $d\delta_l/dk$, once $M_l(k)$ is explicitly determined. Such a determination will constitute our next task.

Actually, to exploit fully the analyticity domain of $S_l(k)$, one should apply the Poisson formula on a curve like (C') (see Fig. 1). This can be done by mapping conformally the k plane onto a k' plane in such a way that the real axis is conserved and that the semicircle plus the unphysical cut maps onto the unit semicircle (see Fig. 2), and then applying the Nevanlinna theorem on a slightly smaller semicircle (Γ) in the k' plane. Such a transformation exists and is defined by

$$U=\frac{[(1+k^2)(4+k^2)]^{1/2}+k^2-2}{3k},$$

with

$$\text{Re}\{[(1+k^2)(4+k^2)]^{1/2}\}>0 \quad \text{for} \quad \text{Re}k>0.$$

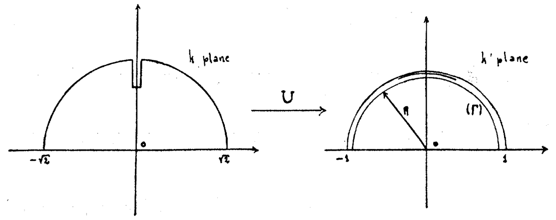

FIG. 2. The conformal mapping of the k plane onto the k' plane.

[6] R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954).

[7] This is no longer true if one considers the combination $S_l(k)=\frac{1}{3}e^{2i\delta_l^{T=0}}+\frac{2}{3}e^{2i\delta_l^{T=2}}$. However, for $k=0$, $|S_l|$ is always 1, since $\delta_l^T(0)=n\pi$.

IV. UPPER BOUND $M_l(k)$ FOR $|S_l(k)|$

According to Eq. (1), the search for the upper limit M_l on (Γ) implies the knowledge of an upper bound for $|F(s,t)|$ in a complex domain containing (Γ) and for t lying on some complex path joining 0 and $\frac{1}{2}(4-s)$. Because of the subtraction function $C(t)$, it seems very difficult to find an upper bound for $|F(s,t)|$. Nevertheless, Martin,[8] by an extensive use of assumptions (ii) and (iii), succeeded in solving the problem for real s, t, u inside the triangle $s<4, t<4, u<4$. However, the method does not apply for physical or complex values of s, t, u (which we need for our purpose).

In fact, using again assumption (iii), we can write

$$F(s,t)=F(s,t)-F(s_0,t)+F(t,s_0)-F(t_0,s_0)+F(s_0,t_0). \quad (7)$$

Now, if s_0, t_0 are chosen inside the triangle $s<4$, $t<4$, $u<4$, an upper bound for $F(s_0,t_0)$ is known. A good choice seems to be $s_0=t_0=2$; then $F(2,2)<37$. On the other hand, the first four terms, taken as differences, no longer contain the arbitrary subtraction constants, and, after some algebraic manipulations, one gets

$$F(s,t)-F(2,t)=\frac{(s-2)(s-2+t)}{\pi}\times\int_4^\infty\frac{A(s',t)(2s'+t-4)ds'}{(s'-s)(s'-4+t+s)(s'-2)(s'-2+t)} \quad (8)$$

and

$$F(t,2)-F(2,2)=\frac{2(t-2)t}{\pi}\times\int_4^\infty\frac{A(s',2)(s'-1)ds'}{(s'-t)(s'-2+t)s'(s'-2)}. \quad (9)$$

We are now left with the problem of finding upper bounds for the moduli of the integrals occurring in Eqs. (8) and (9). Unfortunately, we do not have much information about the absorptive parts $A(s',2)$ and $A(s',t)$. However, if we consider the difference

$$F(3,2)-F(2,2)=\frac{6}{\pi}\int_4^\infty\frac{A(s',2)(s'-1)ds'}{(s'-3)(s'+1)(s'-2)s'}, \quad (10)$$

it is positive. Furthermore, from Martin's results,[8] $|F(3,2)|<150$ and $|F(2,2)|<37$; hence

$$0<\frac{6}{\pi}\int_4^\infty\frac{A(s',2)(s'-1)ds'}{(s'-3)(s'+1)(s'-2)s'}<187. \quad (11)$$

[8] A. Martin, in *Proceedings of the Seminar in High-Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), p. 155.

Now, it is easy to show that

$$\left|\int_4^\infty\frac{A(s',2)(s'-1)ds'}{(s'-t)(s'-2+t)(s'-2)s'}\right|<\operatorname*{Sup}_{4\leqslant s'}\left|\frac{(s'+1)(s'-3)}{(s'-t)(s'-2+t)}\right|\times\int_4^\infty\frac{A(s',2)(s'-1)ds'}{(s'-3)(s'+1)(s'-2)s'}. \quad (12)$$

Putting this in Eq. (9) and taking into account the inequality (11), we find

$$|F(t,2)-F(2,2)|<(187/3)|t(t-2)|\times\operatorname*{Sup}_{4\leqslant s'}\left|\frac{(s'+1)(s'-3)}{(s'-t)(s'-2+t)}\right|. \quad (13)$$

An upper bound for $|F(s,t)-F(2,t)|$ is slightly more difficult to obtain. However, by noticing that $|A(s',t)|\leqslant A(s',|t|)$, we have

$$|F(s,t)-F(2,t)|\leqslant\frac{|(s-2)(s-2+t)|}{\pi}\times\int_4^\infty\frac{A(s',|t|)|2s'+t-4|ds'}{|s'-s||s'-u||s'-2+t|(s'-2)}. \quad (14)$$

Let us consider now the difference

$$F(4-|t|,\,|t|)-F((4-|t|)/2,\,|t|)=\frac{(4-|t|)^2}{\pi}\times\int_4^\infty\frac{A(s',|t|)ds'}{s'(s'-4+|t|)(2s'-4+|t|)}; \quad (15)$$

it is positive. Furthermore, for $|t|<4$ (Ref. 9),

$$F(4-|t|,|t|)=F(|t|,0)\leqslant F(4,0)\leqslant 0$$

and

$$F((4-|t|)/2,|t|)\geqslant F(\tfrac{4}{3},\tfrac{4}{3})\geqslant -100,$$

since the absolute minimum for F is attained at the symmetry point. Therefore, for $|t|<4$, it is clear that

$$0<\frac{(4-|t|)^2}{\pi}\int_4^\infty\frac{A(s',|t|)ds'}{s'(s'-4+|t|)(2s'-4+|t|)}<F(4,0)-F(4/3,4/3)<100. \quad (16)$$

If we compare Eqs. (14) and (16), and use again the same arguments as for deriving formula (13), we can find

$$|F(s,t)-F(2,t)|<\frac{100|s-2||s-2+t|}{(4-|t|)^2}\times\operatorname*{Sup}_{s'\geqslant 4}\left|\frac{(2s'+t-4)(2s'+|t|-4)s'(s'-4+|t|)}{(s'-s)(s'-u)(s'-2+t)(s'-2)}\right|. \quad (17)$$

[9] Here we restrict ourselves to negative values of $F(4,0)$.

Finally, collecting the results given by Eqs. (13) and (17), we obtain

$$|F(s,t)|<37+(187/3)|t(t-2)| \times \operatorname*{Sup}_{s'\geqslant 4}\left|\frac{(s'+1)(s'-3)}{(s'-t)(s'-2+t)}\right|+100\frac{|(s-2)(s-2+t)|}{(4-|t|)^2} \times \operatorname*{Sup}_{s'\geqslant 4}\left|\frac{(2s'+t-4)(2s'+|t|-4)s'(s'-4\times|t|)}{(s'-s)(s'-u)(s'-2+t)(s'-2)}\right|. \quad (18)$$

The right-hand side of this inequality, when introduced in Eq. (1), gives us an upper bound $M_l(k)$ for $|S_l(k)|$.

Evidently, this bound becomes meaningless when s, t, u are such that the denominators of the right-hand side of the inequality (18) vanish. It may occur for $|s'-s|\to 0$ or $|s'-u|\to 0$. These latter singularities are harmless since they lie on the unphysical cut in the k plane or on the unit circle in the k' plane. The $|s'-s|\to 0$ singularities are more serious because they occur for physical s or, equivalently, in the k' plane when $\operatorname{Im}k'\to 0$. However, in formula (6), they are inserted under the logarithm and then multiplied by $\sin\phi$ which precisely behaves like $\operatorname{Im}k'$ on any semicircle (Γ), so that the right-hand side of formula (6) is always finite.

V. APPLICATIONS TO THE $\pi^0\pi^0$ S-WAVE SCATTERING LENGTH

The S-wave scattering length is defined as

$$a_0{}^T=\lim_{k\to 0}\left\{\frac{\exp[i\delta_0{}^T(k)]\sin\delta_0{}^T(k)}{k}\right\}=\lim_{k\to 0}\frac{\tan\delta_0{}^T(k)}{k}.$$

Expanding $\delta_0{}^T$ around $k=0$ and, using the fact that $\delta_0{}^T(0)=n\pi$,

$$a_0{}^T=\lim_{k\to 0}\frac{\tan(kd\delta_0{}^T(0)/dk+\cdots)}{k}=\frac{d\delta_0{}^T(0)}{dk}.$$

For the real $\pi^0\pi^0$ system, we have

$$S_0{}^{00}=\tfrac{1}{3}e^{2i\delta_0{}^0}+\tfrac{2}{3}e^{2i\delta_0{}^2}=e^{2i\delta_0{}^{00}}$$

The derivative of this expression taken at threshold along with the property $\delta_0{}^T(0)=n\pi$, yields

$$d\delta_0{}^{00}(0)/dk=\tfrac{1}{3}a_0{}^0+\tfrac{2}{3}a_0{}^2=a_0{}^{00}.$$

Therefore, formula (6), considered at threshold,[10] gives a lower bound for the $\pi^0\pi^0$ S-wave scattering length. We have computed this lower bound, using Eqs. (18), (1), and (6), and the numerical result is

$$a_0{}^{00}>-4\text{pion Compton wavelengths},$$

while Martin and Lukaszuk[11] obtained $a_0{}^{00}>-18$ from the same assumptions but by a different method.

Let us mention, for comparison, some values or bounds for $a_0{}^{00}$ found by several authors: Goebel,[12] using forward dispersion relations and, neglecting the higher partial waves in a limited energy region, got the value -0.33 as a bound for $a_0{}^{00}$.

The π-N[13] analysis yields $a_0{}^{00}\simeq 0.4$, assuming that $a_0{}^{T=2}=0$. From current algebra and the hypothesis of partially conserved axial-vector current, Weinberg[14] found $a_0{}^{00}\simeq 0.022$.

In conclusion, we think that the result obtained in this work demonstrates once more the severe restriction imposed by analyticity, unitarity, and crossing symmetry on physical observables in spite of the crudeness of the mathematical techniques we have used. Our value might, of course, be improved in different ways and the most obvious one would be to keep the integral over the physical region which gives a positive contribution, but then we have to know something about the inelasticity in $\pi^0\pi^0$ processes.

ACKNOWLEDGMENT

We are deeply indebted to A. Martin for suggesting this investigation and for discussions.

[10] Note that in this threshold case $S_0{}^{00}(0)=1$, so that we do not need to use elastic unitarity and can then allow the competition of the $\pi^0\pi^0\to$ (anything) reactions.

[11] L. Lukaszuk and A. Martin, Cambridge University Report, 1967 (unpublished).

[12] C. J. Goebel, in the Rapporteur talk by F. Low at the *Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, Calif., 1967).

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[14] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).